Math. Proc. Camb. Phil. Soc. (2004), **137**, 209 © 2004 *Cambridge Philosophical Society*
DOI: 10.1017/S0305004103007436 *Printed in the United Kingdom*

A new way of constructing examples in operator ergodic theory

BY YURI TOMILOV†

Department of Mathematics and Informatics, Nicholas Copernicus University, Chopin Str. 12/18, 87-100 *Torun, Poland.*
e-mail: tomilov@mat.uni.torun.pl

AND JAROSLAV ZEMÁNEK

Institute of Mathematics, Polish Academy of Sciences, P.O. Box 21, 00-956 *Warsaw, Poland.*
e-mail: zemanek@impan.gov.pl

(*Received* 17 *July* 2002)

Abstract

We propose a method of constructing examples in operator ergodic theory which unifies and extends some previously known examples. It also allows us to answer several questions that have been open for some time (including a question of Allan [**1**]).

1. *Introduction*

Let T be a bounded linear operator on a complex Banach space X. We say that T is *Cesàro ergodic* if

$$A_n(T) := \frac{1}{n+1}\sum_{k=0}^{n} T^k \quad \text{converge strongly on } X. \tag{1·1}$$

Then $X = \mathrm{Ker}(T-I) \oplus \overline{\mathrm{Im}(T-I)}$, and the limit operator P is the projection of X onto $\mathrm{Ker}(T-I)$ along $\overline{\mathrm{Im}(T-I)}$; see e.g. [**13**, corollary VIII·5·2]. It is well known that T is Cesàro ergodic if it is power bounded, that is,

$$\sup_{n\geqslant 0}\|T^n\| < \infty, \tag{1·2}$$

and the space X is reflexive. However, (1·2) is not necessary for (1·1) even if X is a Hilbert space [**11**, p. 453, remark 1]; see also Example 3·1 below.

On the other hand, the *Cesàro boundedness* of T, that is

$$\sup_{n\geqslant 0}\|A_n(T)\| < \infty, \tag{1·3}$$

as well as the condition

$$\lim_{n\to\infty}\frac{1}{n}\|T^n x\| = 0, \quad \text{for every } x \in X, \tag{1·4}$$

† Partially supported by KBN grant and INTAS Young Scientists Fellowship.

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

are certainly necessary for (1·1). We note that (1·3) and (1·4) are, in general, mutually independent; see e.g. [**17**, remark 4]. If the space X is reflexive, then, conversely, (1·3) together with (1·4) imply (1·1); see [**13**, corollary VIII·5·4]. If only (1·3) is assumed, then (1·1) may fail as the well-known Assani example,

$$\mathcal{T} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{1·5}$$

shows [**15**]. It is not clear to what extent the Cesàro bounded operators share ergodic properties with the power bounded operators. The subtlety of the question can be illustrated by the fact that a certain 'quasi-ergodic' theorem always holds for an arbitrary uniformly bounded family of operators on a Hilbert space, see [**9**, lemma 4]. On the positive side, Émilion proved in [**15**] that (1·1) is true for any positive Cesàro bounded operator on a reflexive Banach lattice (in particular on l_2). Here the reflexivity cannot be dropped in general [**18**]. This result was generalized by Derriennic in [**11**], where the positivity of T was replaced by a certain geometrical condition on $\{T^n : n \geqslant 0\}$. However, in the same paper, Derriennic showed that Cesàro bounded operators (even on Hilbert spaces) may have rather 'pathological' behaviour of Cesàro averages in contrast to the well-studied power bounded operators. We present these examples in a unified and simplified way, strengthen some of them and find new ones.

Suppose T is Cesàro ergodic. The restriction of T to $\operatorname{Ker}(T-I)$ is I, and so has a very simple form. The restriction T_0 of T to $\overline{\operatorname{Im}(T-I)}$ satisfies

$$\lim_{n\to\infty} A_n(T_0)x = 0,$$

for every $x \in \overline{\operatorname{Im}(T-I)}$. In general, nothing more can be said about T_0. However, under additional spectral assumptions, quite strong conclusions can be made. In 1986, Katznelson and Tzafriri [**22**] proved that if T is a power bounded linear operator on a Banach space X, and

$$\sigma(T) \bigcap \{\lambda \in \mathbb{C} : |\lambda| = 1\} \subset \{1\} \tag{1·6}$$

holds, then

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0. \tag{1·7}$$

(If $\sigma(T) = \{1\}$, this was proved earlier by Esterle in [**16**]. Actually, the general case can be reduced to this one as was noticed by Vũ [**34**].) Thus, under (1·2) and (1·6), we have ${T_0}^n \to 0$, $n \to \infty$ strongly on $\overline{\operatorname{Im}(T-I)}$, and if X is reflexive, then the powers of T converge strongly on X, a conclusion stronger than (1·1). The Esterle–Katznelson–Tzafriri theorem substantially influenced not only abstract ergodic theory (e.g. uniform ergodic theorems), but also operator theory (e.g. asymptotics of operator semigroups) and function theory (e.g. Tauberian theorems).

It is natural to want to understand the limits of such a good result. With this in mind, note that (1·7) implies (1·6) in view of the spectral mapping theorem, and it also implies

$$\lim_{n\to\infty} \frac{1}{n}\|T^n\| = 0, \tag{1·8}$$

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

by

$$\left\|\frac{T^n}{n}\right\| \leqslant \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{n} \|T^k - T^{k-1}\|. \tag{1·9}$$

In 1989, Allan asked whether a bounded linear operator T on a Banach space X satisfying (1·8) and

$$\sigma(T) = \{1\},$$

the "extremal" spectral condition, also has property (1·7); see [**1**, p. 7] (the formulation contains a misprint but the right formulation has been communicated by Allan orally). We show that the answer to this question is negative. Moreover, our technique allows us to construct a Banach space X and a bounded linear operator T on X such that the properties (1·3), (1·6) and (1·8) hold, but

$$\lim_{n\to\infty} \|T^n(T-I)^m\| = \infty,$$

for every $m \in \mathbb{N} \cup \{0\}$. Thus, the Esterle–Katznelson–Tzafriri theorem fails *in a dramatic way* if the condition of power boundedness of T is replaced by the nearby conditions (1·3) and (1·8).

We denote by $\mathcal{L}(X)$ the space of bounded linear operators on X, and by $\sigma(T)$ and $R(\lambda, T) := (T - \lambda I)^{-1}$ the spectrum and the resolvent of T, respectively. The identity operator on different Banach spaces will always be denoted by the same symbol I.

2. *A general construction*

All examples in this paper will depend on the following transparent and simple matrix construction. In some sense, it is an 'inverse' of a construction proposed in [**8**]. It emerges naturally if one tries to understand the different approaches to the Ritt condition (the property (2·2) below) studied recently in [**7**, **14**, **24**–**26**, **28**]. Note also [**3**, section 3] where the ergodicity of matrix semigroups was studied.

Consider the Banach space $\mathcal{X} = X \oplus X$ with the norm

$$\|x_1 \oplus x_2\|_{X\oplus X} := \sqrt{\|x_1\|^2 + \|x_2\|^2},\ x_1 \oplus x_2 \in X \oplus X.$$

Note that if X is a Hilbert space, then $\mathcal{X}$ is a Hilbert space too. Let the bounded linear operator $\mathcal{T}$ on $\mathcal{X}$ be defined by the operator matrix

$$\mathcal{T} := \begin{pmatrix} T & T-I \\ O & T \end{pmatrix}, \tag{2·1}$$

where $T \in \mathcal{L}(X)$. The operator $\mathcal{T}$ is an extension of T to the larger space $\mathcal{X} \supset X$ and coincides with T on its invariant subspace X.

Lemma 2·1. *We have*:

(i) $\sigma(\mathcal{T}) = \sigma(T)$;

(ii) $\lim_{n\to\infty} \frac{1}{n}\|\mathcal{T}^n\| = 0$ *if and only if* $\lim_{n\to\infty} \|T^n - T^{n+1}\| = 0$;

(iii) $\sup_{n\geqslant 1} \|A_n(\mathcal{T})\| < \infty$ *if and only if* $\sup_{n\geqslant 1} \|T^n\| < \infty$;

(iv) *the averages* $A_n(\mathcal{T})$ *converge in the strong topology of* $\mathcal{X}$ *if and only if the powers* T^n *converge in the strong topology of* X;

(v) *the averages* $A_n(\mathcal{T})$ *converge in the weak topology of* $\mathcal{X}$ *if and only if the powers* T^n *converge in the weak topology of* X;

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

(vi) $\sup_{n\geqslant 0}\|\mathcal{T}^n\|<\infty$ *if and only if* T *satisfies the Ritt condition*

$$\|R(\lambda,T)\|\leqslant\frac{C}{|\lambda-1|},\quad |\lambda|>1, \tag{2·2}$$

for some $C>0$;

(vii) *for a fixed* $m\in\mathbb{N}$, *we have* $\lim_{n\to\infty}\|\mathcal{T}^n(\mathcal{T}-I)^m\|=0$ *if and only if* $\lim_{n\to\infty}\|T^n(T-I)^m\|=0$ *and* $\lim_{n\to\infty}n\|T^n(T-I)^{m+1}\|=0$;

(viii) *the operator* $\mathcal{T}$ *satisfies the Ritt condition if and only if* T *satisfies the Ritt condition.*

Proof. (i) Obvious.

(ii) Since

$$\mathcal{T}^n=\begin{pmatrix}T^n & nT^{n-1}(T-I)\\ O & T^n\end{pmatrix}, \tag{2·3}$$

it suffices to observe that $\lim_{n\to\infty}\|T^n-T^{n+1}\|=0$ implies $\lim_{n\to\infty}\frac{1}{n}\|T^n\|=0$ by (1·9).

(iii) To prove (iii) note that

$$A_n(\mathcal{T})=\frac{1}{n+1}\sum_{k=0}^{n}\mathcal{T}^k=\begin{pmatrix}\frac{1}{n+1}\sum_{k=0}^{n}T^k & \frac{1}{n+1}\sum_{k=1}^{n}kT^{k-1}(T-I)\\ O & \frac{1}{n+1}\sum_{k=0}^{n}T^k\end{pmatrix}.$$

Since

$$\sum_{k=1}^{n}kT^{k-1}(T-I)=nT^n+\sum_{k=1}^{n}(k-1)T^{k-1}-\sum_{k=1}^{n}kT^{k-1}=nT^n-\sum_{k=1}^{n}T^{k-1}$$

we obtain

$$A_n(\mathcal{T})=\begin{pmatrix}\frac{1}{n+1}\sum_{k=0}^{n}T^k & \frac{n}{n+1}T^n-\frac{1}{n+1}\sum_{k=0}^{n-1}T^k\\ O & \frac{1}{n+1}\sum_{k=0}^{n}T^k\end{pmatrix}. \tag{2·4}$$

If $\|A_n(\mathcal{T})\|$ is bounded, then from (2·4) we first get that $\|A_n(T)\|$ is bounded, and then, looking at the right upper corner of that matrix, we conclude that $\sup_{n\geqslant 0}\|T^n\|<\infty$. The converse is also clear from (2·4).

(iv) and (v) are consequences of the formula (2·4), too. For the proof of necessity note that for any $\mathbf{x}\in\mathcal{X}$ of the form $\mathbf{x}=x\oplus x\in\mathcal{X}$, we have

$$A_n(\mathcal{T})\mathbf{x}=\begin{pmatrix}T^nx\\ \frac{1}{n+1}\sum_{k=0}^{n}T^kx\end{pmatrix}.$$

(vi) This is a 'matrix' reformulation of [**26**, theorem, p. 147]; see also [**24**, theorem 1] and [**28**, theorem 2·1].

(vii) For the proof of (vii), observe that

$$\begin{aligned}\mathcal{T}^n(\mathcal{T}-I)^m&=\begin{pmatrix}T^n & nT^{n-1}(T-I)\\ O & T^n\end{pmatrix}\times\begin{pmatrix}(T-I)^m & m(T-I)^m\\ O & (T-I)^m\end{pmatrix}\\&=\begin{pmatrix}T^n(T-I)^m & mT^n(T-I)^m+nT^{n-1}(T-I)^{m+1}\\ O & T^n(T-I)^m\end{pmatrix}.\end{aligned} \tag{2·5}$$

Then the latter matrix representation of $\mathcal{T}^n(\mathcal{T}-I)^m$ gives the conclusion.

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

(viii) This follows directly from (i) and the matrix representation

$$R(\lambda,\mathcal{T}) = \begin{pmatrix} R(\lambda,T) & R^2(\lambda,T)(I-T) \\ O & R(\lambda,T) \end{pmatrix}$$
$$= \begin{pmatrix} R(\lambda,T) & -R(\lambda,T) - (\lambda-1)R^2(\lambda,T) \\ O & R(\lambda,T) \end{pmatrix}.$$

The gap between Cesàro bounded and Cesàro ergodic operators $\mathcal{T}$ in this construction is the gap between operators T with uniformly bounded powers and with strongly convergent powers. The latter one is quite large.

Remark 2·2. All the statements of the above lemma remain true if one replaces T by T^*, and, correspondingly, $\mathcal{T}$ by

$$\mathcal{T}^* = \begin{pmatrix} T^* & O \\ T^* - I & T^* \end{pmatrix}, \tag{2·6}$$

and $\mathcal{X}$ by $\mathcal{X}^* = X^* \oplus X^*$. The verification is left to the reader.

Remark 2·3. Observe that if there exists $T^{-1} \in \mathcal{L}(X)$, then

$$\mathcal{T}^{-1} = \begin{pmatrix} T^{-1} & T^{-1}(T^{-1} - I) \\ O & T^{-1} \end{pmatrix}. \tag{2·7}$$

A reasoning similar to the proof of Lemma 2·1(iii) shows that $\mathcal{T}^{-1}$ is Cesàro bounded if and only if T^{-1} is power bounded.

Remark 2·4. The Assani example (1·5) corresponds essentially to the choice $T = -I$ in (2·1).

As an immediate illustration of Lemma 2·1 we give the following statement.

Corollary 2·5. *Let K be a compact subset of the unit disc $\{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$ such that the intersection $K \bigcap \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is nonempty. Then there exist a Hilbert space X and a Cesàro bounded but not power bounded linear operator T on X such that*

$$\sigma(T) = K.$$

Proof. Suppose that $1 \notin K$. Let $X_1 = l_2(\mathbb{N}, \mathbb{C})$. Let T_1 be the diagonal operator on X_1,

$$T_1\big((x_n)_{n=1}^{\infty}\big) = (\lambda_n x_n)_{n=1}^{\infty}, \quad (x_n)_{n=1}^{\infty} \in X_1,$$

where $\overline{\{\lambda_n : n \geqslant 1\}} = K$. Note that T_1 is a contraction on X_1, and $\sigma(T_1) = K$. Since the intersection $\sigma(T_1) \bigcap \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is nonempty by assumption, it must contain a point different from 1, consequently, the operator T_1 does not satisfy the Ritt condition. Thus the corresponding matrix operator $T := \mathcal{T}_1$ on the Hilbert space $X := \mathcal{X}_1$ is not power bounded by Lemma 2·1(vi), but it is Cesàro bounded in view of Lemma 2·1(iii). Since $\sigma(T_1) = K$, we have $\sigma(T) = K$ by Lemma 2·1(i).

Now, let $1 \in K$. Consider the Volterra operator

$$(Vf)(t) := \int_0^t f(s)\, ds$$

on $X_2 = L_2((0,1))$. Then $T_2 = I - V$ is power bounded; see e.g. [**26**, p. 145]. The operator

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

T_2 does not satisfy the Ritt condition (2·2). Indeed, using the well-known formula for the resolvent of T_2,

$$(R(\lambda, T_2)f)(t) = -\frac{1}{\lambda - 1} f(t) + \frac{1}{(\lambda - 1)^2} \int_0^t e^{-(t-s)/(\lambda-1)} f(s)\, ds, \quad \lambda \neq 1,$$

we obtain

$$\limsup_{n\to\infty} |(1 + in^{-1}) - 1| \, \|R(1 + in^{-1}, T_2)e^{in\cdot}\| = \infty.$$

(Alternatively, one can apply [**25**, theorem, p. 137].) By Lemma 2·1(i), we have $\sigma(\mathcal{T}_2) = \{1\}$. By Lemma 2·1(iii),(vi), the matrix operator $\mathcal{T}_2$ corresponding to T_2 is Cesàro bounded but not power bounded. Then the operator $T := T_1 \oplus \mathcal{T}_2$ on the Hilbert space $X := X_1 \oplus \mathcal{X}_2$ is Cesàro bounded but not power bounded. Moreover, $\sigma(T) = K$.

Note that the assumptions of Corollary 2·5 are necessary for the conclusion. For Hilbert space nonunitary contractions the spectral conclusion of Corollary 2·5 is essentially well known. See, for example, [**30**, theorem 2] for the proof.

The following simple estimate for the norms of operator matrices will be used frequently.

LEMMA 2·6. *Let $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ be defined by the operator matrix $\mathcal{T} := (T_{ij})_{i,j=1}^2$, $T_{ij} \in \mathcal{L}(X)$. Then*

$$\max_{1\leqslant i,j\leqslant 2} \|T_{ij}\| \leqslant \|\mathcal{T}\| \leqslant \sqrt{\sum_{i,j=1}^{2} \|T_{ij}\|^2}.$$

The proof is straightforward and is, therefore, omitted.

3. *Examples*

In this section we will present a series of examples showing a striking difference between 'ergodic' behaviour of power bounded and Cesàro bounded operators.

Define the Hilbert space $\mathcal{X}$ as the orthogonal sum of two copies of l_2, that is $\mathcal{X} := l_2 \oplus l_2$, where l_2 will mean either $l_2(\mathbb{N}, \mathbb{C})$ or $l_2(\mathbb{Z}, \mathbb{C})$.

According to the mean ergodic theorem [**23**, theorem 2·1·1], an operator is Cesàro ergodic if it is power bounded and its Cesàro averages are weakly convergent. On a reflexive space, a power bounded operator and its adjoint are simultaneously Cesàro ergodic (see the Introduction). Examples 3·1 and 3·3 below show that these two properties may fail if the operator in question is merely Cesàro bounded, even on a Hilbert space.

Example 3·1. There exists a $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ such that:
(a) $A_n(\mathcal{T})$ converge strongly;
(b) $A_n(\mathcal{T}^*)$ do not converge strongly;
(c) $\|\mathcal{T}^n\| \geqslant 2n$ for all $n \in \mathbb{N}$.

Let $X = l_2(\mathbb{N}, \mathbb{C})$. Let $T \in \mathcal{L}(X)$ be the backward shift:

$$T\big((x_n)_{n=1}^\infty\big) = (x_{n+1})_{n=1}^\infty, \quad (x_n)_{n=1}^\infty \in X.$$

Then $T^n \to 0$, $n \to \infty$, strongly on X. So by Lemma 2·1(iv), the corresponding matrix

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

operator $\mathcal{T}$ on $\mathcal{X}$ satisfies (a). However, $T^{*n}x$ do not converge in X for all $x \in X$. Thus $\mathcal{T}^*$ satisfies (b) by Remark 2·2 and Lemma 2·1(iv). Lemma 2·6 and (2·3) yield

$$\|\mathcal{T}^n\| \geqslant n\|T^{n-1}(T-I)\| \geqslant n \sup_{\lambda\in\sigma(T)} |\lambda^{n-1}(\lambda-1)| = 2n, \quad n \geqslant 1,$$

hence T satisfies also (c).

Moreover, we can conclude that $A_n(\mathcal{T}^*)\mathbf{x}$ do not converge for any $\mathbf{x}$ of the form $\mathbf{x} = x \oplus x \in \mathcal{X}, x \neq 0$. Indeed, observe that for such $\mathbf{x}$ we have

$$A_n(\mathcal{T}^*)\mathbf{x} = \begin{pmatrix} \frac{1}{n+1}\sum_{k=0}^{n} T^{*k}x \\ T^{*n}x \end{pmatrix},$$

which is divergent.

Remark 3·2. Note that $\|\mathcal{T}^n\| \geqslant 2n, n \in \mathbb{N}$, is clearly the fastest possible growth for a Cesàro bounded operator.

Example 3·3. There exists a $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ such that:

(a) both $A_n(\mathcal{T})$ and $A_n(\mathcal{T}^*)$ converge weakly;
(b) neither $A_n(\mathcal{T})$, nor $A_n(\mathcal{T}^*)$ converge strongly;
(c) $\|\mathcal{T}^n\| \geqslant 2n$ for all $n \in \mathbb{N}$.

Let X be $l_2(\mathbb{Z}, \mathbb{C})$. Let $T \in \mathcal{L}(X)$ be the backward shift:

$$T((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}, \quad (x_n)_{n\in\mathbb{Z}} \in l_2(\mathbb{Z}, \mathbb{C}).$$

Let $\mathcal{T}$ be the operator corresponding to T by Lemma 2·1. By applying Lemma 2·1(iv),(v) to the contractions T, T^*, respectively, we conclude that $\mathcal{T}$ has the properties (a) and (b). As before, (2·3) yields

$$\|\mathcal{T}^n\| \geqslant 2n, \quad n \geqslant 1.$$

Moreover, $A_n(\mathcal{T}^*)\mathbf{x}$ diverge for any nonzero $\mathbf{x} \in \mathcal{X}$ of the form $\mathbf{x} = x \oplus x$.

Remark 3·4. Operators possessing property (b) of the above example can be constructed already on the two-dimensional space $\mathbb{C}^2$ using the Assani matrix (1·5). However, Lemma 2·1 makes it possible to provide such examples with various additional properties. We shall not go into details here.

The next example concerns mixing properties of Cesàro bounded operators. If T is a power bounded operator on a Banach space, then the averages $n^{-1}\sum_{i=1}^{n} T^{k_i}$ are uniformly bounded for all strictly increasing subsequences $\{k_i : i \geqslant 1\}$ of $\mathbb{N} \cup \{0\}$. This makes it possible to study ergodic properties of T along subsequences (see, for example, [**5**, p. 242] and the references therein). However, if T is merely Cesàro bounded, then the averages $n^{-1}\sum_{i=1}^{n} T^{k_i}$ may be unbounded along some subsequence $\{k_i : i \geqslant 1\}$.

Example 3·5. There exists a $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ such that:

(a) $A_n(\mathcal{T})$ converge strongly;
(b) $\lim_{n\to\infty} \|A_n(\mathcal{T}^2)\| = \infty$.

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

To see this it suffices to consider $\mathcal{T}$ with T being the backward shift on on $l_2(\mathbb{N},\mathbb{C})$. Consider

$$\mathbf{x} = 0 \oplus x,\ x = (x_n)_{n=1}^{\infty},\ x_n = \frac{(-1)^n}{\sqrt{n}\ln(n+1)}, n \geqslant 1,$$

and look at the first coordinate of the first component of the vector $A_n(\mathcal{T}^2)\mathbf{x}$. Using (2·3) and the inequality $x^{\frac{1}{4}} \geqslant \frac{1}{8}\ln(2x+1), x \geqslant 0$, we have

$$\begin{aligned}\liminf_{n\to\infty} \|A_n(\mathcal{T}^2)\mathbf{x}\| &\geqslant \liminf_{n\to\infty} \frac{1}{n+1}\left\|\sum_{k=1}^{n} 2k(T^{2k}-T^{2k-1})x\right\| \\ &\geqslant \liminf_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n}\frac{\sqrt{k}}{\ln(2k+1)} \\ &\geqslant \liminf_{n\to\infty} \frac{1}{8(n+1)}\sum_{k=1}^{n} k^{\frac{1}{4}} = \infty.\end{aligned}$$

Remark 3·6. Note that Examples 3·1, 3·3 and 3·5 are not possible on finite-dimensional spaces. Indeed, the property (a) of each of these examples implies already $\sup_{n\geqslant 0}\|\mathcal{T}^n\| < \infty$; see [**11**, p. 449] and [**34**, p. 43, exercise 6]. For Riesz operators see [**35**, theorem 7].

The classical Sz.–Nagy similarity criterion states that if a bounded linear operator T on a Hilbert space satisfies $\sup_{n\in\mathbb{Z}}\|T^n\| < \infty$, then T is similar to a unitary operator [**33**]. So ergodic properties of T in this case are essentially the same as of the corresponding unitary operator. We show in the next example that the version of the Sz.–Nagy criterion for Cesàro bounded operators is not true. This is in contrast to the Gelfand–Hille theorems where the assumption on the growth of the powers can be replaced by the same assumption on the Cesàro means [**12**].

Example 3·7. There exists a $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ such that:
(a) $\sup_{m,n\in\mathbb{Z}} \frac{1}{|m|+|n|}\|\sum_{k=n}^{m}\mathcal{T}^k\| < \infty$;
(b) $\|\mathcal{T}^n\| \geqslant 2|n|, \quad n \in \mathbb{Z}$.

Take $\mathcal{T}$ from Example 3·3. Then, by Remark 2·3 and Lemma 2·1(iii), $\mathcal{T}$ satisfies (a) since T and $T^{-1} = T^*$ are contractions. However, we have $\|\mathcal{T}^n\| \geqslant 2|n|, n \in \mathbb{Z}$, as before.

Remark 3·8. Another counterexample to the version of the Sz.–Nagy similarity criterion for Cesàro bounded operators was obtained in [**10**, p. 64]. This counterexample assumes Cesàro boundedness even in a stronger sense than our Example 3·7. However, the growth of powers of the operator considered there cannot be made faster than $\sqrt{n}$, while in the above example the growth of powers is extremal.

It would be interesting to find $\mathcal{T}$ (not necessarily of the form (2·1)) such that in Examples 3·1 and 3·3 the divergence takes place at all nonzero elements of $\mathcal{X}$.

4. *Counterexamples to Esterle–Katznelson–Tzafriri type theorems*

First we show the dramatic failure of the Esterle–Katznelson–Tzafriri theorem if the power boundedness assumption is replaced by assumptions of Cesàro boundedness and strict sublinear growth. A reasoning close in spirit was given in [**2**, pp. 77–78]. Our example was inspired by [**27**, example 4·5·2].

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

Example 4·1. There exist a Hilbert space $\mathbf{X}$ and a $\mathbf{T} \in \mathcal{L}(\mathbf{X})$ such that
(a) $\sigma(\mathbf{T}) \bigcap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \{1\}$;
(b) $\lim_{n\to\infty} \frac{1}{n}\|\mathbf{T}^n\| = 0$;
(c) $\sup_{n\geqslant 0}\|A_n(\mathbf{T})\| < \infty$;
and, at the same time,

$$\lim_{n\to\infty} \|\mathbf{T}^n(\mathbf{T} - I)^m\| = \infty \tag{4·1}$$

for every $m \in \mathbb{N} \bigcup \{0\}$.

Let $T(k)$ be a multiplication operator on $L_2((0,1))$ defined by

$$(T(k)x)(t) = te^{i\frac{(1-t)^{\frac{1}{k}}}{k}}x(t), \quad k \geqslant 1,$$

and let $\mathcal{T}(k)$ be the operator corresponding to $T(k)$ by Lemma 2·1. We note that $\sigma(T(k)) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\} \bigcup \{1\}$, $\|T(k)\| \leqslant 1$, and, moreover,

$$\overline{\bigcup_{k=1}^{\infty} \sigma(T(k))} \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\} \bigcup \{1\}. \tag{4·2}$$

Indeed, assume, on the contrary, that there are sequences $\{t_n : n \geqslant 1\} \subset [0,1]$ and $\{k_n : n \geqslant 1\}$ such that

$$\lim_{n\to\infty} t_n e^{i\frac{(1-t_n)^{\frac{1}{k_n}}}{k_n}} = e^{i\phi}$$

for some $\phi \in [0, 2\pi)$. Then $t_n \to 1, n \to \infty$, and

$$\phi = \lim_{n\to\infty} \frac{(1-t_n)^{\frac{1}{k_n}}}{k_n} \leqslant \limsup_{n\to\infty} \frac{1}{k_n} = 0,$$

which proves (4·2).

Now observe that for $m \in \mathbb{N}$ we have

$$\begin{aligned}\|T(k)^n(T(k) - I)^m\| &= \max_{t\in[0,1]} t^n \left|te^{i\frac{(1-t)^{\frac{1}{k}}}{k}} - 1\right|^m \\ &= \max_{t\in[0,1]} t^n \left((t-1)^2 + 4t\sin^2\frac{(1-t)^{\frac{1}{k}}}{2k}\right)^{\frac{m}{2}}.\end{aligned} \tag{4·3}$$

In particular,

$$\begin{aligned}\|T(k)^n(T(k) - I)\| &= \max_{t\in[0,1]} t^n \left((t-1)^2 + 4t\sin^2\frac{(1-t)^{\frac{1}{k}}}{2k}\right)^{\frac{1}{2}} \\ &\leqslant \max_{t\in[0,1]} t^n(1-t) + 2\max_{t\in[0,1]} t^n \sin\frac{(1-t)^{\frac{1}{k}}}{2k} \\ &\leqslant \frac{1}{n} + \frac{1}{k}\left(\frac{n}{n+\frac{1}{k}}\right)^n \left(\frac{\frac{1}{k}}{n+\frac{1}{k}}\right)^{\frac{1}{k}} \leqslant \frac{1}{n} + \frac{\frac{1}{k}}{n^{\frac{1}{k}}}.\end{aligned} \tag{4·4}$$

For a fixed $n \geqslant 3$, the function $x \to x/n^x$ achieves its maximum on $[0,1]$ at $x = 1/\ln n$. So, if $n \geqslant 3$ is fixed, then, by (4·4), we get

$$\sup_{k\geqslant 1}\|T(k)^n(T(k) - I)\| \leqslant \frac{2}{\ln n}. \tag{4·5}$$

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

Therefore, using the reasoning in Lemma 2·1(ii), we conclude that

$$\lim_{n\to\infty}\frac{1}{n}\|\mathcal{T}(k)^n\| = 0, \tag{4·6}$$

uniformly in $k\in\mathbb{N}$.

Fix an $m\in\mathbb{N}$. By (4·3), we have

$$\begin{aligned}\|T(k)^n(T(k)-I))^m\| &= \max_{t\in[0,1]} t^n\left((t-1)^2+4t\sin^2\frac{(1-t)^{\frac{1}{k}}}{2k}\right)^{\frac{m}{2}}\\ &\geqslant 2^m\max_{t\in[0,1]} t^{n+\frac{m}{2}}\left(\sin\frac{(1-t)^{\frac{1}{k}}}{2k}\right)^m\\ &\geqslant \left(\frac{2}{k\pi}\right)^m\max_{t\in[0,1]} t^{n+\frac{m}{2}}(1-t)^{\frac{m}{k}}.\end{aligned}$$

Choosing $t=(n-1)/n, n\geqslant 1$, we obtain

$$n\|T(k)^n(T(k)-I)^m\| \geqslant C(k,m)n^{1-\frac{m}{k}}, \tag{4·7}$$

for some $C(m,k)>0$. So, if $k>m$ is fixed, then

$$\lim_{n\to\infty} n\|T(k)^n(T(k)-I)^m\| = \infty.$$

Moreover, the inequality (4·5) yields

$$\lim_{n\to\infty}\|T(k)^n(T(k)-I)^m\| = 0.$$

Thus, using the formula (2·5) (Lemma 2·1(vii)) and Lemma 2·6, we have

$$\lim_{n\to\infty}\|\mathcal{T}(k)^n(\mathcal{T}(k)-I)^{m-1}\| = \infty, \tag{4·8}$$

for fixed m and k such that $k>m\geqslant 2$.

Now let

$$\mathbf{X}:=l_2-\oplus_{k=1}^{\infty}\mathcal{X}_k,\quad \mathcal{X}_k:=L_2((0,1))\oplus L_2((0,1)),$$

and let

$$\mathbf{T}:=\oplus_{k=1}^{\infty}\mathcal{T}(k).$$

The inequality

$$\|T(k)\|\leqslant 1,\quad k\geqslant 1, \tag{4·9}$$

and Lemma 2·6 imply

$$\|\mathcal{T}(k)\|\leqslant\sqrt{6},\quad k\geqslant 1,$$

so that $\mathbf{T}\in\mathcal{L}(\mathbf{X})$. Observe that

$$\|\mathbf{T}^n(\mathbf{T}-I)^m\|\geqslant\|\mathcal{T}(k)^n(\mathcal{T}(k)-I)^m\|,\quad k\geqslant 1. \tag{4·10}$$

So, by (4·8) and (4·10), we get (4·1).

Note that

$$\sigma(\mathbf{T}) = \overline{\bigcup_{k=1}^{\infty}\sigma(T(k))}. \tag{4·11}$$

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

Indeed, the linear operator

$$J(\lambda) = \bigoplus_{k=1}^{\infty} \begin{pmatrix} R(\lambda, T(k)) & R^2(\lambda, T(k))(I - T(k)) \\ O & R(\lambda, T(k)) \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus \overline{\bigcup_{k=1}^{\infty} \sigma(T(k))},$$

is bounded on $\mathbf{X}$, and satisfies

$$(\mathbf{T} - \lambda I)J(\lambda) = J(\lambda)(\mathbf{T} - \lambda I) = I.$$

Therefore, the inclusion

$$\sigma(\mathbf{T}) \subseteq \overline{\bigcup_{k=1}^{\infty} \sigma(T(k))}$$

holds. The converse inclusion is clear.

The property (a) follows from Lemma 2·1(i), (4·2) and (4·11). From (4·6) we obtain the property (b). Further, using (4·9) and (2·4) (the proof of Lemma 2·1(iii)), we obtain by Lemma 2·6

$$\sup_{n \geqslant 0} \|A_n(\mathcal{T}(k))\| \leqslant \sqrt{6}, \quad k \geqslant 1,$$

so that

$$\sup_{n \geqslant 0} \|A_n(\mathbf{T})\| \leqslant \sqrt{6}. \tag{4·12}$$

Thus (c) is true. Finally, the Esterle–Katznelson–Tzafriri theorem implies that $\mathbf{T}$ is not power bounded, and actually (4·1) holds also for $m = 0$.

Remark 4·2. The construction of Example 4·1 shows that, moreover, the operator $\mathbf{T}$ is *Cesàro ergodic*. Indeed, observe that for every $k \in \mathbb{N}$ the powers $T(k)^n$ converge to zero strongly on $L_2((0,1))$. Therefore, by the reasoning in Lemma 2·1(iv), the averages $A_n(\mathcal{T}(k))$ converge to zero strongly on $\mathcal{X}$. Then $A_n(\mathbf{T})\mathbf{x}$ converge to zero for all $\mathbf{x} \in \mathbf{X}$ with finite support. Since the set of such $\mathbf{x}$ is dense in $\mathbf{X}$ and (4·12) holds, we conclude that $A_n(\mathbf{T})$ converge to zero on the whole of $\mathbf{X}$.

Remark 4·3. The construction of Example 4·1 also shows that for each $m_0 \in \mathbb{N}$ there exist a Hilbert space $\mathcal{X}$ and an operator $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ satisfying (a), (b) and (c) such that

$$\|\mathcal{T}^n(\mathcal{T} - I)^m\| \not\longrightarrow 0, n \longrightarrow \infty, \quad \text{for each} \quad m \leqslant m_0, \quad \text{but} \tag{4·13}$$

$$\|\mathcal{T}^n(\mathcal{T} - I)^m\| \longrightarrow 0, n \longrightarrow \infty, \quad \text{for each} \quad m > m_0. \tag{4·14}$$

Indeed, it suffices to take $\mathcal{T} = \mathcal{T}(m_0 + 1)$ and $\mathcal{X} = L_2((0,1)) \oplus L_2((0,1))$. Then (4·13) follows from (4·7) and Lemma 2·1(vii). The inequality (4·4) implies that for every $0 \leqslant r \leqslant m$,

$$\begin{aligned} &((m+1)n + r)\|T(m_0+1)^{(m+1)n+r}(T(m_0+1) - I)^{m+1}\| \\ &\quad = ((m+1)n + r)n^{-1}n\|T(m_0+1)^{(m+1)n+r}(T(m_0+1) - I)^{m+1}\| \\ &\quad \leqslant 2^{m+1}(2m+1)n^{1-\frac{m+1}{m_0+1}} \max_{0 \leqslant r \leqslant m} \|T(m_0+1)^r\|, \end{aligned} \tag{4·15}$$

so that (4·14) follows from (4·15) and Lemma 2·1(vii).

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

The next example answers in the negative a question of Allan from [**1**, p. 7]. Its construction relies on special properties of the Volterra operator and was influenced by [**19**]. First we shall need several asymptotic properties of the Laguerre polynomials

$$L_n^{(k)}(x) = \sum_{m=0}^{n} \binom{n+k}{n-m} (-1)^m \frac{x^m}{m!}, \quad x \in \mathbb{R}, \tag{4·16}$$

with parameter $k \in \mathbb{N} \cup \{0\}$; see [**32**, pp. 177 and 198].

LEMMA 4·4. *Let $k \in \mathbb{N} \cup \{0\}$. Then*:

(i) $|L_n^{(k)}(x)| \leqslant \begin{cases} C_1(k) n^k, & 0 \leqslant x \leqslant \frac{1}{n}, \\ C_2(k) n^{\frac{k}{2}-\frac{1}{4}} x^{-\frac{k}{2}-\frac{1}{4}}, & \frac{1}{n} \leqslant x \leqslant 1; \end{cases}$

(ii) $L_n^{(k)}(x) = \pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{-\frac{k}{2}-\frac{1}{4}} n^{\frac{k}{2}-\frac{1}{4}} \cos(2\sqrt{nx} - k\frac{\pi}{2} - \frac{\pi}{4}) + O(n^{\frac{k}{2}-\frac{3}{4}}), x > 0,$

as $n \to \infty$, where the bound for the remainder holds uniformly in any $[a, b] \subset (0, \infty)$.

We shall also need the following formula for the norm of an integral operator on $L_1((0,1))$. It follows from a very general result given in [**21**, theorem XI·1·4]. To make the formula more accessible we provide it with a particularly simple proof.

LEMMA 4·5. *Suppose that K is a real-valued continuous function on $[0,1]$. If a bounded linear operator T on $L_1((0,1))$ is defined by*

$$(Tx)(t) := \int_0^t K(t-s)x(s)\,ds,$$

then we have

$$\|T\| = \int_0^1 |K(s)|\,ds.$$

Proof. Note that

$$(Tx)(t) := \int_0^1 K_0(t,s)x(s)\,ds,$$

where

$$K_0(t,s) = \begin{cases} K(t-s), & 1 \geqslant t \geqslant s \geqslant 0, \\ 0, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Then the adjoint operator $T^*: L_\infty((0,1)) \to L_\infty((0,1))$ is given by the formula

$$(T^*x)(t) := \int_0^1 K_0(s,t)x(s)\,ds, \quad x \in L_\infty((0,1)).$$

By the Lebesgue bounded convergence theorem, $(T^*x)(\cdot)$ is continuous on $(0,1)$, for every $x \in L_\infty((0,1))$. For a fixed $t_0 \in (0,1)$, the function $x_{t_0}(\cdot) = \operatorname{sign} K(\cdot, t_0)$ belongs to $L_\infty((0,1))$, and $\|x_{t_0}\|_\infty \leqslant 1$. Therefore, taking into account the continuity of $(T^*x)(\cdot)$, we get

$$\begin{aligned} \|T^*\| &\geqslant \|T^* x_{t_0}\|_\infty = \sup_{t\in(0,1)} |(T^* x_{t_0})(t)| \\ &\geqslant |(T^* x_{t_0})(t_0)| = \int_0^1 |K_0(s,t_0)|\,ds. \end{aligned}$$

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

Since the above inequalities hold for all $t_0 \in (0,1)$, we obtain

$$\|T\| = \|T^*\| \geqslant \sup_{t_0 \in (0,1)} \int_0^1 |K_0(s,t_0)|\,ds = \int_0^1 |K(s)|\,ds.$$

The upper estimate is standard, and can be done in the same way as in [**21**, section V·2·5].

Example 4·6. There exists a Banach space $\mathcal{X}$ and a bounded linear operator $\mathcal{T}$ on $\mathcal{X}$ such that:

(a) $\sigma(\mathcal{T}) = \{1\}$;

(b) $\lim_{n\to\infty} \frac{1}{n}\|\mathcal{T}^n\| = 0$;

but

$$\lim_{n\to\infty} \|\mathcal{T}^n(\mathcal{T} - I)\| = \infty. \tag{4·17}$$

As in the proof of Corollary 2·5 again take the Volterra operator

$$(Vf)(t) := \int_0^t f(s)\,ds,$$

this time on $X = L_1((0,1))$.

Consider the operator $T = I - V$ and the corresponding matrix operator $\mathcal{T}$ on $\mathcal{X}$. By Lemma 2·1(i), we have

$$\sigma(\mathcal{T}) = \{1\}.$$

As in [**29**, p. 292] we can write

$$\begin{aligned}(T^n f)(t) - (T^{n+1} f)(t) &= (V(I-V)^n f)(t)\\ &= \left(\sum_{k=0}^n \binom{n}{k}(-1)^k V^{k+1} f\right)(t)\\ &= \int_0^t \sum_{k=0}^n \binom{n}{k}(-1)^k \frac{(t-s)^k}{k!} f(s)\,ds\\ &= \int_0^t L_n^{(0)}(t-s) f(s)\,ds,\end{aligned}$$

where

$$L_n^{(0)}(x) = \sum_{k=0}^n \binom{n}{k}(-1)^k \frac{x^k}{k!}, \quad n \geqslant 1$$

are the Laguerre polynomials (with parameter 0). Similarly,

$$\begin{aligned}(T^n(I-T)^2 f)(t) &= (V^2(I-V)^n f)(t)\\ &= \left(\sum_{k=0}^n \binom{n}{k}(-1)^k V^{k+2} f\right)(t)\\ &= \int_0^t \sum_{k=0}^n \binom{n}{k}(-1)^k \frac{(t-s)^{k+1}}{(k+1)!} f(s)\,ds\\ &= \frac{1}{n+1}\int_0^t (t-s) L_n^{(1)}(t-s) f(s)\,ds,\end{aligned}$$

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

by (4·16) (see also the formula (30) in [**4**, p. 191]). Using Lemma 4·5 we obtain

$$\|T^{n+1}-T^n\| = \int_0^1 \left|L_n^{(0)}(s)\right| ds,$$

$$\|T^n(I-T)^2\| = \frac{1}{n+1}\int_0^1 \left|sL_n^{(1)}(s)\right| ds \geqslant \frac{1}{2(n+1)}\int_{\frac{1}{2}}^1 \left|L_n^{(1)}(s)\right| ds.$$

By Lemma 4·4(i), for $\alpha = 0$, we get

$$\|T^{n+1}-T^n\| \leqslant \int_0^{\frac{1}{n}} \left|L_n^{(0)}(s)\right| ds + \int_{\frac{1}{n}}^1 \left|L_n^{(0)}(s)\right| ds \leqslant \frac{C_1}{n} + C_2 n^{-\frac{1}{4}},$$

for some $C_1, C_2 > 0$. So

$$\|T^{n+1}-T^n\| = \mathrm{O}\big(n^{-\frac{1}{4}}\big), \quad n \longrightarrow \infty. \tag{4·18}$$

Then, from (4·18) and Lemma 2·1(ii), we obtain

$$\lim_{n\to\infty} \frac{1}{n}\|\mathcal{T}^n\| = 0. \tag{4·19}$$

But at the same time, by Lemma 4·4(ii),

$$\begin{aligned}
&\liminf_{n\to\infty} \|n\,T^n(T-I)^2\| \\
&\quad\geqslant \liminf_{n\to\infty} \frac{n}{2(n+1)}\int_{\frac{1}{2}}^1 \left|L_n^{(1)}(s)\right| ds \\
&\quad\geqslant \frac{1}{2\sqrt{\pi}} \liminf_{n\to\infty} n^{\frac{1}{4}} \int_{\frac{1}{2}}^1 e^{\frac{s}{2}} s^{-\frac{3}{4}} \left|\sin\left(2\sqrt{ns}-\frac{\pi}{4}\right)\right| ds \\
&\quad\geqslant \frac{1}{2\sqrt{\pi}} e^{\frac{1}{4}} \liminf_{n\to\infty} n^{\frac{1}{4}} \int_{\frac{1}{2}}^1 s^{-\frac{3}{4}} \left|\sin\left(2\sqrt{ns}-\frac{\pi}{4}\right)\right| ds \\
&\quad= \left(\frac{e}{4\pi^2}\right)^{\frac{1}{4}} \liminf_{n\to\infty} \int_{\sqrt{2n}-\frac{\pi}{4}}^{2\sqrt{n}-\frac{\pi}{4}} \frac{|\sin s|}{\sqrt{s+\frac{\pi}{4}}}\, ds \\
&\quad\geqslant \left(\frac{e}{4\pi^2}\right)^{\frac{1}{4}} \liminf_{n\to\infty} \left(\int_{\sqrt{2n}-\frac{\pi}{4}}^{2\sqrt{n}-\frac{\pi}{4}} \frac{1}{2\sqrt{s+\frac{\pi}{4}}}\, ds - \int_{\sqrt{2n}-\frac{\pi}{4}}^{2\sqrt{n}-\frac{\pi}{4}} \frac{\cos 2s}{2\sqrt{s+\frac{\pi}{4}}}\, ds\right) \\
&\quad= \infty.
\end{aligned}$$

So, using (2·5) (Lemma 2·1(vii)), Lemma 2·6 and (4·18), we derive

$$\lim_{n\to\infty} \|\mathcal{T}^{n+1}-\mathcal{T}^n\| = \infty.$$

Therefore, the Esterle–Katznelson–Tzafriri conclusion does not hold for $\mathcal{T}$.

Remark 4·7. It follows from (4·18) that $\lim_{n\to\infty} n\|T^n(T-I)^5\| = 0$ so that by (2·5) we have $\lim_{n\to\infty} \|\mathcal{T}^n(\mathcal{T}-I)^4\| = 0$, despite (4·17). A similar phenomenon was observed in Remark 4·3, but this time we have $\sigma(\mathcal{T}) = 1$.

Further, we apply Example 4·6 to the study of Cesàro means of bounded operators with one point spectrum. The next example answers a question from [**12**, remark 6]. It is an iteration of the construction (2·1).

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

Example 4·8. There exist a Banach space $\mathcal{Y}$ and a bounded linear operator $\mathcal{S}$ on $\mathcal{Y}$ with $\sigma(\mathcal{S}) = \{1\}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\|A_n(\mathcal{S})\| = 0, \tag{4·20}$$

but

$$\lim_{n\to\infty} \frac{1}{n}\|\mathcal{S}^n\| = \infty. \tag{4·21}$$

Let $\mathcal{T}$ and $\mathcal{X}$ (thus T and X) be as in Example 4·6. Let $\mathcal{Y} = \mathcal{X} \oplus \mathcal{X}$. Define the bounded linear operator $\mathcal{S}$ on $\mathcal{Y}$ by the operator matrix

$$\mathcal{S} := \begin{pmatrix} \mathcal{T} & \mathcal{T} - I \\ O & \mathcal{T} \end{pmatrix}.$$

By (4·17) and (2·3), the operator $\mathcal{S}$ satisfies (4·21). Note that T is Cesàro bounded by [**19**, theorem 11]. Then, by (4·19), (2·4) and Lemma 2·6, we have

$$\limsup_{n\to\infty} \frac{1}{n}\|A_n(\mathcal{S})\| \leqslant \sqrt{3}\, \limsup_{n\to\infty} \frac{1}{n}\|A_n(\mathcal{T})\| \leqslant 3\, \limsup_{n\to\infty} \frac{1}{n}\|A_n(T)\| = 0.$$

5. *Open problems and final remarks*

(1) It is not clear whether or not one can construct a Banach space X and a Cesàro bounded linear operator T on it such that $\sigma(T) = \{1\}$, but at the same time $\|T^n(T - I)\| \not\longrightarrow 0, n \to \infty$; see also [**35**, p. 378]. If there are no such X and T, then using Lemma 2·1, we can conclude that any power bounded operator T on a Banach space X with $\sigma(T) = \{1\}$ has the property

$$n\|T^n(T - I)^2\| \longrightarrow 0,\ n \longrightarrow \infty.$$

Although that claim seems to be false, we are not able to disprove it. Note that, in view of Lemma 2·1(iii), the operator $\mathcal{T}$ from Example 4·6 is not Cesàro bounded since $T = I - V$ is not power bounded [**19**, theorem 11]. The comments preceding Example 3·7 indicate that the condition $\sigma(T) = \{1\}$ might be significant.

(2) We recall a question raised in [**26**, p. 147] in the present context: does the boundedness of the right upper corner of the matrix (2·3), $nT^{n-1}(T - I)$, imply the boundedness of the diagonal T^n? (*Added in proof*: The answer is 'no' in general, see N. Kalton, S. Montgomery-Smith, K. Oleszkiewicz, and Yu. Tomilov, Power bounded operators and related norm estimates, *J. London Math. Soc.*, to appear. However, the question remains open in the interesting case when $\sigma(T) = \{1\}$.) Observe that if we divide (2·3) by n, then the boundedness of $T^{n+1} - T^n$ does imply the boundedness of T^n/n by (1·9), but not conversely (Examples 4·1 and 4·6).

(3) Another problem related to Allan's question is whether one can find an example similar to Example 4·6 on a Hilbert space. Our main difficulty here is to estimate precisely the norms of integral operators on L_2-spaces (see [**21**, section V·2·6]).

(4) Of course, the examples with positive operators have special interest for ergodic theory (e.g. when an operator on $L_p(X, d\mu)$ is induced by transformation of a σ-finite measure space $(X, d\mu)$). However, Lemma 2·1 seems to have no applications to construction of positive operators with properties similar to those described in the examples here. If X is a Banach lattice, then $\mathcal{X}$ is also a Banach lattice with respect

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

to coordinate-wise order. Moreover, $\mathcal{T}$ is positive on $\mathcal{X}$ if and only if $T \geqslant I$ on X. If, in addition, $\mathcal{T}$ is Cesàro bounded, then $T = I$ by Lemma 2·1(iii) and [**6**, theorem 1]. The consideration of Cesàro unbounded operators $\mathcal{T}$ excludes any (Cesàro) mean ergodic theorems for $\mathcal{T}$. One may replace the element $T - I$ of (2·1) by the element $I - T$. Then the corresponding matrix operator $\mathcal{T}$ is positive on $\mathcal{X}$ whenever $O \leqslant T \leqslant I$. If X has order continuous norm (e.g. if $X = L_p(X, d\mu), 1 \leqslant p < \infty$), then T^n converge strongly on X; see [**31**, pp. 89–92]. Similarly to Lemma 2·1(iv), the operator $\mathcal{T}$ is Cesàro ergodic on $\mathcal{X}$.

(5) It would be interesting to know whether the famous Jacobs–de Leeuw–Glicksberg decomposition theorem is true for Cesàro bounded operators. Recall one of its versions which can be found in [**20**, theorem 9]; see [**23**, theorem 2·4·4] for the general case.

THEOREM. *Let T be a contraction on a reflexive Banach space X. Then $X = X_0 \oplus X_1$, where*

$$X_0 = \left\{ x : \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |x^*(T^k x)| = 0 \quad \textit{for any} \quad x^* \in X^* \right\},$$

and $X_1 = c.l.s.\{x : Tx = \lambda x,\ |\lambda| = 1\}$. Moreover, X_1 is T-invariant, and T restricted to X_1 is an invertible isometry.

Could this statement be true if, instead of $\|T\| \leqslant 1$, we assume that $\|A_n(T)\| \leqslant 1, n \geqslant 0$?

(6) Suppose that every power bounded operator $T \in \mathcal{L}(X)$ is Cesàro ergodic. It is still unknown whether X must be reflexive. Suppose now that even every $T \in \mathcal{L}(X)$ from the a priori larger class of operators satisfying (1·3) and (1·4) is Cesàro ergodic. Does it follow that X is reflexive?

(7) Since the mid-1990s. Atzmon has been claiming the existence of another example (an essentially different approach) answering negatively Allan's question; see [**35**, p. 373]. However, the details have not been available until now.

Acknowledgements. The authors would like to thank R. Askey for consulting on Laguerre polynomials, M. Lin for useful remarks and D. Tsedenbayar for stimulating discussions.

REFERENCES

[**1**] G. R. ALLAN. *Power-bounded elements in a Banach algebra and a theorem of Gelfand*. Conference on Automatic Continuity and Banach Algebras (Canberra, January 1989), Proc. Centre Math. Anal. Austral. Nat. Univ. **21** (1989), 1–12.

[**2**] G. R. ALLAN and T. J. RANSFORD. Power-dominated elements in a Banach algebra. *Studia Math.* **94** (1989), 63–79.

[**3**] W. ARENDT and C. J. K. BATTY. A complex Tauberian theorem and mean ergodic semigroups. *Semigroup Forum* **50** (1995), 351–366.

[**4**] H. BATEMAN and A. ERDÉLYI. *Higher Transcendental Functions*, Vol. II. (McGraw-Hill, 1953).

[**5**] D. BEREND and V. BERGELSON. Mixing sequences in Hilbert spaces. *Proc. Amer. Math. Soc.* **98** (1986), 239–246.

[**6**] S. J. BERNAU and C. B. HUIJSMANS. On the positivity of the unit element in a normed lattice ordered algebra. *Studia Math.* **97** (1990), 143–149.

[**7**] N. BOROVYKH, D. DRISSI and M. N. SPIJKER. A note about Ritt's condition, related resolvent conditions and power bounded operators. *Numer. Funct. Anal. Optim.* **21** (2000), 425–438.

[**8**] A. BRUNEL and R. ÉMILION. Sur les opérateurs positifs à moyennes bornées. *C. R. Acad. Sci. Paris Sér. I Math.* **298** (1984), 103–106.

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press

[**9**] B. Calvert. Convergence sets in reflexive Banach spaces. *Proc. Amer. Math. Soc.* **47** (1975), 423–428.
[**10**] J. A. van Casteren. *Boundedness properties of resolvents and semigroups of operators*. Linear Operators, Banach Center Publ. **38** (1997), 59–74.
[**11**] Y. Derriennic. On the mean ergodic theorem for Cesàro bounded operators. *Colloq. Math.* **84/85** (2000), 443–455.
[**12**] D. Drissi and J. Zemánek. Gelfand–Hille theorems for Cesàro means. *Quaestiones Math.* **23** (2000), 375–381.
[**13**] N. Dunford and J. T. Schwartz. *Linear Operators, Part* I: *General Theory*. (Interscience Publishers, 1958).
[**14**] O. El-Fallah and T. J. Ransford. Extremal growth of powers of operators satisfying resolvent conditions of Kreiss–Ritt type. *J. Funct. Anal.* **196** (2002), 135–154.
[**15**] R. Émilion. Mean bounded operators and mean ergodic theorems. *J. Funct. Anal.* **61** (1985), 1–14.
[**16**] J. Esterle. *Quasimultipliers, representations of H^∞, and the closed ideal problem for commutative Banach algebras*. Radical Banach Algebras and Automatic Continuity (Long Beach, Calif., 1981), Lecture Notes in Math. **975** (Springer, 1983), 66–162.
[**17**] S. Grabiner and J. Zemánek. Ascent, descent, and ergodic properties of linear operators. *J. Operator Theory* **48** (2002), 69–81.
[**18**] H. Heinich. Convergence des moyennes d'un opérateur positif. *C. R. Acad. Sci. Paris Sér. I Math.* **297** (1983), 237–240.
[**19**] E. Hille. Remarks on ergodic theorems. *Trans. Amer. Math. Soc.* **57** (1945), 246–269.
[**20**] L. K. Jones and M. Lin. Unimodular eigenvalues and weak mixing. *J. Funct. Anal.* **35** (1980), 42–48.
[**21**] L. V. Kantorovich and G. P. Akilov. *Functional Analysis*, Second edition. (Pergamon Press, 1982).
[**22**] Y. Katznelson and L. Tzafriri. On power bounded operators. *J. Funct. Anal.* **68** (1986), 313–328.
[**23**] U. Krengel. *Ergodic Theorems*, with a supplement by Antoine Brunel. de Gruyter Studies in Mathematics **6** (Walter de Gruyter, 1985).
[**24**] Yu. Lyubich. Spectral localization, power boundedness and invariant subspaces under Ritt's type condition. *Studia Math.* **134** (1999), 153–167.
[**25**] Yu. Lyubich. The single-point spectrum operators satisfying Ritt's resolvent condition. *Studia Math.* **145** (2001), 135–142.
[**26**] B. Nagy and J. Zemánek. A resolvent condition implying power boundedness. *Studia Math.* **134** (1999), 143–151.
[**27**] O. Nevanlinna. *Convergence of Iterations for Linear Equations*. Lectures in Mathematics (ETH Zürich, Birkhäuser, 1993).
[**28**] O. Nevanlinna. *On the growth of the resolvent operators for power bounded operators*. Linear Operators, Banach Center Publ. **38** (1997), 247–264.
[**29**] T. Pytlik. Analytic semigroups in Banach algebras and a theorem of Hille. *Colloq. Math.* **51** (1987), 287–294.
[**30**] B. Russo. Unimodular contractions in Hilbert spaces. *Pacific J. Math.* **26** (1968), 163–169.
[**31**] H. H. Schaefer. *Banach Lattices and Positive Operators*. Grundlehren math. Wiss. **215**, (Springer, 1974).
[**32**] G. Szegö. *Orthogonal Polynomials*, Fourth edition. Amer. Math. Soc. Colloq. Publ. **23** (Amer. Math. Soc., 1975).
[**33**] B. Sz.-Nagy. On uniformly bounded linear transformations in Hilbert space. *Acta Sci. Math.* (*Szeged*) **11** (1947), 152–157.
[**34**] Vũ Quôc Phóng. A short proof of the Y. Katznelson's and L. Tzafriri's theorem. *Proc. Amer. Math. Soc.* **115** (1992), 1023–1024.
[**35**] J. Zemánek. *On the Gelfand-Hille theorems*. Functional Analysis and Operator Theory, Banach Center Publ. **30** (1994), 369–385.

https://doi.org/10.1017/S0305004103007436 Published online by Cambridge University Press