JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS Vol. 43, No. 4, Dec. 2008, pp. 1055–1090
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https://doi.org/10.1017/S0022109000014447 Published online by Cambridge University Press

Can Tests Based on Option Hedging Errors Correctly Identify Volatility Risk Premia?

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Abstract

Tests for the existence and the sign of the volatility risk premium are often based on expected option hedging errors. When the hedge is performed under the ideal conditions of continuous trading and correct model specification, the sign of the premium is the same as the sign of the mean hedging error for a large class of models. We show that discrete trading and model misspecification may cause the standard test to yield unreliable results. In particular, ignoring jump risk premia can lead to incorrect conclusions. We also show that delta-gamma hedges do not increase the reliability of the test.

I. Introduction

Explaining excess asset returns is at the core of the discipline financial economists call "asset pricing." Naturally, researchers have also become interested in the properties of excess returns on options, see, for example, the paper by Coval and Shumway (2001). There are two main questions in this context: which risk factors are present in the market under consideration, and which premia are associated with the individual sources of risk.

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In the area of option pricing, the two most prominent risk factors besides the fundamental price risk of the underlying are stochastic volatility (SV) and stochastic jumps (SJ). The presence of these two risk factors is documented in a large number of studies using time series of asset prices, cross sections of option prices, and both sources of information simultaneously, see, e.g., Bates (1996), Bakshi, Cao, and Chen (1997) (BCC hereafter), Pan (2002), Carr and Wu (2003), Eraker, Johannes, and Polson (2003), Eraker (2004), and Broadie, Chernov, and Johannes (2007). The evidence on the sign and magnitude of the risk premia for SV and SJ, however, is much less clear. While some papers conclude that there is a negative market price of volatility risk, other papers find a jump risk premium, but no significant volatility risk premium, see, e.g., Guo (1998), Poteshman (1998), Benzoni (2002), Pan (2002), Eraker (2004), and Broadie, Chernov, and Johannes (2007).

It is important to note that it is not a priori clear what sign the market price of volatility risk (MPVR) should have in equilibrium. Whereas risk-averse investors would certainly demand a (positive) premium for bearing stock price risk, this is not so clear in the case of volatility risk. Liu and Pan (2003), e.g., show that the hedging demand for volatility risk can be either negative or positive, depending on the risk aversion of the investor. Even given that their analysis is performed in a partial equilibrium model, intuition nevertheless suggests that the sign of the risk premium (which is set such that some investors are willing to provide insurance) can thus also be positive or negative. As a consequence we do not have a clear null hypothesis for the sign of the MPVR, and its determination becomes a purely empirical matter.

In asset pricing studies, risk factors are usually represented by so-called factor mimicking portfolios, e.g., the size or book-to-market portfolios suggested by Fama and French (1993). This approach cannot be applied directly to a factor like SV or SJ, since volatility and jumps are not traded. As a consequence, volatility and jump sensitive instruments like options have to be employed to infer the properties of the market prices of risk for these factors. Hedging-based tests, studied among others by Bakshi and Kapadia (2003), rely on the error from hedging an option with the stock and the money market account. The basic idea of these tests is that the expected excess return on an option is determined by the expected excess return on the stock and by the premia for other risk factors, like SV or SJ, which the option is exposed to.

If, in an empirical study, the excess option return cannot be explained by the excess stock return only (or if the option cannot be hedged by the stock and the money market account only), the conclusion is that there are other risk factors besides the price risk of the underlying asset. Furthermore, if the *expected* excess return of the option cannot be explained by the expected excess stock return (or if the average hedging error is different from zero), then at least one of these additional risk factors is priced. Buraschi and Jackwerth (2001) apply a similar argument to conclude, from the fact that the pricing kernel cannot be spanned by the money market account and the stock only, that there are other priced risk factors.

Bakshi and Kapadia (2003) furthermore show that the sign of the expected hedging error (EHE) and the sign of the MPVR coincide. This results in what

https://doi.org/10.1017/S0022109000014447 Published online by Cambridge University Press

we call a *standard hedging-based test*, where the sign of the MPVR is inferred from the sign of the EHE. They conclude from an empirically negative mean hedging error that the MPVR is negative. Using a similar approach, Coval and Shumway (2001) and Driessen and Maenhout (2007) conclude that there are additional priced risk factors, and argue that the results are in line with a negative volatility risk premium.

In this paper, we focus on the theoretical properties of standard hedging-based tests aiming at the identification of the MPVR. We analyze the robustness of these tests with respect to model misspecifiation and with respect to discrete instead of continuous trading. These problems cannot be avoided in an empirical implementation, so that the robustness of hedging-based tests with respect to the sources of potential problems would be desirable.

Note that, as opposed to risk management, the ultimate purpose of hedging in our paper is not to minimize the risk of a certain option position, but to use the sign of the EHE as an identification criterion for the sign of the MPVR. However, discrete trading and model mis-specification are problems also relevant in risk management, and our results certainly have implications in that area, too, although a thorough analysis of hedging under these problems is beyond the scope of this paper.

Hedging-based tests rely on several assumptions, one of which is that the relation between the EHE and the market prices of risk is known. In an SV model, Bakshi and Kapadia (2003) show that under ideal conditions, the sign of the EHE and the sign of the MPVR coincide, which is the theoretical basis for the standard hedging-based test. However, the relation between the EHE and the market prices of risk may depend on the characteristics of the option, e.g., on its moneyness, and on current values of the state variables like volatility. This implies that when estimating the EHE from a time series of hedging errors, where all of these variables can change over time, it would be hard to detect such a relation.

Furthermore, the relation between EHE and MPVR may depend on the specific model used. In this case, the problem of model misspecifiation becomes important. In particular, the omission of risk factors that are actually priced may lead to incorrect conclusions, since the risk premium for the omitted factors is then falsely attributed to the remaining risk factors, i.e., in our case to SV. To illustrate the problems caused by omitted risk factors, we consider the model of Bates (1996) with SV and SJ (SVJ model hereafter). In this model, the EHE depends on the MPVR and on the pricing of jump risk. To keep the analysis simple, we assume that only jump intensity risk is priced, i.e., that the distribution of the jump size is the same under the physical and the risk-neutral measure. For empirically relevant scenarios, the contribution of jump risk to the EHE of a call option is then negative. When jumps are ignored, the negative EHE caused by jump risk premia is instead attributed to SV, and the conclusion from the standard test is that the MPVR is negative, while it may in fact be zero or even positive.

A second important assumption of hedging-based tests is that stock price risk can indeed be eliminated from the hedge portfolio, so that the EHE only depends on the market prices of risk for the nontraded risk factors. When trading is continuous, this can be achieved by setting the hedge ratio equal to the partial derivative of the claim price with respect to the stock price. In empirical studies,

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however, the hedging strategy cannot be implemented in continuous time, which leads to a *discretization error*. Furthermore, the option delta is obviously model dependent, so in case the hedge model does not coincide with the true model, the incorrect hedge ratio will be applied. This leads to an error due to *model misspecification*.

If discrete trading and model mis-specification only add noise to the hedging error, the standard test which uses the EHE to learn about the market prices of risk can still be used. In fact, the robustness of hedging-based tests with respect to the choice of the hedge model would be a significant advantage over approaches that estimate the MPVR and its sign by assuming a specific parametric option pricing model. However, if discrete trading and model misspecification have a systematic impact on the EHE, the conclusions drawn from the standard test will be at least unreliable.

We analyze the impact of discrete trading and model misspecification in the SV model of Heston (1993) as one of the most popular approaches to SV option pricing and in the SVJ model of Bates (1996). If hedging-based tests are supposed to work for basically all types of models, they should also work in these two special cases. This argument also motivates the use of the Black and Scholes (1973) model as a further benchmark. If only stock price risk and volatility risk are priced, the EHE can identify the sign of the MPVR under ideal conditions in all three models. Any improper conclusion drawn from the test can then be attributed to discrete trading, the use of a wrong hedge ratio due to model misspecification, and the omission of the premium for jump risk from the analysis.

For the case of continuous trading, the realized hedging error in the model of Black and Scholes (1973) is of course identically equal to zero. If the portfolio can only be rebalanced at discrete points in time, the market becomes incomplete and a perfect hedge is no longer possible. For a convex claim like a call option, the EHE can be shown to be always positive. The standard hedging-based test would then wrongly signal that volatility is stochastic with a positive MPVR. In the models of Heston (1993) and Bates (1996), the use of a stale hedge ratio also leads to a discretization error. The staleness due to stock price changes has a positive contribution to the EHE, while the sign of the discretization error introduced by staleness due to volatility changes and stock price jumps depends on the moneyness of the option.

In the Bates (1996) model, the jump risk premium also contributes to the EHE. For a convex claim like a call option, it can be shown that this contribution is increasing in the difference between the physical and the risk-neutral jump intensity. For equity markets, jumps are on average downward, and the risk-neutral intensity of jumps is commonly larger than its counterpart under the physical measure. In this case, the contribution of jump risk to the EHE is negative. When jumps are ignored in the analysis, this negative contribution is attributed to the MPVR, so that the test may falsely conclude that the MPVR is negative while it is actually zero or even positive.

Even when the hedge portfolio is rebalanced continuously, stock price risk is no longer eliminated in case of model misspecification. To analyze this problem, we assume that the hedge in the model of Heston (1993) is incorrectly determined in the BS model, which is the simplest choice of a hedge model. We show that

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the additional component of the EHE due to model misspecification depends on the equity risk premium and on the slope of the implied volatility curve. Given the empirical evidence on stock and option markets around the world, the most important scenario here is certainly characterized by a downward sloping smile curve and a positive equity risk premium. In this setting, model misspecification tends to increase the EHE. Thus, the standard test may falsely conclude that the volatility risk premium is positive when it is actually zero or negative.

To solve the problems caused by discrete trading, one approach is to adjust the hedge ratio. We show that there are indeed "discrete hedge ratios" such that the discretization error vanishes from the EHE. However, these hedge ratios depend on the expected return on the stock, in case of SV they furthermore depend on the MPVR, and for an SVJ model, we also need to know the jump risk premium. However, once the required MPVR is known, there is no more need to determine its sign via a hedging-based test.

A second approach to solve the problems of discrete trading and model misspecification is to use a more sophisticated delta-gamma hedge instead of a simple delta hedge, as suggested by Doran (2005). The discretization error caused by changes in delta due to changes in the stock price will then be reduced significantly. In a good scenario, the discretization errors due to changes in volatility and due to jumps are also significantly reduced. If the same holds true for the impact of model misspecification and the jump risk premium, then the EHE could again be used to identify the sign of the MPVR. However, it seems questionable that all these conditions would be met simultaneously, and back-of-the-envelope calculations indeed suggest that this is not the case. This conjecture is confirmed by a simulation study, where we compare hedging-based tests using delta hedges and delta-gamma hedges.

In summary, our analyses show that tests based on option hedging errors should be handled with care. Although they offer a theoretically elegant way of identifying the MPVR, the researcher has to be very careful when applying these ideas empirically. Discretization error, model misspecification, and the omission of priced risk factors may all lead to incorrect conclusions about the MPVR.

The remainder of the paper is organized as follows. In Section II the main idea of hedging-based tests is described in more detail. The model setup is then presented in Section III. Section IV analyzes the properties of hedging-based tests under ideal conditions, in case of discrete trading and in case of model misspecification. Two approaches to correct for the problems are discussed in Section V. Section VI concludes.

II. An Overview of Tests for Volatility Risk Premia

A. Review of the Literature

One approach to answering the question of which factors present in a given model are priced is to estimate the model both under the physical and under the risk-neutral measure. The risk premia can then be inferred from the differences between the model parameters under the two measures. Guo (1998), Poteshman (1998), and Benzoni (2002) estimate SV models and report a negative MPVR. Pan

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(2002) estimates a model with SV and jumps in the stock price. She concludes that the MPVR is not significantly different from zero, while there is a large premium for jump size risk in the stock. Eraker (2004) additionally includes jump risk in the volatility dynamics. He finds that, depending on the model specification, volatility risk is either not priced or has a negative risk premium, and that the same is true for stochastic jumps in returns. Broadie, Chernov, and Johannes (2007) also find that the significance of the volatility risk premium depends on the choice of the model. The most striking feature of these empirical tests is the strong dependence of the results on the models, so that model misspecification may lead to incorrect conclusions concerning the sign and size of premia for risk factors.

A second group of studies relies on hedging (or spanning) based tests. The main idea of these tests for the existence and magnitude of risk premia is that the expected excess return of an option can be decomposed into the sum of the risk premia for stock price risk, volatility risk, and other types of risk. If, in an empirical study, the expected excess option return can be perfectly explained by the expected excess stock return only, two scenarios are possible. Either there are no other risk factors besides stock price risk (as is the case in the BS model) or there are other risk factors, but they are not priced, i.e., investors do not require additional compensation for bearing those types of risk. The converse is that if expected excess option returns cannot be explained by expected excess stock returns, there are additional risk factors, and at least one of these risk factors is priced.

This line of argument leads to two testable implications. First, one can investigate whether excess option returns can be explained by excess returns on the underlying, as is done in Jones (2005) and in Coval and Shumway (2001). This yields the generic test equation

$$\frac{C_{t+\tau}}{C_t} - e^{r\tau} = \beta_t \left(\frac{S_{t+\tau}}{S_t} - e^{r\tau} \right) + \epsilon_{t+\tau}, \tag{1}$$

with C_t and S_t representing the price of the option and the stock at time t, respectively, $\epsilon_{t+\tau}$ an error term, and β_t denoting some measure of risk. As argued above, if the expectation of $\epsilon_{t+\tau}$ is different from zero, there are other risk factors besides stock price risk, and at least one of these factors is priced.

An alternative to the spanning approach described by equation (1) is hedging-based tests. The basic idea is to set up a hedge portfolio with zero initial investment and no stock price risk. Then any change in the portfolio value can be attributed to other risk factors besides stock price risk, and can thus be used to draw inference about these other risk factors. The test equation is

$$C_{t+\tau} - C_t e^{r\tau} - \widehat{H}_t \left(S_{t+\tau} - e^{r\tau} S_t \right) = \widehat{\epsilon}_{t+\tau}, \tag{2}$$

where $\widehat{H}_t := \beta_t C_t / S_t$ is the hedge ratio and $\widehat{\epsilon}_{t+\tau} := C_t \epsilon_{t+\tau}$ is the hedging error. Again, we conclude that if the expected hedging error (EHE) is zero, only stock price risk carries a premium, whereas in the opposite case there must be some other priced risk factor.

As one can see, equations (1) and (2) are basically just multiples of each other. The researcher can thus choose which specification should be used for the

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test, and for the rest of the paper we will focus on hedging errors of the type shown in equation (2).

B. Hedging-Based Tests

We now consider the design of hedging-based tests in more detail. A hedging strategy for a contingent claim consists of investments in the two traded basis assets, the stock and the money market account. The number of shares held at time t, the hedge ratio, is denoted by H_t, and the investment in the money market account simply represents the residual and serves to make the strategy self-financing.

The hedging error is defined as the difference between the claim price and the value of the hedge portfolio. For the hedging error $D(t, t+\tau)$ over the interval $[t, t+\tau]$, one obtains the expression

$$(3) \quad D(t,t+\tau) = C_{t+\tau} - \left(e^{r\tau} C_t + \int_t^{t+\tau} e^{r(t+\tau-u)} H_u (dS_u - rS_u du) \right)$$

$$= e^{r(t+\tau)} \left(\int_t^{t+\tau} e^{-ru} (dC_u - rC_u du) - \int_t^{t+\tau} e^{-ru} H_u (dS_u - rS_u du) \right).$$

In terms of notation, the hedging error over some finite interval $[t, t+\tau]$ is denoted by $D^c(t, t+\tau)$ for continuous trading and by $D^d(t, t+\tau)$ for discrete trading. When trading is discrete, the hedge ratio has to be held constant over every subperiod of discrete length, i.e., the hedge ratio from time t_i to time t_{i+1} is held constant at the value H_{t_i}. This implies that the discrete hedging error is given by

$$(4) \quad D^d(t_i, t_{i+1}) = C_{t_{i+1}} - C_{t_i} e^{r(t_{i+1}-t_i)} - H_{t_i} \left(S_{t_{i+1}} - S_{t_i} e^{r(t_{i+1}-t_i)} \right).$$

In the following, we will be interested in the EHE, i.e., in the expectation of $D^c(t, t+\tau)$ or $D^d(t_i, t_{i+1})$. As explained above, a non-zero hedging error implies that there are other risk factors like SV and SJ, and a non-zero EHE implies that these risk factors are priced. If we furthermore know the sensitivity of the hedge portfolio with respect to these risk factors, we can infer the sign and perhaps even the magnitude of the market prices of risk.

The assumption that the hedge portfolio is not exposed to stock price risk holds under the ideal conditions of continuous trading and the hedge ratio being equal to the delta of the claim under the true model. However, by applying continuous-time hedge ratios or risk measures in discrete time, a discretization error is introduced. Furthermore, the test may suffer from model misspecification, since the true data generating process is unknown. The hedge ratio depends on the option pricing model, and when an improper hedge model is used, the hedge ratio will in general be no longer correct. If discretization error or model misspecification do not just represent additional noise, but have a systematic impact on the properties of the EHE, the test will become invalid.

A second assumption is that we know the dependence of the EHE on the market prices of risk and can invert this relation to draw inference about the market prices of risk. The sensitivities of the hedge portfolio with respect to these risk

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factors depend on the model, thus again introducing the risk of model misspecification into the test. They may also depend on the current values of state variables like volatility, and they may depend on characteristics of the option, like its time to maturity or its moneyness. This dependence can be problematic when the EHE is estimated from the time series of hedging errors for different options, since the sensitivities change over time. Furthermore, note that in case only one risk factor is involved, the sign of the sensitivity is enough to infer the sign of the market price of risk from the sign of the EHE. If two or more risk factors are involved, we need EHEs for options that differ in moneyness or time to maturity, and it is no longer sufficient to know the sign of the sensitivities only, but we also need to know how the sensitivities differ between these options.

A third assumption is that we know which risk factors are involved. If, again due to model misspecification, one of the priced risk factors is omitted, the premium for this factor is incorrectly attributed to the other risk factors, resulting in a wrong inference about their market prices of risk. In the standard test for the MPVR, a natural candidate for such an omitted risk factor is jump risk, and we will discuss this case in detail below.

III. Model Setup

Our analysis relies on the models of Heston (1993) and Bates (1996) as two of the most prominent SV and SVJ models, respectively. The BCC (1997) model is slightly more general than the setup suggested by Bates (1996), since it also includes stochastic interest rates. However, this component is not central to our analysis, so we opt for the simpler version of an SVJ model.

As noted above, if the standard test works in general for large classes of models with SV and jumps, it should also give the correct results in these two special cases. Systematic problems for these two models make the applicability in the larger class questionable.

Under the physical measure P, the stock price and the local variance in the SVJ model are governed by the stochastic differential equations

$$dS_t = \left(\mu - \mu_X h^P\right) S_t dt + \sqrt{V_t} S_t dW_t^S + S_{t-}\left(e^{X_t} - 1\right) dN_t$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t}\left(\rho dW_t^S + \sqrt{1-\rho^2} dW_t^V\right).$$

In this equation, dW_t^S and dW_t^V are the increments of two independent Brownian motions, and ρ represents the correlation coefficient between stock returns and volatility changes. N is a Poisson process with constant intensity h^P, and X is the stochastic jump size in the logarithm of the stock price. Like BCC (1997), we assume that X is normally distributed with mean $\ln(1+\mu_X) - 0.5\sigma_X^2$ and variance σ_X^2, so that the mean percentage jump size in the stock price is μ_X.

The market price of volatility risk is assumed to be equal to $\lambda_V \sqrt{V_t}$. For the jumps, we make the simplifying assumption that only jump intensity risk is priced, i.e., the only difference between the physical and the risk-neutral measure is in the jump intensity, while the jump size distribution remains unchanged. The jump intensity under the risk neutral measure is denoted by h^Q. While a complete

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analysis of hedging errors under jump risk is beyond the scope of this paper, this setup still allows us to analyze the impact of jump risk on the properties of standard tests for the MPVR. The processes under the risk-neutral measure Q are then given by

$$dS_t = \left(r - \mu_X h^Q\right) S_t dt + \sqrt{V_t} S_t d\widetilde{W}_t^S + S_{t-}\left(e^{X_t} - 1\right) dN_t \quad \text{and}$$

$$dV_t = \left[\kappa(\theta - V_t) - \sigma_V \lambda_V V_t\right] dt + \sigma_V \sqrt{V_t}\left(\rho d\widetilde{W}_t^S + \sqrt{1-\rho^2} d\widetilde{W}_t^V\right).$$

The SVJ model nests the models of Black and Scholes (1973) (for $V_t = \theta = \sigma^2$, $\sigma_V = 0$, and $h^P = h^Q = \mu_X = 0$, where σ is the constant volatility of the stock price) as well as the SV model of Heston (1993) (for $h^P = h^Q = \mu_X = 0$) and the SJ model of Merton (1976) (for $V_t = \theta = \sigma^2$, $\sigma_V = 0$).

The stochastic process for the price $C_t = c(t, S_t, V_t)$ of a contingent claim follows from Ito's lemma and the partial differential equation the claim price must fulfill. It is given by

$$\begin{aligned} dC_t = {} & rC_t dt + \frac{\partial c}{\partial s}\left(dS_t - rS_t dt\right) + \frac{\partial c}{\partial v}\left\{dV_t - \kappa(\theta - V_t)dt + \sigma_V \lambda_V V_t dt\right\} \\ & + \left\{f(t, S_{t-}, V_t, X_t) dN_t - E^Q\left[f(t, S_{t-}, V_t, X_t) | \mathcal{F}_{t-}\right] h^Q dt\right\}, \end{aligned} \tag{5}$$

where the additional jump risk f of the claim is defined by

$$f(u, s, v, x) = c(u, se^x, v) - c(u, s, v) - \frac{\partial c}{\partial s}(u, s, v)\, s\,(e^x - 1). \tag{6}$$

For each jump size x in the stock price, f captures the part of the change in the claim price that is not explained by the delta. The local expected excess return on the claim over the next infinitesimal interval is

$$\frac{1}{C_t}\left\{\frac{\partial c}{\partial s}(\mu - r) S_t + \frac{\partial c}{\partial v}\sigma_V \lambda_V V_t + E^Q\left[f(t, S_{t-}, V_t, X_t) | \mathcal{F}_{t-}\right]\left(h^P - h^Q\right)\right\}. \tag{7}$$

It depends on the sensitivity of the claim with respect to stock price risk and volatility risk as well as on the additional jump risk f.

Equations (5) and (7) give the basic intuition for hedging based tests. The idea is to eliminate the exposure with respect to stock price risk. Then, the expected excess return (or the EHE) only depends on the market prices of risk for the remaining risk factors.

IV. Hedging-Based Tests: Delta Hedges

A. General Remarks

Hedging-based tests rely on the expectation of the hedging error from equation (3) to draw inference about the market prices of risk. The general structure of the EHE, which we will derive in the following, is

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$$\begin{aligned}(8)\qquad \text{EHE} &= \left\{\text{IE}^S(\mu - r) + \text{IE}^V\lambda_V + \text{IE}^J(h^P - h^Q)\right\} \\ &\quad + (\mu - r)\left\{\text{DE}^S(\mu - r) + \text{DE}^V\lambda_V + \text{DE}^J(h^P - h^Q)\right\} \\ &\quad + (\mu - r)\text{ME},\end{aligned}$$

where the dependence on the time interval and on the state variables has been omitted. A key property of the EHE is that it is additive. The IE terms denote the expected hedging error under ideal conditions, i.e., for continuous trading and without model mis-specification. DE describe the additional expected hedging errors due to discrete trading, and ME is the part caused by the use of an improper hedge model. A detailed derivation with more specific expression for the different terms of this equation will be given in the next sections.

Under ideal conditions, the EHE is a linear function of the market prices of risk for stock price risk, volatility risk, and jump risk, where the coefficients IE^S, IE^V, and IE^J capture the sensitivity of the hedge portfolio with respect to these risk factors. When the hedge ratio is set equal to the delta of the claim, IE^S is equal to zero, and the part of the ideal expected hedging error that is due to stock price risk vanishes.

Under discrete trading, the use of a stale hedge ratio leads to a nonvanishing exposure to stock price risk. The average of this exposure, multiplied by the equity risk premium, is given by the second summand on the right hand side of Equation (8). The staleness in delta can be caused by changes in the stock price, changes in volatility, and by jumps. It is captured by the terms DE^S, DE^V, and DE^J, which are multiplied by the respective risk premia. The impact of model misspecification on the EHE is captured by the average remaining exposure ME, which is again multiplied by the equity risk premium.

In the following, we will derive explicit formulas for the components of the EHE. We start with the ideal case of continuous trading, and we discuss the consequences of omitting the market price for jump risk from the analysis. Then we turn to the case of discrete trading, where we will compare the size of the discretization error to the ideal hedging error. Finally, we consider the impact of model misspecification and thus the use of a wrong hedge ratio on the hedging error. Hedging-based tests are usually implemented using highly liquid options, and to simplify the exposition, we focus on the case where a European call is used. Most results hold for puts, too.

B. Ideal Case

We first analyze the ideal case of continuous trading. The EHE for a general claim and for a general hedge ratio is given in the following proposition:

Proposition 1 (*SVJ: EHE for continuous trading*). The true model is Bates (1996). The price of the claim is denoted by $C_t = c(t, S_t, V_t)$. The hedge portfolio is rebalanced continuously, the hedge ratio at time t is denoted by H_t. Then the EHE over the interval $[t, t + \tau]$ is given by

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$$(9) \quad E^P\left[D^c(t,t+\tau) \mid \mathcal{F}_t\right]$$
$$= \mathrm{IE}^S(t,t+\tau)(\mu - r) + \mathrm{IE}^V(t,t+\tau)\lambda_V + \mathrm{IE}^J(t,t+\tau)(h^P - h^Q),$$

where

$$\mathrm{IE}^S(t,t+\tau) = \int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[\left(\frac{\partial c}{\partial s}(u,S_u,V_u) - H_u\right) S_u \mid \mathcal{F}_t\right] du$$

$$\mathrm{IE}^V(t,t+\tau) = \int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[\frac{\partial c}{\partial v}(u,S_u,V_u)\, \sigma_V V_u \mid \mathcal{F}_t\right] du$$

$$\mathrm{IE}^J(t,t+\tau) = \int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[f(u,S_{u-},V_u,X) \mid \mathcal{F}_t\right] du,$$

and the additional jump risk exposure f is given in equation (6).

Proof. See Appendix A.

Note that this proposition corrects an error in equation (24) in Bakshi and Kapadia (2003), where the term $\mathrm{IE}^J(t,t+\tau)$ is incorrect.

To eliminate the exposure of the hedge portfolio to stock price risk, we set the hedge ratio H equal to the partial derivative of the claim price with respect to the stock price, i.e.,

$$H_u = \frac{\partial c}{\partial s}(u,S_u,V_u).$$

Then, the first term $\mathrm{IE}^S(t,t+\tau)$ in equation (9) vanishes, and the EHE only depends on the MPVR and the jump risk premium.

As indicated above, we do not aim at solving a risk management problem here, since our objective is not to minimize the remaining risk exposure of the hedge portfolio, but rather to use this remaining risk to identify the market prices of risk.

We now discuss the properties of the ideal EHE in our benchmark models. In the model of Black and Scholes (1973), there is neither SV nor SJ. Every claim satisfying certain regularity conditions can be hedged perfectly, the EHE is identically equal to zero, and the standard hedging-based test leads to the correct conclusion that there are no other priced risk factors besides the stock.

For the SV model of Heston (1993), only the second term on the right hand side of equation (9) remains, and the EHE is proportional to the MPVR. To identify the sign of the proportionality factor $\mathrm{IE}^V(t,t+\tau)$, note that, for a European call option, the partial derivative with respect to volatility is positive, i.e., $\partial c^{\mathrm{SV}}/\partial v > 0$. The sign of the EHE is thus equal to the sign of the MPVR. Furthermore, the EHE is largest in absolute terms for ATM options with a medium to long time to maturity. Long-term ATM options are thus best suited to identify the sign of the MPVR in an empirical study, which creates a trade-off with the lower liquidity of these long-term options. Additionally, the absolute EHE is increasing in current volatility, so that the MPVR can be better identified in times of high volatility.

In the SVJ model of Bates (1996), the third term on the right hand side of equation (9) also contributes to the EHE. It depends on the expected additional

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jump risk of the claim, which is positive for a convex contract like a call option both for upward and downward jumps. Thus, $\mathrm{IE}^J(t, t+\tau)$ is also positive. In empirically relevant scenarios, the jump intensity is larger under the risk-neutral measure than under the true measure, leading to a negative contribution of jump risk to the EHE.

For the SVJ model, the EHE is thus the sum of two terms that capture the impact of SV and SJ. A negative EHE can be explained by a negative MPVR, but also by a jump risk premium where the risk-neutral intensity is larger than the physical intensity. If one assumes that the EHE is caused by SV only and ignores the pricing of jump risk, any negative contribution of jump risk to the EHE is incorrectly attributed to SV. This model misspecification induces a downward bias into the assessment of the MPVR. In particular, if volatility risk is not priced, the omission of the jump risk premium from the analysis will lead to the wrong conclusion that volatility risk carries a negative premium.

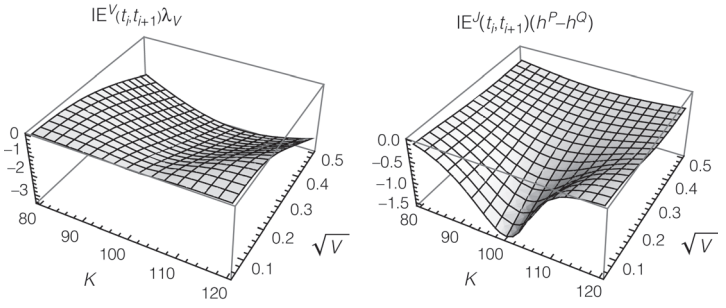
Figure 1 shows the contribution of stochastic volatility and jumps to the EHE for options with three months to maturity as a function of strike price and current volatility $\sqrt{V}$. The MPVR is negative, and the jump intensity is larger under the risk-neutral measure than under the true measure. As argued above, both components of the EHE are negative. The figure further shows that they are of the same order of magnitude. So it would be very hard to distinguish between the jump and the volatility impact on the EHE, which could easily lead to incorrect conclusions about the MPVR.

The figure also shows a crucial difference between the ideal EHEs due to volatility risk and jump risk. While the contribution of the MPVR is in absolute terms increasing in current volatility, the opposite is true for the contribution of the jump risk premium. This suggests that the dependence of the EHE on the level of volatility can be used to distinguish between the pricing of jump risk and the pricing of SV. However, there are at least two problems with this idea. First, the EHE is usually estimated from a time series, so that the dependence on volatility

FIGURE 1

Impact of SV and SJ on EHE (Ideal Case)

The graphs show the impact of SV and SJ on the ideal EHE as a function of the strike price and the current level of volatility $\sqrt{V}$. The left graph shows the term $\mathrm{IE}^V(t_i, t_{i+1})\lambda_V$, the right graph shows the term $\mathrm{IE}^J(t_i, t_{i+1})(h^P - h^Q)$. The MPVR is given by $\lambda_V = -2$, the difference of the jump intensities is $h^Q - h^P = 0.75$. The time to maturity of the options is three months, the current stock price is 100, and the length $t_{i+1} - t_i$ of the hedge interval is one day, i.e., 1/365. The other parameters are $r = 0$, $\kappa^Q = 2.0$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.5$, $h^Q = 1.5$, $\mu_X = -0.05$, $\sigma_X = 0.07$.



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is (at least partly) removed. Second, the monotonicity of the EHE with respect to volatility is driven by the dependence of the risk premium on volatility. The ideal EHE due to SJ decreases in volatility if we assume, as above, that the jump intensities under the true and the risk-neutral measure are constant. If we consider jump intensities that are proportional to the level of V, like in Bates (2000), the contribution of jump risk to the EHE is also proportional to the level of V, and it is in absolute terms larger when the current volatility is higher. In this case, the dependence of the EHE on volatility can no longer be used to distinguish between the pricing of SV and SJ.

Finally, note that IE^V is increasing in the time to maturity of the option, while IE^J is decreasing in time to maturity. This reflects the fact that jump risk is more important for pricing short-term options, while stochastic volatility becomes dominant for medium- to long-term options. The EHE from long-term options is thus better suited to pick up the MPVR than the EHE from short-term options. In an empirical test, this would again create a trade-off with the lower liquidity of options with longer maturity.

The EHE we consider in our analysis is derived from the hedging errors of options with a given time to maturity and a given moneyness, and it is conditional on a certain value for the stochastic volatility. In an empirical study like the one performed by Bakshi and Kapadia (2003), however, the EHE is estimated from a time series of hedging errors. The options used will in most cases vary in terms of moneyness and time to maturity, and in case of an SV model, volatility as a state variable will also change over time. One can try to minimize the impact of the first problem by choosing the options accordingly, but there is no way to really get rid of the second problem. The mean hedging error in the empirical study thus corresponds to an average over the EHE considered here, where this average is taken with respect to volatility, time to maturity, and moneyness. By this averaging, we lose at least part of the information embedded in the dependence of the EHE on these factors.

In the preceding analysis we were mainly interested in the sign of the EHE. Fortunately, the relation between the sign of the ideal EHE and the sign of the premia for volatility risk and jump risk does not depend on any of the characteristics discussed in the previous paragraph, so that the properties of the EHE derived above and in the following sections also apply to an empirically observed average hedging error. In an SV model and under the ideal conditions of continuous trading and no model misspecification, this average hedging error can thus still be used to identify the sign of the MPVR.

C. Discretization Error

1. General Results

In reality, trading is not continuous, and the hedge portfolio chosen at time t_i cannot be adjusted over the interval (t_i, t_{i+1}). If the induced discretization error has no systematic impact, this will result in additional noise, but the standard test may still be viable. If, on the other hand, the discretization error has a systematic impact on the EHE, then inferences based on the EHE in the ideal case may be

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misleading. The general formula for the EHE in case of discrete trading is given in the next proposition.

Proposition 2 (*SVJ: discretization error*). The true model is Bates (1996). Trading takes place at the discrete dates $0 = t_0, t_1, t_2, \ldots, t_n = T$, the hedge ratio at time t_i is set equal to the delta of the claim at time t_i. Then the EHE over the interval $[t_i, t_{i+1}]$ is

$$
\begin{aligned}
(10) \qquad & \mathrm{E}^P\left[D^d(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}\right] \\
& = \mathrm{IE}^V(t_i, t_{i+1})\lambda_V + \mathrm{IE}^J(t_i, t_{i+1})(h^P - h^Q) \\
& \quad + (\mu - r)\left[\mathrm{DE}^S(t_i, t_{i+1})(\mu - r) + \mathrm{DE}^V(t_i, t_{i+1})\lambda_V\right. \\
& \quad \left. + \mathrm{DE}^J(t_i, t_{i+1})(h^P - h^Q)\right],
\end{aligned}
$$

where IE^V and IE^J are defined in Proposition 1, and where

$$
\mathrm{DE}^S(t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} E^P\left[S_t \int_{t_i}^{t} \frac{\partial^2 c}{\partial s^2}(u, S_u, V_u)\, S_u du \mid \mathcal{F}_{t_i}\right] dt
$$

$$
\mathrm{DE}^V(t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} E^P\left[S_t \int_{t_i}^{t} \frac{\partial^2 c}{\partial s \partial v}(u, S_u, V_u)\, \sigma_V V_u du \mid \mathcal{F}_{t_i}\right] dt
$$

$$
\mathrm{DE}^J(t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)}\, \mathrm{E}^P\left[S_{t-} \int_{t_i}^{t} \mathrm{E}^Q\left[g(u, S_{u-}, V_{u-}, X_t) \mid F_{u-}\right] du \;\middle|\; \mathcal{F}_{t_i}\right] dt
$$

$$
g(t, s, v, x) = \left[\frac{\partial c}{\partial s}(t, se^x, v) - \frac{\partial c}{\partial s}(t, s, v)\right] e^x - \frac{\partial^2 c}{\partial s^2}(u, s, v)\, s\left(e^x - 1\right).
$$

Proof. See Appendix B.

The first two terms on the right hand side of equation (10) are the ideal EHE as given in Proposition 1. They represent the EHE if trading is continuous and if stock price risk is eliminated by setting the hedge ratio equal to the delta of the claim. The remaining terms capture the impact of discrete trading. Since the hedge portfolio is not adjusted continuously, a stale hedge ratio is used. This creates an additional exposure to stock price risk on which the equity risk premium is earned. The resulting discretization error depends on the average stock risk exposure caused by the staleness of the hedge ratio. The factors DE^S, DE^V, and DE^J, multiplied with the respective risk premia, capture the staleness due to random changes in the hedge ratio that are caused by stock price changes, volatility changes, and jumps.

2. BS Model

To obtain the model of BS, we set $h^P = h^Q = 0$, $\sigma_V = 0$, and $V_0 = \theta = \sigma^2$, where σ is the constant volatility. As shown in Section IV.B, applying a hedge ratio equal

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to the partial derivative of the claim price with respect to the stock price generates a zero hedging error in the ideal case of continuous trading. This is no longer true with discrete trading, since technically speaking, the market becomes incomplete.

From Proposition 2, we see that the EHE is given by

$$E^P\left[D^d(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}\right] = (\mu - r)^2 \, \mathrm{DE}^S(t_i, t_{i+1}).$$

$\mathrm{DE}^S(t_i, t_{i+1})$ depends on the sensitivity of delta with respect to changes in the stock price, i.e., on the gamma of the claim. For any path-independent claim with a convex payoff function, Bergman, Grundy, and Wiener (1996) show that the price is also a convex function of the stock price, so that gamma is positive. The discretization error thus induces a positive EHE (see also Dudenhausen (2003)), which is the largest for short term ATM options and for a low volatility.

This result has important consequences for the test described in Section II. This test is based on the idea that the EHE is zero if and only if equity risk is the only priced risk factor. The (wrong) conclusion derived from the standard hedging based test in the BS model must be that equity risk is not the only priced factor, but that there is some other source of risk with a positive premium (given that the number of observations is large enough to make the mean significantly positive in an empirical application).

The absolute amount of the EHE of course depends on the length of the time interval between two consecutive trading dates (see also Bertsimas, Kogan, and Lo (2000)). It becomes smaller the shorter this period is, but, irrespective of the length of the trading interval, it always remains analytically strictly positive for non-zero equity risk premia. Note that our result differs from Boyle and Emanuel (1980), who use limiting arguments improperly to arrive at the conclusion that the discretization error vanishes even in discrete time.

3. SV Model

Given that standard hedging-based tests can no longer be applied in the model of BS, we expect problems to be even worse in an SV model. Delta is changing over time not only due to changes in the stock price, but also due to varying volatility. The average remaining exposure to stock price risk is given by DE^S and DE^V in equation (10).

The structure of DE^S is the same as in the model of BS. For a convex claim like the call, DE^S is strictly positive, and if the equity risk premium does not vanish, the discretization error caused by a stale hedge ratio is positive.

The term DE^V depends on the sensitivity of delta with respect to volatility. This sensitivity is positive for high strike prices and negative for low strike prices. It is multiplied with the MPVR and the equity risk premium, so that there is now a second term besides the ideal EHE that depends on the market price of volatility risk. For a positive equity risk premium, the sum $\mathrm{IE}^V + (\mu - r)\mathrm{DE}^V$ is positive when the strike price of the call is high, while for a low strike price, the sign cannot be determined uniquely. We thus expect the test to be more reliable for options with high strike prices. However, in both cases the presence of DE^S may lead to incorrect conclusions about the MPVR.

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The key question is how severe the problems caused by discrete trading actually are. The larger the discretization error compared to the ideal EHE, the higher its impact on the EHE, and the more often our conclusion about the sign of the MPVR will not be based on the ideal EHE, but (incorrectly) on the discretization error. Since the expressions for the EHE in the preceding proposition are only semi-closed, we have to calculate the EHE for this and the following scenarios using a Monte Carlo simulation. We use 10,000 runs over a hedging horizon of one week (or one day) for every run. The parameters are chosen similar to the empirical results shown in BCC (1997). The stochastic differential equations for the dynamics of the stock price and the local variance are discretized using a standard Euler scheme. To avoid problems with possibly negative values for the variance and to come closer to a continuous representation of the stochastic differential equations of the state variables, we divide the day into 100 steps of length $\Delta t = 1/36{,}525$ each. As described in Section II, the hedge portfolio is composed at time t_i (which may be taken to be $t = 0$ without any loss of generality) by selling the option short, buying the number of stocks indicated by the delta, and borrowing the necessary funds at the risk-free rate. The hedging error at time t_{i+1}, i.e., after one week (one day), is then given by equation (4).

To increase the accuracy of our estimates we use the expected value of V at the end of the hedging interval as a control variate. We apply a two-sided t-test with significance level $\alpha = 0.05$ to test the null hypothesis that the mean hedging error is zero, whereas in the cases when the null states a positive or a negative premium, the test is one-sided. Note that this does not mean that we consider the t-test appropriate. It is merely that this test has been used in papers trying to identify the sign of λ_V using expected option hedging errors, for example, Bakshi and Kapadia (2003).

The results of our simulation analysis are shown in Table 1, which presents the average hedging error for different maturity classes, strike prices, current values of V, and risk premia of the stock. The MPVR is set equal to zero to isolate the discretization error. The hedging interval is chosen to be one week, which coincides with the hedging interval used by Coval and Shumway (2001) in their empirical study of option returns. This period length allows an analysis of discretization errors under a magnifying glass, since we expect the hedging error to increase with the duration of the hedge. The qualitative properties of the EHE should remain unchanged compared to shorter hedging intervals.

As one can see from the table, the EHE is either not significantly different from zero or significantly positive. In the latter case, we would deduce from the simple discrete hedging-based test that the volatility risk premium is positive when it is truly zero. This misinterpretation of the test result occurs mostly for a low current volatility of $\sqrt{V_0} = 0.1$ and a high equity risk premium of $\mu - r = 0.2$. As argued above, the discretization error is largest for these scenarios.

We also performed a simulation experiment for the shorter hedging interval of one day and the scenarios considered above. The results (not shown here to save space) show that the average hedging error is not statistically different from zero any more for the case of a zero volatility risk premium. For a positive or a negative MPVR, there are still some scenarios when a test based on the EHE

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TABLE 1

SV Model: Impact of Discretization Error ($\lambda_V = 0$)

Average hedging error (multiplied by 1,000) for a European call option as a function of the strike price K, the equity risk premium $\mu - r$, and the time to maturity TTM in months. The hedge ratio is given by the Heston delta, the hedging interval is one week ($\tau - t = 1/52$). The parameters of the risk-neutral process are given by $S_0 = 100$, $r = 0.0$, $\kappa^Q = 1.15$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.65$. The symbol "+" behind an average hedging error entry indicates that it is significantly positive. One would thus wrongly reject the null hypothesis of a zero MPVR.

		$V_0 = 0.01$			$V_0 = 0.04$			$V_0 = 0.09$		
TTM	$\mu - r$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
1	0	–0.069	–2.181	–0.001	–1.257	–3.485	–0.255	–3.368	–4.757	–1.598
3	0	–0.260	–1.337	–0.040	–1.145	–2.311	–1.267	–2.245	–3.114	–2.523
6	0	–0.281	–0.850	–0.386	–0.928	–1.740	–1.624	–1.705	–2.403	–2.471
12	0	–0.228	–0.483	–0.617	–0.707	–1.204	–1.476	–1.290	–1.829	–2.139
1	0.1	–0.027	–0.081	0.005	–1.040	–2.436	0.033	–3.067	–4.135	–1.212
3	0.1	–0.117	–0.275	0.248	–0.910	–1.698	–0.664	–2.012	–2.728	–2.126
6	0.1	–0.123	–0.209	0.480	–0.736	–1.318	–1.032	–1.526	–2.123	–2.133
12	0.1	–0.086	–0.116	0.103	–0.563	–0.926	–1.041	–1.160	–1.628	–1.874
1	0.2	0.087	6.350 +	0.020 +	–0.442	1.098	1.092	–2.142	–1.811	0.351
3	0.2	0.306	2.934 +	1.192 +	–0.176	0.303	1.464	–1.223	–1.352	–0.629
6	0.2	0.355	1.733 +	3.258 +	–0.123	0.046	0.987	–0.912	–1.138	–0.896
12	0.2	0.354	1.007 +	2.339 +	–0.094	–0.022	0.400	–0.706	–0.926	–0.927

produces a misleading result, but compared to a hedging interval of one week, the number of incorrect results is reduced significantly.

4. SVJ Model

In case of the SVJ model, the EHE is

$$\mathrm{DE}^S (\mu - r)^2 + \left[\mathrm{IE}^V + \mathrm{DE}^V(\mu - r)\right] \lambda_V + \left[\mathrm{IE}^J + \mathrm{DE}^J(\mu - r)\right] (h^P - h^Q).$$

The first two terms are structurally equal to the EHE in an SV model, while the third and last term captures the contribution of jump risk. It depends on the ideal EHE, and, additionally, on the staleness of the hedge ratio caused by jumps in the stock price.

If jump risk is not priced, that is, if $h^Q = h^P$, the staleness of the hedge ratio due to jump risk does not contribute to the EHE, but only increases the variance of the hedging error. There will thus be more cases in which, for a non-zero MPVR, the mean hedging error is not statistically significantly different from zero, and it will be more difficult to identify the sign of the MPVR.

If jump risk is priced, the contribution of the jump risk premium to the EHE is given by the ideal EHE and the discretization error. For a convex claim like a call option, IE^J is positive, as argued above. DE^J does not have a unique sign, and numerical calculations show that it is negative for low strike prices and positive for high strike prices. The EHE is thus increasing in the difference $h^P - h^Q$ for high strike prices, while it may be increasing or decreasing for low strike prices. If the staleness of the hedge ratio indeed changes the monotonicity of the EHE with respect to the difference $h^P - h^Q$, we will draw a possibly incorrect inference about the pricing of jump risk.

Again, the question is how severe the problems caused by discrete trading actually are when there is also jump risk in addition to volatility risk. To answer this question, we again perform a simulation study. The setup is similar to the one

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TABLE 2

SVJ Model: Impact of Discretization Error ($\lambda_V = -2$, $h^Q = h^P$)

Average hedging error (multiplied by 1,000) for a European call option as a function of the strike price K, the equity risk premium $\mu - r$, and the time to maturity TTM in months. The hedge ratio is given by the SVJ delta, the hedging interval is one week ($\tau - t = 1/52$). The parameters of the risk-neutral process are given by $S_0 = 100$, $r = 0.0$, $\kappa^Q = 2.0$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.5$, $h^Q = 1.5$, $\mu_X = -0.05$, $\sigma_X = 0.07$. A "**0**" behind an average hedging error entry indicates that it is not significantly different from zero. One would thus wrongly reject the null hypothesis of a negative MPVR.

		$V_0 = 0.01$			$V_0 = 0.04$			$V_0 = 0.09$		
TTM	$\mu - r$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
1	0	−0.039 0	−7.294 0	−0.684 0	−3.288 0	−15.90	−4.322	−10.82	−24.63	−14.51
3	0	−3.255 0	−8.227 0	−3.099	−12.72	−22.12	−15.39	−26.06	−36.67	−31.56
6	0	−5.267	−7.899	−6.075	−17.88	−24.24	−21.08	−34.34	−42.80	−40.43
12	0	−5.572	−6.808	−6.297	−19.04	−22.66	−21.58	−37.12	−43.05	−42.43
1	0.1	0.053 0	−5.136	−0.663 0	−2.907 0	−14.58	−4.177	−10.20	−23.70	−14.32
3	0.1	−2.933 0	−7.207 0	−2.732	−12.23	−21.37	−15.08	−25.52	−36.13	−31.35
6	0.1	−4.921	−7.273	−5.524	−17.46	−23.73	−20.75	−33.91	−42.41	−40.24
12	0.1	−5.284	−6.405	−5.885	−18.73	−22.31	−21.30	−36.81	−42.77	−42.25
1	0.2	0.346 0	0.977 0	−0.557 0	−2.080 0	−10.86 0	−3.241	−8.934 0	−21.09	−13.01
3	0.2	−2.097 0	−4.332 0	−1.348 0	−11.12	−19.28	−13.54	−24.36	−34.58	−30.15
6	0.2	−3.993 0	−5.453 0	−3.688 0	−16.48	−22.26	−19.38	−32.96	−41.27	−39.24
12	0.2	−4.471	−5.215	−4.599	−17.93	−21.28	−20.25	−36.07	−41.92	−41.44

in Section IV.C.3. To include jumps, we set the number of paths with one, two or more jumps equal to the expected number of paths with this number of jumps, thus eliminating the noise from the simulated distribution of the number of jumps. Furthermore, if there are n paths with a jump, we set the jump size x_i such that $P(X \leq x_i) = (2i - 1)(2n - 1)^{-1}$, thus also eliminating the simulation error from the jump size distribution.

Table 2 gives the results of the test in case of a negative MPVR and a zero risk premium for jump risk when the holding period is one week. The simulations show that the test is more likely to reject the null of a negative MPVR the lower current volatility. Intuitively, this can be explained by the fact that the ideal EHE due to volatility risk is (in absolute) terms increasing in volatility, so that a non-zero EHE is more difficult to identify when volatility is low.

If jump risk is priced, both IE^J and DE^J contribute to the EHE. Further simulation results not presented here suggest that the ideal EHE due to jump risk is much larger than the discretization error. Thus we have to worry more about the negative bias induced by the jump risk premium than about the discretization error.

D. Model Misspecification

1. Model Misspecification Under Continuous Trading

The term model misspecification refers to the use of an incorrect hedge model, caused by the fact that the true data generating process is unknown. In this paper, all hedges under model misspecification are based on the BS model. This seems a natural choice, since this model represents the simplest approach to option pricing, and it has proved to be rather robust in empirical applications

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(see, e.g., BCC (1997) or Dumas, Fleming, and Whaley (1998)). The hedge ratio applied in the test for volatility risk premia is thus given by

$$H_t \equiv \frac{\partial c^{\mathrm{BS}}}{\partial s}(t, S_t).$$

To compute this partial derivative, a volatility input is needed. We use the implied BS volatility of the option for this purpose, although other studies use historical measures. Nevertheless, the implied volatility of an option seems preferable to us as the more natural choice, given the way practitioners usually handle liquid options.

An expression for the EHE in case of continuous trading and model misspecification in the SVJ model is given in the following proposition.

Proposition 3 (*SVJ: EHE under model misspecification*). The true model is Bates (1996). The hedge portfolio is rebalanced continuously, the hedge ratio is the BS delta based on the implied volatility. $\sigma^{\mathrm{BS}}(t, M_t)$ denotes the BS implied volatility at time t for an option with moneyness $M_t = K/S_t$, where K is the strike price. Then the EHE over the interval $[t, t+\tau]$ is given by

$$\begin{aligned}(11)\quad & \mathrm{E}^P\left[D^c(t, t+\tau) \mid \mathcal{F}_t\right] \\ & = \mathrm{IE}^V(t, t+\tau)\lambda_V + \mathrm{IE}^J(t, t+\tau)(h^P - h^Q) + \mathrm{ME}^c(t, t+\tau)(\mu - r),\end{aligned}$$

where IE^V and IE^J are defined in Proposition 1, and where

$$\mathrm{ME}^c(t, t+\tau) = -\int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[\frac{\partial c^{\mathrm{BS}}}{\partial \sigma^{\mathrm{BS}}}(u, S_u)\,\frac{\partial \sigma^{\mathrm{BS}}}{\partial m}(u, M_u)\,M_u \mid \mathcal{F}_t\right] du.$$

Proof. The first two terms on the right-hand side of equation (11) are the EHE if the correct hedge ratio is used in continuous time (see Proposition 1). The third term originates from the use of the BS delta instead of the correct partial derivative from the SVJ model. It therefore represents the impact of model misspecification, and it depends on the remaining stock risk exposure and on the equity risk premium. The expected remaining stock risk exposure can be rewritten as

$$\mathrm{ME}^c(t, t+\tau) = \int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[\left(\frac{\partial c^{\mathrm{SVJ}}}{\partial s}(u, S_u, V_u) - \frac{\partial c^{\mathrm{BS}}}{\partial s}(u, S_u)\right) S_u \mid \mathcal{F}_t\right] du.$$

Based on a result in Bates (2005), one can show that the difference between the partial derivatives in the SVJ model and the model of BS is related to the slope of the implied volatility function via

$$\frac{\partial c^{\mathrm{SVJ}}}{\partial s}(u, S_u, V_u) - \frac{\partial c^{\mathrm{BS}}}{\partial s}(u, S_u) = -\frac{\partial c^{\mathrm{BS}}}{\partial \sigma^{\mathrm{BS}}}(u, S_u) \cdot \frac{\partial \sigma^{\mathrm{BS}}}{\partial m}(u, M_u)\frac{M_u}{S_u}.$$

Combining the results gives the expression for ME^c from Proposition 3. □

Empirically, the smile for the major stock market indices decreases in the strike price of the options. Furthermore, the equity risk premium is generally found

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to be positive. These stylized facts imply that the term due to model risk is positive, so that the EHE under model risk is larger than the EHE when the correct SVJ delta is used. This introduces an upward bias into the estimate of the MPVR.

2. Model Misspecification and Discretization Error

In an empirical study, the problems caused by discrete trading and model misspecification occur simultaneously, and both of them make the results of the standard test questionable. Before we turn to the detailed simulation study, we give the general expression for the EHE. The hedge ratio is set equal to the Black-Scholes delta, so that

$$H_{t_i} \equiv \frac{\partial c^{\mathrm{BS}}}{\partial s}(t_i, S_{t_i}).$$

The volatility used to compute the partial derivative is the implied volatility for which the BS option price is equal to the price under the correct model. We obtain:

Proposition 4 (*SVJ: EHE with model error and discretization error*). The true model is Bates (1996), and trading is discrete. The hedge ratio H_{t_i} at time t_i is the BS delta based on the implied volatility of the claim. Then the EHE over the interval $[t_i, t_{i+1}]$ is given by

$$\begin{aligned}(12)\qquad & \mathrm{E}^P\left[D^d(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}\right] \\ &= \mathrm{IE}^V(t_i, t_{i+1})\lambda_V + \mathrm{IE}^J(t_i, t_{i+1})(h^P - h^Q) \\ &\quad + (\mu - r)[\mathrm{DE}^S(t_i, t_{i+1})(\mu - r) + \mathrm{DE}^V(t_i, t_{i+1})\lambda_V \\ &\qquad + \mathrm{DE}^J(t_i, t_{i+1})(h^P - h^Q)] + \mathrm{ME}^d(t_i, t_{i+1})(\mu - r),\end{aligned}$$

where IE^V and IE^{J} are defined in Proposition 1, DE^S, DE^V, and DE^J are defined in Proposition 2, and

$$\mathrm{ME}^d(t_i, t_{i+1}) = -\frac{\partial c^{\mathrm{BS}}}{\partial \sigma^{\mathrm{BS}}}(t_i, S_{t_i})\,\frac{\partial \sigma^{\mathrm{BS}}}{\partial m}(t_i, M_{t_i})\,\frac{M_{t_i}}{S_{t_i}}\int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)}E^P\left[S_t \mid \mathcal{F}_{t_i}\right]dt.$$

Proof. The proof proceeds along the same lines as the proof for Proposition 3. Note that for discrete trading, we do not have to consider the average difference of the hedge ratios in the two models over the interval $[t_i, t_{i+1}]$, but only the difference at time t_i. □

Proposition 4 gives the EHE we deal with in an empirical study. The first two terms in equation (12) are the ideal EHE. The next three terms represent the additional EHE due to discrete trading if the hedge ratio is the correct partial derivative from the SVJ model (see Proposition 2). The last term is an additional component due to the use of an incorrect hedge model. Following the arguments

https://doi.org/10.1017/S0022109000014447 Published online by Cambridge University Press

for the case of continuous hedging, this term is positive if the equity risk premium is positive and if implied volatility is a decreasing function of the strike price.

The empirically most relevant scenario is given by a positive equity risk premium and a downward sloping smile. Furthermore, we assume that the hedging interval is so small that the contribution of the discretization error to the EHE can be ignored. If neither jump risk nor volatility risk are priced, the EHE is positive due to model misspecification. The wrong conclusion from the standard hedging-based test will then be that the MPVR is positive. If volatility risk is priced, while jump risk is not, we have to distinguish two cases. For a positive MPVR, the EHE is also positive, both due to model risk and due to the volatility risk premium, and the conclusion from the standard hedging-based test is correct. For a negative MPVR, however, the EHE may be negative or positive. To end up with a negative EHE and thus the correct conclusion from the hedging-based test, the ideal EHE due to SV has to be larger in absolute terms than the contribution of model misspecification. This is more likely for a large MPVR and a small equity risk premium, as well as for high values of current volatility (since the ideal EHE due to SV is increasing in volatility) and for a long time to maturity (for which the smile becomes quite flat, which decreases the impact of model risk). Finally, if jump risk is also priced with $h^Q > h^P$, the contribution of the jump risk premium to the EHE is negative. This will bias the test toward the conclusion that the MPVR is negative. In particular, if the MPVR is zero, the jump risk premium might be misinterpreted as a negative MPVR.

The most important result that we can derive from this discussion is that the sign of the EHE is in general not equal to the sign of the MPVR, but that there are additional terms due to model misspecification and the presence of jump risk, so that the standard hedging-based test becomes unreliable.

SV Model. To assess the severity of the problems caused by discrete trading and model misspecification, we first perform a simulation study for the SV model. This case is ideal in the sense that the only risk factor besides stock price risk is SV, so that we do not have to worry about jump risk premia. The setup is the same as in Section IV.C.3, except that the hedge ratio is now calculated in the model of BS. Tables 3 and 4 present the average hedging errors for different maturities, strike prices, current values of V, and equity risk premia. The market price of volatility risk λ_V is chosen to be 0 and -2, respectively, and the hedging interval is one day. As discussed above, for this period length the discretization error can basically be neglected, so that all of the error of the test procedure can be attributed to model misspecification.

If the MPVR is zero, the mean hedging error is in most cases larger than under ideal conditions, as we can see from the entries in Table 3. This is perfectly in line with our theoretical arguments presented above. The test from Section II is not valid in general, and the user tends to conclude that the MPVR is positive, although it is actually zero.

The most critical scenario in terms of the validity of the standard test procedure is of course the situation when the true MPVR is negative. Since average hedging errors under the assumed scenario of a downward sloping smile and a positive equity risk premium tend to be positive, the test is likely to indicate a

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TABLE 3

SV Model: Impact of Model Misspecification ($\lambda_V = 0$)

Average hedging error (multiplied by 1,000) for a European call option as a function of the strike K, the equity risk premium $\mu - r$, and the time to maturity TTM in months. The hedge ratio is given by the BS delta, the hedging interval is one day ($\tau - t = 1/365$). The parameters of the risk-neutral process are given by $S_0 = 100$, $r = 0.0$, $\kappa^Q = 1.15$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.65$. The symbol "**+**" ("**–**") behind an average hedging error entry indicates that it is significantly positive (negative). One would thus wrongly reject the null hypothesis of a zero MPVR.

		$V_0 = 0.01$			$V_0 = 0.04$			$V_0 = 0.09$		
TTM	$\mu - r$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
1	0	–0.027	–0.272	0.00	–0.193	–0.588	–0.078	–0.476	–0.892	–0.427
3	0	–0.085	–0.164	–0.008	–0.257	–0.368	–0.152	–0.447	–0.559	–0.416
6	0	–0.091	–0.13	–0.017	–0.237	–0.291	–0.168	–0.386	–0.444	–0.373
12	0	–0.088	–0.109	–0.048	–0.21	–0.243	–0.181	–0.337	–0.378	–0.344
1	0.1	0.04	1.686 +	0.002 +	0.05	0.454	0.032	–0.151	–0.193	–0.087
3	0.1	0.645 +	2.445 +	–0.071 –	0.663 +	1.352 +	0.454	0.396	0.63	0.503
6	0.1	1.372 +	2.623 +	0.011	1.273 +	1.901 +	0.959 +	0.907	1.188	1.024
12	0.1	1.951 +	2.604 +	0.959 +	1.797 +	2.201 +	1.512 +	1.425	1.668	1.484
1	0.2	0.108 +	3.745 +	0.005 +	0.301	1.549 +	0.153	0.187	0.542	0.276
3	0.2	1.382 +	5.111 +	–0.128 –	1.595 +	3.104 +	1.087 +	1.251 +	1.841 +	1.444 +
6	0.2	2.843 +	5.415 +	0.06	2.793 +	4.117 +	2.113 +	2.209 +	2.836 +	2.441 +
12	0.2	4.00 +	5.341 +	1.994 +	3.814 +	4.662 +	3.227 +	3.196 +	3.728 +	3.327 +

TABLE 4

SV Model: Impact of Model Misspecification ($\lambda_V = -2$)

Average hedging error (multiplied by 1,000) for a European call option as a function of the strike price K, the equity risk premium $\mu - r$, and the time to maturity TTM in months. The hedge ratio is given by the BS delta, the hedging interval is one day ($8\tau - t = 1/365$), and the hedge is not rebalanced in between. The parameters of the risk-neutral process are given by $S_0 = 100$, $r = 0.0$, $\kappa^Q = 1.15$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.65$. The symbol "**+**" ("**0**") behind an average hedging error indicates that it is significantly positive (not significantly different from zero). One would thus wrongly reject the null hypothesis of a negative market price of volatility risk.

		$V_0 = 0.01$			$V_0 = 0.04$			$V_0 = 0.09$		
TTM	$\mu - r$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
1	0	–0.074	–1.417	–0.001	–0.744	–2.972	–0.471	–2.105	–4.489	–2.357
3	0	–0.580	–1.793	–0.138	–2.295	–4.219	–2.316	–4.535	–6.463	–5.472
6	0	–1.035	–1.872	–0.586	–3.421	–5.060	–3.804	–6.242	–8.039	–7.646
12	0	–1.269	–1.730	–1.177	–4.178	–5.345	–4.757	–7.598	–9.176	–9.142
1	0.1	–0.007 0	0.542 0	0.002 +	–0.499	–1.929	–0.367	–1.775	–3.789	–2.025
3	0.1	0.151 0	0.817 +	–0.203	–1.371	–2.498	–1.720	–3.687	–5.273	–4.562
6	0.1	0.429 0	0.881 +	–0.561	–1.908	–2.867	–2.687	–4.946	–6.407	–6.255
12	0.1	0.771 +	0.983 +	–0.174 0	–2.168	–2.901	–3.071	–5.833	–7.129	–7.320
1	0.2	0.062 +	2.603 +	0.004 +	–0.244 0	–0.833 0	–0.250	–1.431	–3.054	–1.671
3	0.2	0.890 +	3.484 +	–0.261	–0.434 0	–0.745 0	–1.098	–2.826	–4.062	–3.630
6	0.2	1.903 +	3.673 +	–0.516	–0.384 0	–0.651 0	–1.542	–3.639	–4.759	–4.845
12	0.2	2.821 +	3.720 +	0.858 +	–0.148 0	–0.439 0	–1.362	–4.059	–5.070	–5.481

positive or zero volatility premium, whereas it is indeed negative. As becomes clear from Table 4, the problem is again especially pronounced for a low current volatility and medium or high equity risk premia.

As one may expect, for a positive MPVR, the EHE is also almost always significantly positive (the results are omitted to save space). However, this is not due to an improved quality of the test per se, but an immediate consequence of the fact that the ideal EHE and the hedging errors caused by discretization and model

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TABLE 5

SVJ Model: Impact of Model Misspecification and Omission of Jump Risk
($\lambda_V = 0$, $h^Q = 3h^P$)

Average hedging error (multiplied by 1,000) for a European call option as a function of the strike price K, the equity risk premium $\mu - r$, and the time to maturity TTM in months. The hedge ratio is given by the BS delta, the hedging interval is one day ($\tau - t = 1/365$), and the hedge is not rebalanced in between. The parameters of the risk-neutral process are given by $S_0 = 100$, $r = 0.0$, $\kappa^Q = 2.0$, $\theta^Q = 0.04$, $\sigma_v = 0.4$, $\rho = -0.5$, $h^Q = 1.5$, $\mu_X = -0.05$, $\sigma_X = 0.07$. The symbol "+" ("−") behind an average hedging error entry indicates that it is significantly positive (negative). One would thus wrongly reject the null hypothesis of a zero MPVR.

		$V_0 = 0.01$			$V_0 = 0.04$			$V_0 = 0.09$		
TTM	$\mu - r$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
1	0	−2.419 −	−6.534 −	−0.420 −	−2.671 −	−4.775 −	−0.888 −	−2.403 −	−3.393 −	−1.419 −
3	0	−2.247 −	−4.400 −	−1.354 −	−2.130 −	−3.289 −	−1.934 −	−1.802 −	−2.374 −	−1.864 −
6	0	−1.849 −	−2.938 −	−2.110 −	−1.704 −	−2.416 −	−2.059 −	−1.484 −	−1.886 −	−1.761 −
12	0	−1.356 −	−1.852 −	−1.816 −	−1.320 −	−1.708 −	−1.704 −	−1.233 −	−1.494 −	−1.505 −
1	0.1	−1.862 −	−4.103 −	−0.540 −	−2.157 −	−3.629 −	−0.824 −	−1.987 −	−2.709 −	−1.146 −
3	0.1	−1.387 −	−2.245 −	−1.325 −	−1.304 −	−1.949 −	−1.353 −	−1.098	−1.452	−1.194
6	0.1	−0.790	−1.280 −	−1.313 −	−0.663	−1.070	−1.133 −	−0.560	−0.805	−0.866
12	0.1	−0.241	−0.539	−0.757 −	−0.209	−0.467	−0.662	−0.185	−0.360	−0.497
1	0.2	−1.301	−1.588	−0.658 −	−1.633 −	−2.434 −	−0.745 −	−1.556	−1.991	−0.850
3	0.2	−0.515	−0.051	−1.276 −	−0.465	−0.581	−0.748	−0.381	−0.509	−0.505
6	0.2	0.281	0.403	−0.490	0.389	0.296	−0.186	0.375	0.292	0.045
12	0.2	0.885 +	0.791	0.320	0.913	0.788	0.396	0.873	0.786	0.524

misspecification tend to be positive here so that the test tends to detect a positive volatility risk premium.

SVJ Model. Next we perform a similar simulation study for the SVJ model. The setup is the same as the one described in Section IV.C.4, except that the hedge ratio is now calculated in the model of BS. Table 5 presents the average hedging error. The market price of volatility risk λ_V is set equal to zero, and the hedging interval is one day. Jump risk is priced, and we assume $h^Q = 3h^P$.

Since the MPVR is zero, the EHE depends on the jump risk premium and on the impact of model misspecification. The latter vanishes if the equity risk premium is zero, and in this case, the mean hedging error is significantly negative, reflecting the impact of jump risk. The wrong conclusion from the standard hedging-based test will then be that the MPVR is negative, since SV is confused with jump risk. The larger the equity risk premium, the larger the impact of model risk on the EHE, and the higher the mean hedging error. If the equity risk premium is large enough, the mean hedging error is no longer significantly negative, and the conclusion drawn from the hedging-based test is again correct. However, in this case we are simply lucky in that the negative EHE due to jump risk is offset by the positive EHE due to model misspecification, but we can in no way rely on this to happen every time we apply the test.

V. Hedging-Based Tests: Beyond Delta

A. Alternatives to Standard Delta Hedges

The results in the previous section show that the applicability of standard hedging-based tests is questionable. The question is thus whether any modifica-

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tions of these tests overcome the problems caused by discrete trading and model misspecification. In this section, we give a brief discussion of two straightforward approaches.

The first is to adjust the hedge ratio. Up to now, we have used the hedge ratio derived under the assumptions of continuous trading in a discrete time setup. Now, we try to find a discrete hedge ratio such that the discretization error vanishes. The problem of model misspecification, however, cannot be solved by adjusting the hedge ratio. We assume that the model used to calculate the hedge ratio is already the best alternative available, and while the model of BS is certainly a very simple choice, it illustrates the problems that arise whenever there is model risk.

The second approach, which is also suggested in Doran (2005), is to use yet another option to implement a delta-gamma hedge instead of just a delta hedge. Setting gamma equal to zero should reduce the discretization error caused by stock price risk. The impact on the other terms, i.e., the discretization errors due to volatility risk and jump risk, the term due to model misspecification, and the ideal hedging errors due to volatility risk and jump risk, however, is less clear.

B. Discrete Hedge Ratio

We first consider the adjustment of the hedge ratio to the discreteness of the hedging interval. In the model of BS, this discrete hedge ratio for a general claim is given in the next proposition.

Proposition 5 (*BS: discrete hedge ratio*). The true model is BS. Trading takes place at the discrete dates $0 = t_0, t_1, t_2, \ldots, t_n = T$. The discrete hedge ratio is then given by

$$H_{t_i}^{\mathrm{BS,disc.}}(t_{i+1}) = \frac{\partial c^{\mathrm{BS}}}{\partial s}(t_i, S_{t_i}) + \frac{\int_{t_i}^{t_{i+1}} e^{-rt} E^P \left[S_t \int_{t_i}^{t} \frac{\partial^2 c^{\mathrm{BS}}}{\partial s^2}(u, S_u) S_u du \mid \mathcal{F}_{t_i} \right] dt}{\int_{t_i}^{t_{i+1}} e^{-rt} E^P \left[S_t \mid \mathcal{F}_{t_i} \right] dt} (\mu - r).$$

Proof. From Propositions 1 and 2, we know that the EHE in the BS model is

$$\mathrm{E}^P \left[D^d(t_i, t_{i+1}) \mid \mathcal{F}_t \right]$$

$$= \left(\frac{\partial c^{\mathrm{BS}}}{\partial s}(t_i, S_{t_i}) - H_{t_i} \right) \int_{t_i}^{t_{i+1}} e^{-rt} E^P \left[S_t \mid \mathcal{F}_{t_i} \right] dt(\mu - r) + \mathrm{DE}^S(t_i, t_{i+1})(\mu - r)^2.$$

Setting the EHE equal to zero and solving for H_{t_i} gives the discrete hedge ratio. □

If the claim to be hedged is a European call or put, we can also give a closed form solution for the EHE and for the discrete hedge ratio that is based on a formula for the expected price of a call option (see Rubinstein (1984)).

The discrete hedge ratio differs from the partial derivative by the second term. For a convex claim, gamma is positive, which implies that the fraction is positive. In case the equity risk premium is positive, the discrete hedge ratio is

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greater than the partial derivative. This result is intuitively clear. If the hedge ratio is equal to the partial derivative, the EHE is positive. This implies that the expected terminal value of the hedge portfolio for the option is too small. To get an EHE of zero, we have to adjust the number of shares such that the expected terminal value of the hedge portfolio increases. For a positive (negative) risk premium, this means to increase (decrease) the number of shares.

The most important property of the hedge ratio given in Proposition 5 is that based on $H_{t_i}^{\mathrm{BS,disc.}}(t_{i+1})$, the standard test works properly. We would draw the correct conclusion that there is no other priced factor besides the stock. However, to achieve this objective we have to specify the dynamics of the underlying asset both under the risk-neutral and the physical measure, as opposed to the case where we only use the partial derivative, which is independent of μ.

The same idea can also be applied to the SVJ model:

Proposition 6 (*SVJ: discrete hedge ratio*). The true model is Bates (1996) and trading is discrete. The discrete hedge ratio is given by

$$
\begin{aligned}
H_{t_i}^{\mathrm{SVJ,disc.}}(t_{i+1}) &= \frac{\partial c^{\mathrm{SV}}}{\partial s}(t_i, S_{t_i}, V_{t_i}) \\
&+ \frac{\int_{t_i}^{t_{i+1}} e^{-rt} E^P\left[S_t \int_{t_i}^{t} \frac{\partial^2 c^{\mathrm{SV}}}{\partial s^2}(u, S_u, V_u) S_u du \mid \mathcal{F}_{t_i}\right] dt}{\int_{t_i}^{t_{i+1}} e^{-rt} E^P\left[S_t \mid \mathcal{F}_{t_i}\right] dt} (\mu - r) \\
&+ \frac{\int_{t_i}^{t_{i+1}} e^{-rt} E^P\left[S_t \int_{t_i}^{t} \frac{\partial^2 c^{\mathrm{SV}}}{\partial s \partial v}(u, S_u, V_u)\, \sigma_V V_u du \mid \mathcal{F}_{t_i}\right] dt}{\int_{t_i}^{t_{i+1}} e^{-rt} E^P\left[S_t \mid \mathcal{F}_{t_i}\right] dt} \lambda_V \\
&+ \frac{\int_{t_i}^{t_{i+1}} e^{-rt} E^P\left[S_t \int_{t_i}^{t} \mathrm{E}^Q\left[g(u, S_{u-}, V_{u-}) \mid \mathcal{F}_{u-}\right] du \mid \mathcal{F}_{t_i}\right] dt}{\int_{t_i}^{t_{i+1}} e^{-rt} E^P\left[S_t \mid \mathcal{F}_{t_i}\right] dt} (h^P - h^Q).
\end{aligned}
$$

The EHE over the interval $[t_i, t_{i+1}]$ generated by this hedge ratio is given by

$$
E^P\left[D^d(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}\right] = \lambda_V \mathrm{IE}^V(t_i, t_{i+1}) + (h^P - h^Q)\mathrm{IE}^J(t_i, t_{i+1}).
$$

Proof. The proof is analogous to the proof of Proposition 5. □

If the discrete hedge ratio is used, the EHE coincides with the EHE under ideal conditions. If jump risk is not priced or if there is no jump risk, its sign is equal to the sign of the MPVR. If jump risk is priced, however, the EHE depends on two risk premia, and again, the jump risk premium may be wrongly interpreted as a volatility risk premium.

The main problem, however, is that the discrete hedge ratio not only depends on the equity risk premium, but also on the MPVR and the pricing of jump risk. Thus, it can only be calculated once these quantities are known, but then there is no more need to identify the sign of the MPVR via a hedging-based test.

https://doi.org/10.1017/S0022109000014447 Published online by Cambridge University Press

C. Delta-Gamma Hedge

1. General Results

By setting the gamma of the hedge portfolio equal to zero when the hedge is initiated, the sensitivity of delta with respect to stock price changes is reduced, which should mitigate the adverse effects of a stale hedge ratio. Furthermore, one hopes that the discretization errors due to volatility risk and jump risk are also reduced. This would leave only the ideal hedging errors due to volatility risk and jump risk and the term due to model misspecification.

For the delta-gamma hedge, two call options with strike prices K_1 and K_2 are used. We fix the position in the first call to one. The position in the second call and in the stock is chosen such that the delta and the gamma of the portfolio are zero, and the portfolio is made self-financing by an appropriate position in the money market account. This portfolio can be interpreted as a long position in a delta-hedged portfolio for the first call and a short position in the delta-hedged portfolio for the second call, which allows us to build on our results for delta hedges. The hedging error in case of discrete trading is equal to

$$
\begin{aligned}
(13) \qquad D^d(t_i, t_{i+1}) &= \left\{ C^{(1)}_{t_{i+1}} - C^{(1)}_{t_i} e^{r(t_{i+1}-t_i)} - H^{(1)}_{t_i} \left(S_{t_{i+1}} - S_{t_i} e^{r(t_{i+1}-t_i)} \right) \right\} \\
&\quad - \alpha_{t_i} \left\{ C^{(2)}_{t_{i+1}} - C^{(2)}_{t_i} e^{r(t_{i+1}-t_i)} - H^{(2)}_{t_i} \left(S_{t_{i+1}} - S_{t_i} e^{r(t_{i+1}-t_i)} \right) \right\},
\end{aligned}
$$

where $H^{(i)}$ is the hedge ratio for claim i, and

$$
(14) \qquad \alpha_{t_i} = \frac{\frac{\partial^2 c^{(1)}}{\partial s^2}(t_i, S_{t_i}, V_{t_i})}{\frac{\partial^2 c^{(2)}}{\partial s^2}(t_i, S_{t_i}, V_{t_i})}.
$$

For this delta-gamma hedge, the EHE is given in the next proposition, where we include both the discretization error and the error due to model misspecification.

Proposition 7 (*SVJ: EHE delta-gamma hedge*). The true model is Bates (1996) and trading is discrete. Consider a portfolio of one call $C^{(1)}$ with strike K_1 long and α_{t_i} calls $C^{(2)}$ with strike K_2 short, where α_{t_i} is given in equation (14). For each claim, the hedge ratio H_{t_i} (gamma) at time t_i is the BS delta (gamma) based on the implied volatility of the claim. Then the EHE over the interval $[t_i, t_{i+1}]$ is given by

$$
\begin{aligned}
(15) \qquad & \mathrm{E}^P \left[D^d(t_i, t_{i+1}) \mid \mathcal{F}_{t_i} \right] \\
&= \left[\mathrm{IE}^{V,1}(t_i, t_{i+1}) - \alpha_{t_i} \mathrm{IE}^{V,2}(t_i, t_{i+1}) \right] \lambda_V \\
&\quad + \left[\mathrm{IE}^{J,1}(t_i, t_{i+1}) - \alpha_{t_i} \mathrm{IE}^{J,2}(t_i, t_{i+1}) \right] (h^P - h^Q) \\
&\quad + (\mu - r) \left\{ \left[\mathrm{DE}^{S,1}(t_i, t_{i+1}) - \alpha_{t_i} \mathrm{DE}^{S,2}(t_i, t_{i+1}) \right] (\mu - r) \right. \\
&\qquad \left. + \left[\mathrm{DE}^{V,1}(t_i, t_{i+1}) - \alpha_{t_i} \mathrm{DE}^{V,2}(t_i, t_{i+1}) \right] \lambda_V \right.
\end{aligned}
$$

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$$+\left[\mathrm{DE}^{J,1}(t_i,t_{i+1})-\alpha_{t_i}\mathrm{DE}^{J,2}(t_i,t_{i+1})\right](h^P-h^Q)\Big\}$$

$$+\left[\mathrm{ME}^{d,1}(t_i,t_{i+1})-\alpha_{t_i}\mathrm{ME}^{d,2}(t_i,t_{i+1})\right](\mu-r),$$

where IE^V and IE^J are defined in Proposition 1, DE^S, DE^V, and DE^J are defined in Proposition 2, ME^d is defined in Proposition 4, and the second superscript denotes the respective call.

Proof. Equation (13) shows that the hedging error in case of a delta-gamma hedge depends on the hedging errors for the delta hedges of the two options. With the expression for the EHE in case of a delta hedge given in Proposition 4, the result follows immediately. □

Like for a delta hedge, the EHE is additive. It consists of the ideal components, the contributions of the discretization error, and the impact of model misspecification. Model misspecification, however, also shows up in the ideal EHE and the discretization error, since they depend on the ratio of the gammas of the two options. If the wrong model is used, this ratio also deviates from the correct value. The main question is now which term dominates the EHE and how this term depends on the MPVR.

2. Discretization Error Only

First, we consider the model of BS as a benchmark. Here, there is only a discretization error caused by a stale hedge ratio due to stock price changes. This discretization error is significantly reduced for a delta-gamma hedge compared to a delta hedge. However, since gamma is zero only when the hedge is initiated, there will still be a nonvanishing discretization error. Intuitively, the number of scenarios where the hedging error is statistically significantly different from zero should be smaller compared to the case of a delta hedge. In these scenarios, however, the test based on the delta-gamma hedge still leads to incorrect conclusions.

For the SV model, we expect the discretization error due to stock price changes to be rather small, following the argument from the BS model. This leaves us with the discretization error and the ideal hedging error due to volatility risk. Since closed-form expressions for these terms are not available, we rely on a numerical analysis.

The terms $\mathrm{IE}^{V,1}(t_i,t_{i+1})-\alpha_{t_i}\mathrm{IE}^{V,2}(t_i,t_{i+1})$ and $\mathrm{DE}^{V,1}(t_i,t_{i+1})-\alpha_{t_i}\mathrm{DE}^{V,2}(t_i,t_{i+1})$ in equation (15) depend on the relation between the two strike prices. For $K_1 > K_2$, the first difference is negative, while the second one is positive. The ideal EHE is thus decreasing in the MPVR, and for a positive equity risk premium, the discretization error due to volatility is increasing in the MPVR. The properties of the EHE then depend on the relative size of these two error terms. The numerical results suggest that the ideal term is much larger than the discretization error, so that the EHE is decreasing in the MPVR. This also holds for the less realistic case of a negative equity risk premium, where both the ideal EHE and the discretization error are decreasing in the MPVR. For $K_2 > K_1$, an analogous argument shows that the EHE tends to increase in the MPVR.

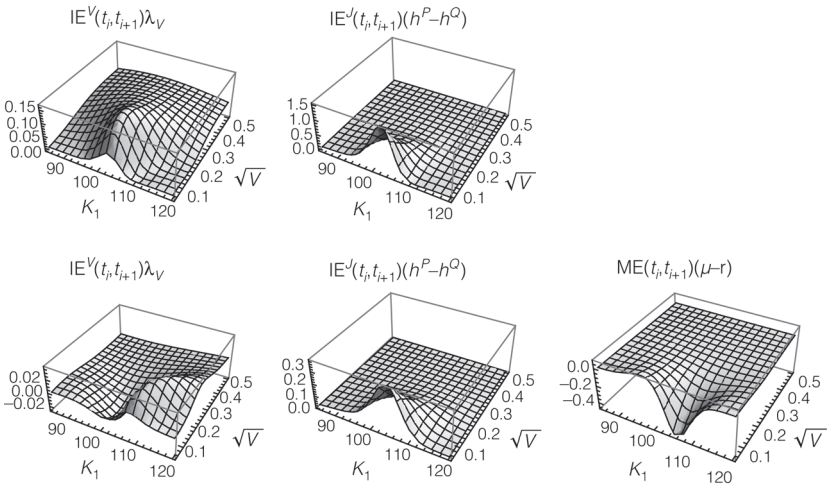
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FIGURE 2

Delta-Gamma Hedge: Impact of SV and SJ on EHE

The graphs show the impact of SV and SJ on the ideal EHE as a function of the strike price K_1 (with $K_2 = K_1 - 5$) and the current level of volatility $\sqrt{V}$ as well as the impact of model misspecification. In the upper panel, the gamma hedge is based on the correct model, while in the lower panel, the model of BS is used to set up the hedge.

The terms $\mathrm{IE}^V(t_i, t_{i+1})\lambda_V$ and $\mathrm{IE}^J(t_i, t_{i+1})(h^P - h^Q)$ are shown in the left and middle graph, respectively. The right graph in the lower panel shows the term $\mathrm{ME}(t_i, t_{i+1})(\mu - r)$. The MPVR is given by $\lambda_V = -2$, the difference of the jump intensities is $h^Q - h^P = 0.75$. The time to maturity of the options is three months, the current stock price is 100, and the length $t_{i+1} - t_i$ of the hedge interval is one day, i.e., 1/365. The other parameters are $r = 0$, $\kappa^Q = 2.0$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.5$, $h^Q = 1.5$, $\mu_X = -0.05$, $\sigma_X = 0.07$.



For the SV model, we thus know the relation between the EHE and the MPVR, and the delta-gamma hedge seems a promising alternative to the delta-hedge. To ultimately compare the two tests, we basically repeat the simulation study from Section IV.C.3, where we now also implement a delta-gamma hedge. The conclusion drawn from a test based on delta-gamma hedges is correct in most cases, although the results (not given here to save space) also show that it is less reliable than a simple delta hedge, which should therefore be the preferred alternative.

Next we turn to the SVJ model and consider the impact of jump risk. The properties of the differences $\mathrm{IE}^{J,1}(t_i, t_{i+1}) - \alpha_{t_i}\mathrm{IE}^{J,2}(t_i, t_{i+1})$ and $\mathrm{DE}^{J,1}(t_i, t_{i+1}) - \alpha_{t_i}\mathrm{DE}^{J,2}(t_i, t_{i+1})$ are very similar to the properties of the respective terms for SV. For $K_1 > K_2$ and a positive equity risk premium, the EHE decreases in the difference $h^P - h^Q$ as long as the discretization error is not too large, and in our numerical examples, this condition certainly holds for hedging intervals of one day. For $K_2 > K_1$, on the other hand, the EHE tends to increase in the difference $h^P - h^Q$. A comparison of the contributions of jump risk and volatility risk to the ideal EHE in the upper panel of Figure 2 shows that they are of comparable size. Thus, the delta-gamma hedge is exposed to the same problem as the delta hedge: a positive EHE for $K_1 > K_2$ can be attributed to a negative MPVR, but also to a risk premium for jump risk with $h^Q > h^P$. In particular, the delta-gamma hedge does not eliminate the impact of jump risk.

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3. Discretization Error and Model Misspecification

In case of model misspecification, delta and gamma for both options are computed in the model of BS, based on implied volatilities. First, we focus on the impact on the ideal EHE and the discretization error. The lower panel of Figure 2 shows that the ideal EHEs due to volatility risk and jumps become smaller in case of model misspecification, compared to the respective values shown in the upper panel. The use of the wrong gammas may even lead to a sign change, as can be seen from the behavior of the ideal EHE due to volatility risk, and in these cases, the standard hedging-based test will fail. Furthermore, if jump risk is priced, then the MPVR may again be confused with a jump risk premium.

Second, model misspecification adds a further term to the EHE. This term is also shown in Figure 2 for the case $K_1 > K_2$, assuming as usual a downward sloping smile and a positive equity risk premium. As can be seen, it is in most cases negative and of the same order of magnitude as the ideal EHE due to jump risk. In general, the sum of these two terms can have any sign, and the resulting EHE thus not only depends on the MPVR, but also on the pricing of jump risk and the contribution of model misspecification.

To compare the delta hedge and the delta-gamma hedge, we again perform a simulation study. For the SV model, we use the same parameters as in Section IV.C.3. The results for a negative MPVR are shown in Table 6. In most scenarios, the test fails. Back-of-the-envelope calculations show that the sign of the EHE is mainly determined by the sign of the hedging error due to model misspecification, which does not depend on the MPVR. The conclusion is again that the EHE from a delta-gamma hedge cannot be used to draw reliable inference about the sign of the MPVR. A comparison with Table 4 shows that the performance of the delta-gamma hedge is even worse than that of a simple delta hedge.

TABLE 6

SV Model: Impact of Model Misspecification on the Delta-Gamma Hedge ($\lambda_V = -2$)

Average hedging error (multiplied by 1,000) for a delta-gamma hedge with one option long (strike price K_1) and an appropriate number of options short (strike price K_2) as a function of the strike prices (K_1/K_2), of the equity risk premium $\mu - r$, and of the time to maturity TTM in months. For each option, delta and gamma are calculated in the BS model. The hedging interval is one day ($\tau - t = 1/365$), and the hedge is not rebalanced in between. The parameters of the risk-neutral process are given by $S_0 = 100$, $r = 0.0$, $\kappa^Q = 1.15$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.65$. The symbol "**+**" ("**0**") behind an average hedging error indicates that it is significantly positive (not significantly different from zero). One would thus wrongly reject the null hypothesis of a negative MPVR.

		$V_0 = 0.01$			$V_0 = 0.04$			$V_0 = 0.09$		
TTM	$\mu - r$	100/90	100/110	90/110	100/90	100/110	90/110	100/90	100/110	90/110
1	0	0.602 0	−1.417	−0.074	0.271 0	−0.388 0	−0.151 0	0.163 0	−0.151 0	−0.142 0
3	0	0.448	−0.813 0	−0.326	0.173 0	−0.526 0	−0.365 0	0.094 0	−0.156 0	−0.173 0
6	0	0.420	−0.662 0	−0.489	0.237	−0.606	−0.545	0.096 0	−0.216 0	−0.240 0
12	0	0.279	−0.357 0	−0.402	0.351	−0.533	−0.648	0.205	−0.347	−0.448
1	0.1	0.735 0	0.542 0	−0.007 0	0.243 0	0.084 0	−0.037 0	0.133 0	−0.062 0	−0.088 0
3	0.1	0.232 0	2.258 +	0.524 +	0.125 0	0.245 0	0.063 0	0.058 0	−0.016 0	−0.051 0
6	0.1	−0.069 0	2.041 +	0.953 +	0.087 0	0.279 0	0.124 0	0.039 0	−0.007 0	−0.035 0
12	0.1	−0.238 +	1.185 +	0.899 +	0.054 0	0.206 0	0.111 0	0.073 0	−0.061 0	−0.108 0
1	0.2	0.917	2.603 +	0.062 +	0.229 0	0.540 0	0.071 0	0.108 0	0.022 0	−0.039 0
3	0.2	0.045 0	5.341 +	1.371 +	0.086 0	1.006 +	0.481 +	0.025 0	0.121 0	0.067 0
6	0.2	−0.540 +	4.740 +	2.384 +	−0.057 0	1.155 +	0.783 +	−0.016 0	0.198 0	0.164 0
12	0.2	−0.746 +	2.719 +	2.188 +	−0.237 +	0.938 +	0.863 +	−0.058 0	0.224 0	0.228 0

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TABLE 7

SVJ Model: Impact of Model Misspecification on the Delta-Gamma Hedge
($\lambda_V = 0$, $h^Q = 3h^P$)

Average hedging error (multiplied by 1,000) for a delta-gamma hedge with one option long (strike price K_1) and an appropriate number of options short (strike price K_2) as a function of the strike prices (K_1/K_2), of the equity risk premium $\mu - r$, and of the time to maturity TTM in months. For each option, delta and gamma are calculated in the BS model. The hedging interval is one day ($\tau - t = 1/365$), and the hedge is not rebalanced in between. The parameters of the risk-neutral process are given by $S_0 = 100$, $r = 0.0$, $\kappa^Q = 2.0$, $\theta^Q = 0.04$, $\sigma_V = 0.4$, $\rho = -0.5$, $h^Q = 1.5$, $\mu_X = -0.05$, $\sigma_X = 0.07$. The symbol"**+**" ("**−**") behind an average hedging error entry indicates that it is significantly positive (negative). One would thus wrongly reject the null hypothesis of a zero MPVR.

		$V_0 = 0.01$			$V_0 = 0.04$			$V_0 = 0.09$		
TTM	$\mu - r$	100/90	100/110	90/110	100/90	100/110	90/110	100/90	100/110	90/110
1	0	9.424 −	0.327	−1.379 −	4.145 −	−1.619 −	−1.726 −	1.616 −	−0.950 −	−1.231 −
3	0	0.883	−1.363 −	−0.955 −	0.342	−0.642 −	−0.578	0.156	−0.248	−0.288
6	0	0.011	−0.300	−0.195	−0.039	−0.140	−0.073	−0.011	−0.066	−0.043
12	0	−0.098 +	0.033	0.102	−0.073	0.012	0.069	−0.030	−0.009	0.018
1	0.1	8.182 −	4.718 +	−0.525	3.573 −	−0.700	−1.280	1.432	−0.737	−1.040
3	0.1	1.015	0.726	−0.123	0.274	−0.097	−0.217	0.089	−0.090	−0.127
6	0.1	−0.020	0.362	0.239	−0.145	0.182	0.235	−0.098	0.090	0.149
12	0.1	−0.227 +	0.247 +	0.366 +	−0.209 +	0.201 +	0.331 +	−0.141 +	0.131	0.229
1	0.2	6.994	9.167 +	0.329	3.017	0.214	−0.839	1.253	−0.527	−0.854
3	0.2	1.160 −	2.811 +	0.702	0.211	0.443	0.136	0.025	0.067	0.030
6	0.2	−0.045	1.015 +	0.665 +	−0.247 +	0.502 +	0.537 +	−0.182 +	0.245 +	0.338 +
12	0.2	−0.353 +	0.459 +	0.628 +	−0.343 +	0.388 +	0.590 +	−0.251 +	0.269 +	0.437 +

We also implement the delta-gamma hedge in the SVJ model where we use the same parameters as in Section IV.C.4. The MPVR is set equal to zero, and the risk-neutral jump intensity is larger than the physical jump intensity. The results are shown in Table 7. Compared to the delta hedge shown in Table 5, there are fewer cases in which the test wrongly signals a negative MPVR, but now the opposite problem occurs when the test falsely rejects the null against the alternative $\lambda_V > 0$. Thus, we can again conclude that the delta-gamma hedge is not superior to the delta hedge.

VI. Conclusion

Hedging-based tests rely on the expected hedging errors for options to draw inference about the sign (and possibly also the magnitude) of market prices of risk for state variables like SV. In this paper, we have analyzed the robustness of these tests with respect to discrete trading and model misspecification. Such a robustness would mean a significant advantage of hedging-based tests over the estimation of parametric option pricing models, where the results on the pricing of risk factors seem to depend heavily on the chosen model specification. Our results, however, show that the use of standard hedging-based tests to identify the sign of the MPVR may lead to wrong conclusions about both the existence and the sign of the MPVR.

Under the ideal conditions of continuous trading, correct model specification, and SV as the only additional risk factor, the sign of the EHE is indeed equal to the sign of the MPVR. If there is also jump risk, the EHE due to jump risk is negative for empirically relevent scenarios, and ignoring jump risk results in a downward bias in the assessment of the MPVR. In case of discrete trading, the

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use of a stale hedge ratio results in discretization errors which have a systematic impact on the EHE and in most cases lead to an upward bias in the assessment of the MPVR. Finally, the impact of model mis-specification depends on the slope of the smile. For a positive equity risk premium and a downward sloping smile, the impact is positive, and ignoring the model error again leads to an upward bias in the assessment of the MPVR.

A negative EHE can thus be caused by a negative MPVR, but also by a premium for jump risk. A positive EHE may be caused by a positive MPVR, but also by model mis-specification or by discrete trading. Assuming that it is only the MPVR that influences the EHE may thus lead to wrong conclusions.

We further show that the hedge ratio could basically be adjusted for time discreteness. However, to compute this adjusted hedge ratio one would have to know all market prices of risk, and the knowledge of these market prices of risk would make the test redundant. Finally, delta-gamma hedges, sometimes considered superior to delta hedges, do not outperform the simpler alternative.

The analysis performed in this paper constitutes a first step toward a more detailed investigation of the characteristics of option hedging errors. A next step might be a detailed analysis of the impact of jump risk on the hedging error. Furthermore, there is a need for more robust tests, which are less sensitive to discretization error and model misspecification.

Appendix

A. Proof of Proposition 1

The dynamics of the price $C_t = c(t, S_t, V_t)$ in the SVJ model can be derived by applying Ito's lemma to the price function c and by plugging in the fundamental partial differential equation

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial s} rs + \frac{\partial c}{\partial v} \left[\kappa(\theta - v) - \sigma_V \lambda_V v\right] + \frac{1}{2} \frac{\partial^2 c}{\partial s^2} vs^2 + \frac{1}{2} \frac{\partial^2 c}{\partial v^2} \sigma_V^2 v + \frac{\partial^2 c}{\partial s \partial v} \rho \sigma_V v s + \mathrm{E}^Q \left[f(t, s, v, X) | \mathcal{F}_{t-} \right] h^Q = rc,$$

where the additional jump risk of the claim is given by

$$f(t, s, v, x) = c(t, se^x, v) - c(t, s, v) - \frac{\partial c}{\partial s}(t, s, v)\, s\, (e^x - 1)\,.$$

Under the usual mild technical conditions, the dynamics of the claim price are given by

$$\text{(A1)} \qquad dC_t = rC_t dt + \frac{\partial c}{\partial s}\left(dS_t - rS_t dt\right) + \frac{\partial c}{\partial v}\left[dV_t - \kappa(\theta - V_t)dt + \sigma_V \lambda_V V_t dt\right] + \left\{ f(t, S_{t-}, V_t, X_t) dN_t - \mathrm{E}^Q \left[f(t, S_{t-}, V_t, X_t) | \mathcal{F}_{t-} \right] h^Q dt \right\}$$

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Plugging equation (A1) into the general expression (3) for the hedging error gives

$$\begin{aligned} D^c(t,t+\tau) &= \int_t^{t+\tau} e^{r(t+\tau-u)} \left(\frac{\partial c}{\partial s}(u,S_u,V_u) - H_u \right) (dS_u - rS_u du) \\ &\quad + \int_t^{t+\tau} e^{r(t+\tau-u)} \frac{\partial c}{\partial v}(u,S_u,V_u) \big(dV_u - \kappa(\theta - V_u)du + \lambda_V \sigma_V V_u du \big) \\ &\quad + \int_t^{t+\tau} e^{r(t+\tau-u)} \{ f(u,S_{u-},V_u,X_{u-}) dN_u \\ &\qquad - \mathrm{E}^Q [f(t,S_{u-},V_u,X_u) | \mathcal{F}_{u-}] h^Q du \Big\} dt. \end{aligned}$$

Taking expectations gives the proposition. Note that we have assumed that the distribution of X is identical under P and under Q. □

B. Proof of Proposition 2

In case of discrete trading, the hedging error over the interval $[t_i, t_{i+1}]$ is given by

$$\begin{aligned} D^d(t_i,t_{i+1}) &= \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} (dC_t - rC_t dt) - H_{t_i} \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} (dS_t - rS_t dt) \\ &= \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} (dC_t - rC_t dt) - \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} H_t (dS_t - rS_t dt) \\ &\quad + \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} (H_t - H_{t_i}) (dS_t - rS_t dt) \\ &= D^c(t_i,t_{i+1}) + \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} (H_t - H_{t_i}) (dS_t - rS_t dt), \end{aligned}$$

where H_t is the delta of the claim at time t. For the EHE, we get

$$\begin{aligned} &\mathrm{E}^P \left[D^d(t_i,t_{i+1}) \mid \mathcal{F}_{t_i} \right] \\ &= \mathrm{E}^P \left[D^c(t_i,t_{i+1}) \mid \mathcal{F}_{t_i} \right] + \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} \mathrm{E}^P \left[(H_t - H_{t_i}) (dS_t - rS_t dt) \mid \mathcal{F}_{t_i} \right] \\ &= \mathrm{E}^P \left[D^c(t_i,t_{i+1}) \mid \mathcal{F}_{t_i} \right] + (\mu - r) \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)} \mathrm{E}^P [S_t | \mathcal{F}_{t_i}] \mathrm{E}^{P^S} \left[H_t - H_{t_i} \mid \mathcal{F}_{t_i} \right], \end{aligned}$$

where the measure P^S is defined by

$$\frac{dP^S}{dP} = \frac{S_T}{\mathrm{E}^P[S_T]}$$

for some $T \geq t_{i+1}$. Applying Ito to the hedge ratio gives

$$\begin{aligned} d\frac{\partial c}{\partial s}(t,S_t,V_t) &= \frac{\partial^2 c}{\partial t \partial s}(t,S_t,V_t)\, dt + \frac{\partial^2 c}{\partial s^2}(t,S_t,V_t) \left(dS_t - S_{t-} \left(e^{X_t} - 1 \right) dN_t \right) \\ &\quad + \frac{\partial^2 c}{\partial s \partial v}(t,S_t,V_t)\, dV_u + \frac{1}{2} \frac{\partial^3 c}{\partial s^3}(t,S_t,V_t) V_t S_t^2 dt \end{aligned}$$

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$$+\frac{1}{2}\frac{\partial^3 c}{\partial s\partial v^2}(t,S_t,V_t)\sigma_V^2 V_t dt+\frac{\partial^3 c}{\partial s^2\partial v}(t,S_t,V_t)\sigma_V\rho S_t V_t dt$$

$$+\left[\frac{\partial c}{\partial s}(t,se^x,v)-\frac{\partial c}{\partial s}(t,s,v)\right]dN_t.$$

To simplify this expression, we differentiate the fundamental partial differential equation for the claim price with respect to s:

$$\frac{\partial^2 c}{\partial t\partial s}(t,s,v)+\frac{\partial^2 c}{\partial s^2}(t,s,v)\left(r-h^Q\mu_X\right)s+\frac{\partial c}{\partial s}(t,s,v)\left(r-h^Q\mu_X\right)$$

$$+\frac{\partial^2 c}{\partial s\partial v}(t,s,v)\left[\kappa(\theta-v)-\lambda\sigma_V v\right]+\frac{1}{2}\frac{\partial^3 c}{\partial s^3}(t,s,v)vs^2+\frac{\partial^2 c}{\partial s^2}(t,s,v)vs$$

$$+\frac{1}{2}\frac{\partial^3 c}{\partial s\partial v^2}(t,s,v)\sigma_V^2 v+\frac{\partial^3 c}{\partial s^2\partial v}(t,s,v)\rho\sigma_V vs+\frac{\partial^2 c}{\partial s\partial v}(t,s,v)\rho\sigma_V v$$

$$+\mathrm{E}^Q\left[\frac{\partial c}{\partial s}(t,se^X,v)e^X-\frac{\partial c}{\partial s}(t,s,v)|\mathcal{F}_{t-}\right]h^Q=r\frac{\partial c}{\partial s}(t,s,v).$$

The difference between the hedge ratios at time t and time t_i is thus

$$\frac{\partial c}{\partial s}(t,S_t,V_t)-\frac{\partial c}{\partial s}(t_i,S_{t_i},V_{t_i})$$

$$=\int_{t_i}^t\frac{\partial^2 c}{\partial s^2}(u,S_u,V_u)\left[dS_u-S_{u-}\left(e^{X_u}-1\right)dN_t-(r+V_u-h^Q\mu_X)S_u du\right]$$

$$+\int_{t_i}^t\frac{\partial^2 c}{\partial s\partial v}(u,S_u,V_u)\left[dV_u-\kappa(\theta-V_u)du+\lambda_V\sigma_V V_u du-\sigma_V\rho V_u du\right]$$

$$+\int_{t_i}^t\left\{\left[\frac{\partial c}{\partial s}(t,S_{u-}e^{X_u},V_{u-})-\frac{\partial c}{\partial s}(t,S_{u-},V_{u-})\right]N_u\right.$$

$$\left.-\mathrm{E}^Q\left[\frac{\partial c}{\partial s}(t,S_{u-}e^{X_u},V_{u-})e^{X_u}-\frac{\partial c}{\partial s}(t,S_{u-},V_{u-})e^{X_u}|\mathcal{F}_{u-}\right]h^Q\right\}du.$$

For the expected change in the hedge ratio under the measure P^S we get

$$\mathrm{E}^{P^S}\left[H_t-H_{t_i}\mid\mathcal{F}_{t_i}\right]$$

$$=\mathrm{E}^{P^S}\left[\int_{t_i}^t\frac{\partial^2 c}{\partial s^2}(u,S_u,V_u)\left[dS_u-S_{u-}\left(e^{X_u}-1\right)dN_t\right.\right.$$

$$\left.\left.-(r+V_u-h^Q\mu_X)S_u du\right]\mid\mathcal{F}_{t_i}\frac{\partial^2 C}{\partial S^2}\right]$$

$$+\mathrm{E}^{P^S}\left[\int_{t_i}^t\frac{\partial^2 c}{\partial s\partial v}(u,S_u,V_u)\left[dV_u-\kappa(\theta-V_u)du\right.\right.$$

$$\left.\left.+\lambda_V\sigma_V V_u du-\sigma_V\rho V_u du\right]\mid\mathcal{F}_{t_i}\right]$$

$$+\mathrm{E}^{P^S}\left[\int_{t_i}^t\left\{\left[\frac{\partial c}{\partial s}(t,S_{u-}e^{X_u},V_{u-})-\frac{\partial c}{\partial s}(t,S_{u-},V_{u-})\right]dN_u\right.\right.$$

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$$- \mathrm{E}^{Q}\left[\left(\frac{\partial c}{\partial s}(t, S_{u-}e^{X_u}, V_{u-}) - \frac{\partial c}{\partial s}(t, S_{u-}, V_{u-})\right) \times e^{X_u} \,\middle|\, \mathcal{F}_{u-}\right] h^{Q}\Big\} du \,\middle|\, \mathcal{F}_{t_i}\Big].$$

The Wiener processes W^{S,P^S} and W^{V,P^S} are given by

$$dW_u^{S,P^S} = dW_u^S - \sqrt{V_u}du \quad \text{and} \quad dW_u^{V,P^S} = dW_u^V.$$

From the formulas for the measure transform in case of jumps, it follows that the intensity of the Poisson process under P^S, h^{P^S}, is equal to $h^P E^P[e^X]$. The distribution of the jump size has the Radon-Nikodym-derivative

$$\frac{P^S(dx)}{P(dx)} = \frac{e^x}{\mathrm{E}^P[e^X]}.$$

For the expected change in the hedge ratio this results in

$$\begin{aligned}
&\mathrm{E}^{P^S}\left[H_t - H_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right] \\
&= \mathrm{E}^{P^S}\left[\int_{t_i}^{t} \frac{\partial^2 c}{\partial s^2}(u, S_u, V_u)\left(\mu - r - \mu_X(h^P - h^Q)\right) S_u du \,\middle|\, \mathcal{F}_{t_i}\right] \\
&\quad + \mathrm{E}^{P^S}\left[\int_{t_i}^{t} \frac{\partial^2 c}{\partial s \partial v}(u, S_u, V_u)\lambda_V \sigma_V V_u du \,\middle|\, \mathcal{F}_{t_i}\right] \\
&\quad + \mathrm{E}^{P^S}\left[\int_{t_i}^{t} \mathrm{E}^{Q}\left[\left(\frac{\partial c}{\partial s}(t, S_{u-}e^{X_u}, V_{u-}) - \frac{\partial c}{\partial s}(t, S_{u-}, V_{u-})\right) e^{X_u} | \mathcal{F}_{u-}\right]\right. \\
&\qquad \left. \times (h^P - h^Q) du \,\middle|\, \mathcal{F}_{t_i}\right] \\
&= \mathrm{E}^{P^S}\left[\int_{t_i}^{t} \frac{\partial^2 c}{\partial s^2}(u, S_u, V_u)(\mu - r) S_u du \,\middle|\, \mathcal{F}_{t_i}\right] \\
&\quad + \mathrm{E}^{P^S}\left[\int_{t_i}^{t} \frac{\partial^2 c}{\partial s \partial v}(u, S_u, V_u)\lambda_V \sigma_V V_u du \,\middle|\, \mathcal{F}_{t_i}\right] \\
&\quad + \mathrm{E}^{P^S}\left[\int_{t_i}^{t} \mathrm{E}^{Q}\left[g(u, S_{u-}, V_{u-}, X_u)\right](h^P - h^Q) du \,\middle|\, \mathcal{F}_{t_i}\right],
\end{aligned}$$

where

$$g(t, s, v, x) = \left[\frac{\partial c}{\partial s}(t, se^x, v) - \frac{\partial c}{\partial s}(t, s, v)\right] e^x - \frac{\partial^2 c}{\partial s \partial v}(u, s, v)\, s\,(e^x - 1).$$

The EHE is thus equal to

$$\begin{aligned}
&\mathrm{E}^{P}\left[D^d(t_i, t_{i+1}) \,\middle|\, \mathcal{F}_{t_i}\right] \\
&= \mathrm{E}^{P}\left[D^c(t_i, t_{i+1}) \,\middle|\, \mathcal{F}_{t_i}\right] \\
&\quad + (\mu - r)\left\{(\mu - r)\int_{t_i}^{t_{i+1}} e^{r(t_{i+1} - t)}\, \mathrm{E}^{P}\left[S_t \int_{t_i}^{t} \frac{\partial^2 c}{\partial s^2}(u, S_u, V_u) S_u du \,\middle|\, \mathcal{F}_{t_i}\right] dt\right.
\end{aligned}$$

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$$+\lambda_V \int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)}\,\mathrm{E}^P\left[S_t \int_{t_i}^{t} \frac{\partial^2 c}{\partial s \partial v}(u, S_u, V_u)\sigma_V V_u du \mid \mathcal{F}_{t_i}\right] dt$$

$$+(h^P - h^Q)\int_{t_i}^{t_{i+1}} e^{r(t_{i+1}-t)}\,\mathrm{E}^P$$

$$\times\left[S_t \int_{t_i}^{t} \mathrm{E}^Q\left[g(u, S_{u-}, V_{u-}, X_u)|\mathcal{F}_{u-}\right] du \mid \mathcal{F}_{t_i}\right] dt\Bigg\}.$$

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