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Dynamics of functions meromorphic outside a small set

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Abstract. The theory of Fatou and Julia is extended to include the dynamics of functions f which are meromorphic in $\widehat{\mathbb{C}}$ outside a totally disconnected compact set $E(f)$ at whose points the cluster set of f is $\widehat{\mathbb{C}}$. The Julia set is defined not only by the standard approach but is also characterized in terms of the set of points whose orbits approach a point of $E(f)$. For the subclass where $E(f)$ has a complement of class O_{AD} and the inverse of f has a finite set of singular points it is shown that neither wandering components nor Baker domains occur in $F(f)$. As an application, functions of a certain general class are shown to have a totally disconnected Julia set.

1. *Introduction*

Iteration of rational maps of $\widehat{\mathbb{C}}$ originated essentially with the work of Fatou [**22**] and Julia [**27**], nearly a century ago.

Fatou [**23**] extended the discussion to transcendental entire functions. Fatou remarks that for a transcendental meromorphic function f the iterates f^n, $n \in \mathbb{N}$, defined by $f^1 = f$, $f^n = f \circ f^{n-1}$ in general have infinitely many essential singularities, whereas rational and entire functions are analytic maps of $\widehat{\mathbb{C}}$ or $\mathbb{C}$, respectively, into itself, whose iterates are of the same type. This was a discouragement from considering the meromorphic case. It is, however, natural to consider the iteration of meromorphic functions, which arises in such practical problems as the attempt to solve an equation $g(z) = 0$ with transcendental entire g by Newton's method. Indeed in the last decade or so a theory of the dynamics of transcendental meromorphic functions in $\mathbb{C}$ has developed. The main features of this theory are described in the survey [**9**]: many results turn out to be similar to those for rational or entire maps but require different proofs. The iteration of meromorphic functions at once leads us out of a closed system: if we iterate meromorphic f in $\mathbb{C}$ we are also iterating f^2, which is not in general meromorphic, so there must be a more general version of the theory.

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This suggests trying to extend the 'classical' theory of Julia and Fatou as far as possible while retaining its basic results. An early step in this direction was taken by Rådström [**39**] who also pointed out that the Fatou–Julia approach is appropriate for analytic maps of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to itself. The case of the punctured plane has been developed by various authors, for example in [**4, 12, 28, 29, 30, 31, 33**].

It is of more than purely aesthetic interest that the class of functions to be studied in an extended theory should be closed under iteration or even arbitrary functional composition. One reason is that in the classical theory it is often very convenient in proofs to replace a function by its iterate of some higher order. For example points of period p of f can be discussed as fixed points of f^p. Also f, f^p have the same Fatou and Julia sets (see below for definitions).

The smallest class which obviously includes the meromorphic functions with one essential singularity and which is closed under composition is $\mathbf{K} = \{f$: there is a compact countable set $E(f) \subset \widehat{\mathbb{C}}$ such that f is meromorphic in $\widehat{\mathbb{C}} \setminus E(f)$ but in no larger set$\}$. This class has been studied by Bolsch [**13, 14**]. Independently, broader generalizations have been given in [**26**].

This paper will give a simplified presentation of some parts of [**26**]. §2 discusses the classes of functions considered, which are functions meromorphic outside a totally disconnected set, E of essential singularities, subject to some further assumptions. At this stage the Fatou set $F(f)$ and the Julia set $J(f)$ of such a function are defined. These are the basic objects of the theory and in §3 the standard properties are obtained for functions of our classes.

In §4 we extend a result obtained first by Eremenko [**18**] for transcendental entire functions and later in [**17**] for functions meromorphic in $\mathbb{C}$: under fairly general conditions on the functions treated in §2, for any $e \in E$ the set $I(f, e) = \{z : f^n(z)$ is defined and different from e for all $n \in \mathbb{N}$ and $f^n(z) \to e$ as $n \to \infty\}$ is non-empty and $J(f) = \partial I(f, e)$. This gives an alternative way to define $J(f)$.

In §5 we note the classification of the different types of components of the open set $F(f)$ according to the behaviour of the sequence of iterates of f in them. In particular a component G such that for each $n \in \mathbb{N}$, $f^n(G)$ is contained in a different component $F(f)$ is called wandering.

The result of Sullivan [**42**] that a rational function cannot have wandering domains solved a major problem which had remained open from the time of Fatou and Julia. It was a major cause of the subsequent progress in rational iteration. A number of authors perceived that the finiteness of the number of critical values of a rational function is a crucial ingredient of Sullivan's proof and obtained extensions of the no wandering domains theorem to classes of transcendental entire functions [**20, 24**], maps of $\mathbb{C}^*$ [**10, 28, 30, 33**] or even transcendental functions meromorphic in $\mathbb{C}$ [**8**], such that the inverse function has a finite set of singularities.

In §6 we introduce the subclass **MSR** of functions f whose totally disconnected set of singularities $E(f)$ satisfies a technical condition (the complement of $E(f)$ is a set of type O_{AD}, defined in Appendix A) and which are such that the inverse f^{-1} has singularities over a finite set of points $S(f)$. This class is closed under composition and contains many members of the class **K** mentioned above, for example $\tan(e^z)$, $R(e^z)$ where R is rational.

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We extend Sullivan's theorem to show that for functions in **MSR** the Fatou set has no wandering components.

In §7 we note that the method of Eremenko and Lyubich [**20**] gives conditions under which the set $I(f, e)$ defined above has empty interior. In particular we show that for the functions in **MSR** there are no Fatou components in which a limit function of iterates belongs to $E(f)$ (no 'Baker domains'). Hence for such functions all Fatou components are eventually periodic and these cyclic components are those related to an attracting or parabolic cycle of points or are cycles of Siegel discs or Herman rings.

In §8 we note the relation between singular orbits and Fatou domains. This allows us to determine the dynamics of a map of the class **MSR** in many cases. In particular, in §9, Theorem G we give a class of functions for which the Julia set is totally disconnected, that is all the components of $J(f)$ are singletons. These provide interesting new examples even in the class of transcendental meromorphic functions in $\mathbb{C}$.

Finally an appendix contains a number of definitions and results from complex analysis which are used in the paper.

Notation. We shall frequently use $D(a, r)$ for an open disc of centre a, radius r and $C(a, r)$ for its circumference.

2. *The class* **M** *of functions*

If E is any compact totally disconnected set in $\widehat{\mathbb{C}}$, if $z_0 \in E$ and f is a function meromorphic in $E^c = \widehat{\mathbb{C}} \setminus E$, the cluster set $C(f, E^c, z_0)$ is defined as $\{w : w = \lim_n f(z_n)$ for some $z_n \in E^c$ with $z_n \to z_0\}$.

Denote by $\mathbf{M}(E)$ the class of functions f which are meromorphic in E^c and have an essential singularity at every point of E, that is f has no meromorphic continuation over any point in E. Further we introduce the class $\mathbf{M} = \{f$: there is a compact totally disconnected set $E = E(f)$ such that f is meromorphic in E^c and $C(f, E^c, z_0) = \widehat{\mathbb{C}}$ for for all $z_0 \in E$. If $E = \emptyset$ we make the further assumption that f is neither constant nor univalent in $\widehat{\mathbb{C}}\}$.

For any choice of E we may construct functions in **M** for which $E(f) = E$. For example, take a sequence $\{w_k\}$ dense in $\mathbb{C}$, and sequences $A_1, \ldots A_n, \ldots$ in E^c such that the A_n are mutually disjoint, and the set of limit points of each A_n and of $B = \cup_n A_n$ is precisely E. (One may construct the A_n inductively to be mutually disjoint and $d(A_n, E) < 1/n$). Mittag–Leffler's theorem [**25**, p. 196] allows the construction of two functions g, h meromorphic in E^c and such that the poles of g are at the points $b \in B$ where the principal part is $(z - b)^{-1}$, while those of h are at points $b \in B$ such that when $b \in A_n$ the principal part is $w_n(z - b)^{-1}$. Then $f = h/g$ is meromorphic in E^c and $f(z) = w_n$ when $z \in A_n$. Hence $C(f, E^c, z_0) = \widehat{\mathbb{C}}$ for every $z_0 \in E$. Of course, $f \in \mathbf{M}$ has essential singularities precisely at the points of $E(f)$ so that $f \in \mathbf{M}(E(f))$. As a partial converse of this statement we note that for any compact totally disconnected set E whose complement is of class O_{AB} defined in Definition A.2 and in particular any compact E whose linear measure is zero, a function in $\mathbf{M}(E)$ necessarily belongs to **M**, see Theorem A.7.

LEMMA 1. *Let $f \in \mathbf{M}$, $E = E(f)$ and let B be compact totally disconnected. Then $A = E \cup f^{-1}(B)$ is compact and totally disconnected.*

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Proof. We know that E is compact. Take a sequence z_n in $f^{-1}(B)$. We may, without loss of generality, assume that z_n converges to z, say. Then $f(z_n) = t_n \in B$ and again we may assume that $t_n \to t$, where $t \in B$ since B is compact. Either $z \in E$ or $z \in E^c$. In the latter case f is continuous at z so that $f(z) = t$ and $z \in f^{-1}(B)$. Thus in all cases $z_n \to z \in A$. Hence A is compact.

Take any $\alpha \in A$. If $\alpha \in f^{-1}(B)$ then $\alpha \in \widehat{\mathbb{C}} \setminus E$ and $f(\alpha) = \beta \in B$. Then there is a disc $U = D(\alpha, \rho)$, $\rho > 0$, such that $U \subset \widehat{\mathbb{C}} \setminus E$ and $U \to f(U)$ is a d to 1 branched cover, branched at most at α and with d equal to the valency of f at α.

We may find a Jordan curve γ in $D(\beta, \sigma) \subset f(U)$ which does not meet B or pass through β but winds around β. Then $d\gamma$ lifts by f to a closed Jordan curve γ' in U which does not meet A and winds around α. Since ρ is arbitrarily small, $\{\alpha\}$ is a component of A. Now if A has some component K which is not a singleton then $K \subset E$. But the only connected subsets of E are singletons by assumption. The proof of Lemma 1 is complete. □

LEMMA 2. *If f and g belong to* **M**, *then $f \circ g \in \mathbf{M}$ and $E(f \circ g) = E(g) \cup g^{-1}(E(f))$.*

Proof. Write $A = E(g) \cup g^{-1}(E(f))$. Then A is compact and totally disconnected. Further, $f \circ g$ is meromorphic in A^c: clearly analytic continuation within A^c is single-valued.

If $z_0 \in g^{-1}(E(f)) \setminus E(g)$, then for every sufficiently small neighbourhood U of z_0 the set $f \circ g(U \cap A^c) = f\{g(U) \setminus E(f)\}$ is dense in $\mathbb{C}$.

On the other hand, suppose that $z_0 \in E(g)$ and note that for $t \in \mathbb{C}$, $\epsilon > 0$, there is some w_0 in $\mathbb{C} \setminus E(f)$ and $\delta > 0$ such that for every w with $|w - w_0| < \delta$ we have that $f(w)$ is defined and $|f(w) - t| < \epsilon$. By assumption there is some z_1 in $\mathbb{C} \setminus E(g)$ which is arbitrarily close to z_0 and such that $|g(z_1) - w_0| < \delta$, which implies $|f(g(z_1)) - t| < \epsilon$. Thus for all $z_0 \in A$ we have shown that $C(f \circ g, A^c, z_0) = \widehat{\mathbb{C}}$ and the proof is complete. (We remark that in the case when $A = \emptyset$ neither f nor g is univalent so that $f \circ g$ is a non-univalent function.) □

For any $f \in \mathbf{M}$ we may now define f^0 to be the identity function with $E_0 = \emptyset$, and, inductively, $f^1 = f$, $f^n = f \circ f^{n-1}$. We obtain $f^n \in \mathbf{M}$, $n \in \mathbb{N}$, with $E_n = E(f^n) = \cup_{j=0}^{n-1} f^{-j}(E) = \{\text{singularities of } f^{-n}\}$. Clearly if we set

$$J_1(f) = \{\overline{\cup_{n=0}^{\infty} E_n}\} \tag{1}$$

and

$$F_1(f) = \widehat{\mathbb{C}} \setminus J_1(f), \tag{2}$$

then $F_1(f)$ is the largest open set in which all f^n are defined. Further, $f(F_1) \subset F_1$.

We obtain an extension of standard Fatou and Julia theories by defining the Fatou set $F(f) = \{$the largest open set in which (i) all f^n, $n \in \mathbb{N}$, are meromorphic and (ii) $\{f^n\}$ is a normal family$\}$, and the Julia set $J(f) = \widehat{\mathbb{C}} \setminus F(f)$.

If the set $J_1(f)$ is either empty or contains one point (without loss of generality, $J_1(f) = E_1(f) = \{\infty\}$) or two points (without loss of generality, $J_1(f) = \{0, \infty\}$) we are dealing with maps which are rational or (conjugate to) entire functions or analytic

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maps of the punctured plane $\mathbb{C}^*$. In this case the condition (i) above is trivial and the Fatou and Julia sets are determined by (ii). In all the other cases Montel's theorem shows us that $F(f) = F_1(f)$ and $J(f) = J_1(f)$ as defined above.

It will be useful to consider two subclasses of **M**, both of which are easily seen to be closed under composition. The first class comprises the 'Picard' functions. Let k be an integer not less than two and set:

$$\mathbf{MP}_k = \{f \in \mathbf{M} : E(f) \neq \emptyset \text{ and for each } z_0 \in E(f) \text{ and open } U \text{ which contains } z_0,$$
$$f \text{ takes in } U \setminus E(f) \text{ every value in } \widehat{\mathbb{C}} \text{ with at most } k \text{ exceptions}\}.$$

Next we say that $f \in \mathbf{M}$ has the k-islands property at $z_0 \in E(f)$ if, given any neighbourhood U of z_0 and k simply-connected domains Δ_i in $\widehat{\mathbb{C}}$ which have disjoint closures and which are bounded by sectionally analytic Jordan curves, there is a simply-connected subdomain D in $U \setminus E(f)$ which forms a schlicht island over one of the Δ_i, that is f maps D univalently onto Δ_i. We define the 'Ahlfors' ' functions:

$$\mathbf{MA}_k = \{f \in \mathbf{M} : E(f) \neq \emptyset \text{ and for each } z_0 \in E(f)$$
$$\text{the function } f \text{ has the } k \text{ islands property at } z_0\}.$$

Clearly $\mathbf{MA}_k \subseteq \mathbf{MP}_{k-1}$.

Example 1. If the compact totally disconnected set E contains a dense set of isolated points then it follows from Picard's theorem and the Ahlfors five islands theorem (see, e.g., [**44**, Ch. VI]) respectively that every function in $\mathbf{M}(E)$ belongs to $\mathbf{MA}_5$ and $\mathbf{MP}_2$.

In the particular case when E is a countable compact set category arguments show that isolated points are dense in E. Thus Bolsch's class $\mathbf{K} \subset \mathbf{MP}_2 \cap \mathbf{MA}_5$.

Example 2. Given a sequence of positive numbers (ξ_n), $n \in \mathbb{N}$, such that $0 < \xi_n < 1/3$ and the unit interval $I_{0,1}$, construct a Cantor set E by excluding at the nth subdivision an open segment of length $(1 - 2\xi_n)\Pi_{p=1}^{n-1}\xi_p$ from the middle of each segment $I_{n-1,k}$, $k = 1, 2, \ldots, 2^{n-1}$, to get 2^n closed segments $I_{n,j}$, each of length $\xi_1\xi_2 \ldots \xi_n$.

Let $E = \cap_{n=1}^{\infty}(\cup_k I_{n,k})$. Matsumoto [**34**] showed that if $\xi_{n+1} = o(\xi_n^2)$ as $n \to \infty$ then every $f \in \mathbf{M}(E)$ is in $\mathbf{MP}_2$. This result was further improved by Toppila [**43**, p. 14] who showed that if $\xi_{n+1} = o(\xi_n)$ then E has the following *local Picard property*: there exist no open set V with $E \cap V \neq \emptyset$ and no function f meromorphic in $V \setminus E$ with an essential singularity at each point of $E \cap V$ such that f omits three values in $V \setminus E$. The condition $\xi_{n+1} = o(\xi_n)$ allows E to have positive capacity. On the other hand, Cap $E = 0$ does not by itself guarantee the Picard property.

Example 3. Toppila [**43**, Theorem 3] showed that for any totally disconnected compact set A there exists a perfect totally disconnected compact set B such that $B \supset A$ and $E(f) = B$, $f \in \mathbf{M}$ imply that $f \in \mathbf{MP}_2$. Note that it is possible for such B to have positive two-dimensional Lebesgue measure.

Example 4. The authors in [**38**] describe for each $k \geq 2$ a geometric condition on the totally disconnected set E in terms of exhaustions of E^c which imply that $\mathbf{M}(E) \subset \mathbf{MP}_k$.

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Example 5. Herring [**26**] obtains analogous conditions to those of Example 4 which imply that $\mathbf{M}(E) \subset \mathbf{MA}_6$.

The somewhat complicated conditions under which Examples 4 and 5 are valid include the special case when E is a k-fold arbitrary linear Cantor set $B = B(k, \xi_n)$, where k is an integer greater than one and ξ_n is a sequence of real numbers in $(0, 1/k)$, and we suppose further that $\xi_n \to 0$ as $n \to \infty$. We recall that B is constructed from collections B_n, $n \in \mathbb{N} \cup \{0\}$, of k^n closed-disjoint subintervals I_i of $[0, 1] = B_0$, the length of I_i is $d_n = \xi_1 \ldots \xi_n$, $d_0 = 1$. B_n is constructed inductively: if B_n has already been constructed, then from each I_i remove $(k-1)$ disjoint equal intervals so as to leave k closed intervals $I_i(1), \ldots, I_i(k)$, each of length $d_{n+1} = \xi_{n+1} d_n$. Define $B_{n+1} = \cup_{i=1}^{k^n} \cup_{j=1}^{k} I_i(j)$ and $B(k, \xi_n) = \cap_0^\infty B_n$.

In the next section certain properties of Julia sets will be proved under the assumptions $\mathbf{MP}_k$ or $\mathbf{MA}_k$. The above examples show that these are fulfilled for many classes $\mathbf{M}(E)$.

3. *Standard properties of Julia sets*

We have defined $J(f)$ and $F(f)$ for $f \in \mathbf{M}$ in the preceding section. We collect the main properties of these sets in the following Theorem A. First we introduce the backward orbit $O^-(w)$ of a point $w \in \widehat{\mathbb{C}}$, defined by $\{z : f^n(z) = w \text{ for some } n \in \mathbb{N}\}$ and the α-limit set $\alpha(w)$ of w which is the set of limit points of $O^-(w)$. Recall that a generic property of points in a complete metric space is one which holds for all points outside a set of first category.

THEOREM A. *Suppose that $f \in \mathbf{M}$ as defined in §2. Then*

(1) *$F(f)$ is completely invariant in the sense that $z \in F(f)$ if and only if $f(z) \in F(f)$. Thus $z \in J(f) \setminus E$ if and only if $f(z) \in J(f)$;*
(2) *for every positive integer p, $F(f^p) = F(f)$ and $J(f^p) = J(f)$;*
(3) *if ψ is a Möbius transformation and $f_\psi = \psi f \psi^{-1}$, then $f_\psi \in \mathbf{M}$, $F(f_\psi) = \psi(F(f))$ and $J(f_\psi) = \psi(J(f))$;*
(4) *$J(f)$ is perfect;*
(5) *for a generic point $w \in \widehat{\mathbb{C}}$, the set $\alpha(w)$ contains $J(f)$. If $f \in \mathbf{MP}_k$, then the exceptional set $X(f) = \{z : O^-(z)$ is a finite set$\}$ contains at most k elements and for w not in $X(f)$, $\alpha(w)$ contains $J(f)$, while if $w \in J(f) \setminus X(f)$ we have $\overline{O^-(w)} = J(f)$;*
(6) *if $J(f)$ has a non-empty interior then $F(f) = \emptyset$;*
(7) *if $f \in \mathbf{MA}_k$ for some $k(\geq 5) \in \mathbb{N}$, then repelling periodic points are dense in $J(f)$;*
(8) *if $E(f)$ has the local Picard property in the sense of Example 2 of §2 then every point of $J(f)$ is a limit point of periodic points of f. (We do not assert that the periodic points are repelling.)*

Proof. We note here that Bolsch's class $\mathbf{K} \subset \mathbf{MA}_5$ so that repelling periodic cycles are dense in $J(f)$ for $f \in \mathbf{K}$. In [**13**] this result is proved by an elegant argument which avoids the use of Ahlfors' islands theorem and uses results on normal families and consequences of Nevanlinna's fundamental theorem. For an interesting discussion of relations between these theorems see [**11**]. In our general context where $E(f)$ may be perfect and totally

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disconnected we do not see how to achieve the result of property (7) without using a theorem of Ahlfors' type. It seems of some interest to prove the result (8) by different means.

Since the results of Theorem A are known for nonlinear rational functions, transcendental entire functions and for maps of the punctured plane we ignore these cases and assume that $J(f) = J_1(f)$ is given by equation (1) and it contains at least three points. From (1) and Picard's theorem it then follows that $J(f)$ is in fact infinite. We remark that the general meromorphic functions discussed in [**5, 6**] and [**7**] are a special case. Property (1) follows at once from the fact that $F(f)$ is the largest open set where all f^n are meromorphic. Property (2) follows from equation (1).

By Lemma 2, $f_\psi \in \mathbf{M}$ with $E(f_\psi) = \psi(E(f))$. From $f_\psi^p = \psi f^p \psi^{-1}$ we get $E(f_\psi^p) = \psi(E(f^p))$ and from equation (1) it follows that $J(f_\psi) = \psi J(f)$.

To prove property (4) we need to show that a point z_0 of $J(f)$ cannot be isolated. Suppose that z_0 is an isolated point of $J(f)$. Then $z_0 \in E_n$ and it is an isolated essential singularity for f^n for some smallest $n \in \mathbb{N}$. By our assumptions, $\cup E_m$ contains infinitely many points and so there is some $\alpha \in E_k$, say, such that $f^n(z) = \alpha$ has a sequence of solutions z_m such that $z_m \to z_0$, $z_m \neq z_0$. But $z_m \in J(f)$ and z_0 cannot therefore be isolated.

To prove property (5) note that since $J(f)$ is compact in $\widehat{\mathbb{C}}$ we can find a countable set of discs $D_n = D_n(z_n, \rho_n)$, $z_n \in J(f)$, $\rho_n \to 0$ such that every point of $J(f)$ belongs to arbitrarily small D_n. Now D_n meets some $E_{k(n)}(f)$ and $V_n = f^{k(n)}(D_n \setminus E_{k(n)})$ is open and dense in $\widehat{\mathbb{C}}$, since the cluster set of $f^{k(n)}$ at z_n is $\widehat{\mathbb{C}}$. Hence $V = \cap_{n=1}^{\infty} V_n$ is a dense G_δ set whose complement is a set of first category. Suppose that $w \in V$. Now for each $n \in \mathbb{N}$ we have $w \in V_n$ and some point $O^-(w)$ lies in D_n, so that $O^-(w)$ contains all points of $J(f)$ among its accumulation points.

Now suppose that $f \in \mathbf{MP}_k$ and that $a \in X(f)$. Then a must be a Picard value at each point of $E(f)$ so $X(f)$ contains at most k elements. If D_n, V_n are as above then V_n^c contains at most k points. Thus $f(V_n \setminus E)$ contains $\widehat{\mathbb{C}} \setminus X(f)$. It follows that for $w \notin X(f)$ the set $O^-(w)$ meets each D_n and $\overline{O^-(w)} \supset J(f)$. If $w \in J(f) \setminus X(f)$ then $\overline{O^-(w)} = J(f)$.

To prove property (6) note that if $J(f)$ has a non-empty interior and if $F(f) \neq \emptyset$ then by property (5) we may choose $v \in F(f)$ so that $O^-(v)$ contains points in the interior of $J(f)$, which contradicts the complete invariance of $F(f)$.

In proving property (7) first notice that $f \in \mathbf{MA}_k \subset \mathbf{MP}_{k-1}$ so that by property (5) the set $X(f)$ contains at most $(k-1)$ elements. Since we are considering only the case when in equation (1) $J(f) = J_1(f) = \overline{O^-(E)}$ is a non-empty perfect set we have some $e \in E(f) \setminus X(f)$ and by property (5) we have $\overline{O^-(e)} = J(f)$. (For, if $E(f)$ is finite it clearly cannot be contained in $X(f)$.)

Now take $p \in J(f)$ and an open set T with $p \in T$. There exists a positive integer N and a $z \in T$ such that $f^N(z) = e$. There is a neighbourhood S of z such that $S \subset T$ and f^N is meromorphic in S with $(f^N)' \neq 0$ except perhaps at z. We shall show that S (and hence T) contains some repelling periodic points of f.

Now z belongs to the perfect set $J(f) = \overline{O^-(e)}$, so S contains infinitely many points of $O^-(e)$ and in particular some $w \in O^-(e)$ which is not one of the (at most $(k-1)$) Picard

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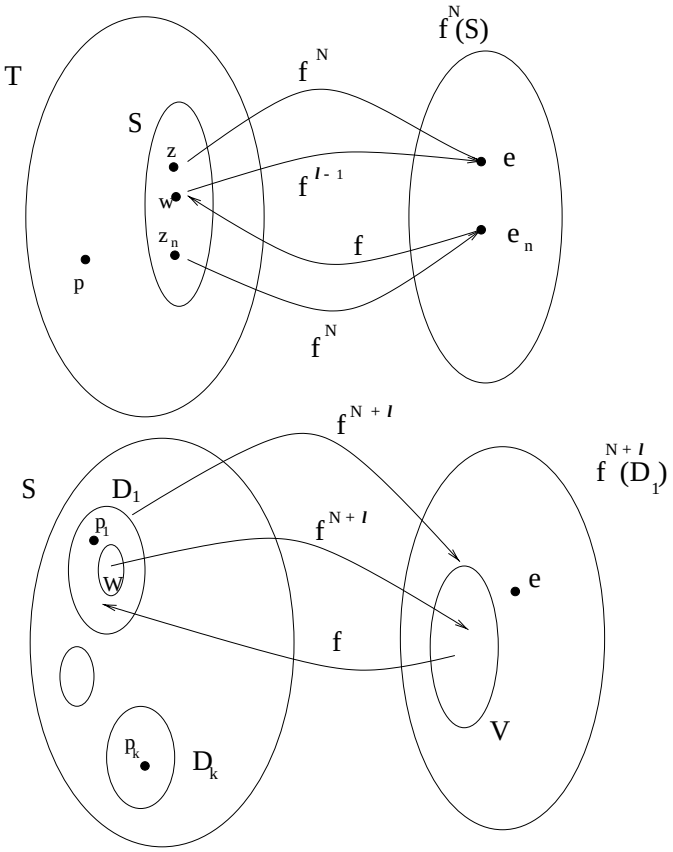


FIGURE 1. The neighbourhood S contains some repelling periodic points.

values of f at e. Thus in the neighbourhood $f^N(S)$ of e there is a sequence of different values e_n such that $e_n \to e$ and $f(e_n) = w$. There is a corresponding sequence z_n in S such that $z_n \to z$ and $f^N(z_n) = e_n$. There is an integer l such that $f^{l-1}(w) = e$. Thus we have $f^{N+l}(z_n) = e$, see Figure 1.

Choose k of the points z_n which lie in S and label them as $p_1, p_2, \ldots, p_k$. Let D_i be a disc centred at p_i such that $\overline{D}_i \subset S$, $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $i \neq j$, and let D_i be so small that $(f^{N+l})' \neq 0$ and $f^{N+l} \neq e$ on $\overline{D}_i \setminus \{p_i\}$. Since $f \in \mathbf{MA}_k$ there is at least one i, $1 \leq i \leq k$, such that in any neighbourhood of e there is a simply-connected domain V_i which is mapped univalently by f onto one of the D_i. We may assume that $i = 1$.

Since $f^{N+l}(D_1)$ is a neighbourhood of e there is a domain of the type V_i, which we call V with $\overline{V} \subset f^{N+l}(D_1)$. Recall that $f^{N+l}(p_1) = e$, that f^{N+l} is locally univalent in D_1 except perhaps at p_1 and that $e \notin V$. There is some simply-connected subdomain W with $\overline{W} \subset D_1$ such that $f^{N+l} : W \to V$ is a univalent cover, so that $f^{N+l+1} : W \to D_1$ is a univalent cover. If g is the inverse of f^{N+l+1} such that $g : D_1 \to W$, using Rouché's theorem and Schwarz's lemma shows that g has a fixed point t in $W \subset S$ with $|g'(t)| < 1$. Thus t is a repelling fixed point of f^{N+l+1}.

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Now prove property (8). If f satisfies the stated assumptions then in particular $f \in \mathbf{MP}_2$ so that by property (5) $X(f)$ contains at most two points. We may assume that $E = E(f)$ is uncountably infinite, since if E is countable we have $f \in \mathbf{MA}_5$ and the result follows from property (7). (If we wish to avoid the use of (7) we may refer to the 'elementary' proof of Bolsch [**13**] for the class **K**.) Since the critical values of the iterates form a countable set we may choose $e \in E$ so that e is not exceptional and is not a critical value of any iterate. From (5) we have $J(f) = \overline{O^-(e)}$.

If $p \in J(f)$ and T is an open set which contains p, then there is $z \in T$ and $N \in \mathbb{N}$ such that $f^N(z) = e$. Since $J(f)$ is perfect we may ensure that z is not a Picard value of f at e. Since z is not a critical point of f^N there is a disc $S = D(z, r) \subset T$ such that f^N maps S univalently to a neighbourhood $V = f^N(S)$ of e with inverse $g : V \to S$. We take two different solutions t_i in V of $f(t_i) = z$, where $f'(t_i) \neq 0$. Then there is a disc $S_1 = D(z, r_1) \subset S$ such that the branches ϕ_i of f^{-1} with $\phi_i(z) = t_i$ are analytic in S_1 and for all u in S_1 we have $\phi_1(u) \neq \phi_2(u)$, $\phi_i(u) \neq u$, $\phi_i(S_1) \subset V$. This is possible since $z \neq e$.

The function defined in $S_1 \setminus f^{-N}(E)$ by

$$h(u) = \frac{(f^{N+1}(u) - \phi_1(u))(u - \phi_2(u))}{(f^{N+1}(u) - \phi_2(u))(u - \phi_1(u))}$$

is meromorphic. If $h(u) = 0$ or ∞, then u is a point where $f^{N+2}(u) = f\phi_i(u) = u$. If $h(u) = 1$, then $f^{N+1}(u) = u$. Thus, if T contains no periodic points of f, it follows that h omits $0, 1, \infty$ in $S_1 \setminus f^{-N}(E)$ and $h \circ g$ omits $0, 1, \infty$ in $f^N(S_1) \setminus E$. But

$$h \circ g = \frac{(f - \phi_1 \circ g)(g - \phi_2 \circ g)}{(f - \phi_2 \circ g)(g - \phi_1 \circ g)}$$

is meromorphic in $f^N(S_1) \setminus E$ and has essential singularities at all the points of $f^N(S_1) \cap E$. Since E has the local Picard property $h \circ g$ cannot omit $0, 1, \infty$ and T must contain periodic points of f, as required. □

4. *Orbits which approach a singular point*

For transcendental entire functions f, Eremenko [**18**] studied the set

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}. \tag{3}$$

He proved that for such functions (a) $I(f) \neq \emptyset$ and indeed (b) $J(f) = \partial I(f)$ and (c) $J(f) \cap I(f) \neq \emptyset$. It has been observed [**9, 21**] that if f is a map of $\mathbb{C}^*$ to itself with essential singularities at 0 and ∞ then the same results hold, and by symmetry we could replace ∞ by 0 in the definition of $I(f)$.

In [**17**] it is shown that Eremenko's result also applies to transcendental meromorphic functions if one replaces the definition (3) by

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \text{ and } f^n(z) \neq \infty\}.$$

See also [**41**].

Consider now $f \in \mathbf{M}$ and for $e \in E(f)$ define

$$I(f, e) = \{z \in \mathbb{C} : f^n(z) \text{ is defined for all } n \text{ and } f^n(z) \to e \text{ as } n \to \infty\}.$$

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THEOREM B. *If for some $k \in \mathbb{N}$ we have $f \in \mathbf{MP}_k$, then for $e \in E(f)$ we have:*
(1) $I(f, e) \neq \emptyset$;
(2) $J(f) = \partial I(f, e)$;
(3) $J(f) \cap I(f, e) \neq \emptyset$.

The proof will use the following result proved in [**17**].

LEMMA 3. *There is an absolute constant c such that if f is analytic in $0 < R \leq |z| < \infty$ and if $M(r) = \mathrm{Max}_{|z|=r} |f(z)| \to \infty$ as $r \to \infty$, then for all sufficiently large ρ the image $f(A_\rho)$ of $A_\rho = \{z : R \leq |z| \leq \rho\}$ contains some circle $C(0, r) = \{z : |z| = r\}$, where $r \geq cM(\rho/2)$.*

Proof of Theorem B(1). (a) First we consider the case when $e \in E(f)$ is an isolated point of $E(f)$ and is also a Picard exceptional value of f at e. By conjugating with a Möbius transformation we may assume that $e = \infty$ and that f is analytic in $0 < R \leq |z| < \infty$ with an isolated essential singularity at ∞. In this case $M(r)/r^2 \to \infty$ as $r \to \infty$ so that there is some $\sigma > 0$ such that for all $\rho > \sigma$ we have $cM(\rho/2) > 2\rho$, where c is the constant in Lemma 3. The argument in this case is as in [**17**] but we give it for completeness.

Now let r_1 be greater than $\mathrm{Max}(\sigma, M(R))$, $\gamma_1 = C(0, r_1)$ and B the region between $C(0, R)$ and $C(0, r_1)$. It follows from Lemma 3 that $f(B)$ contains $C(0, r)$ for some r such that $r \geq cM(r_1/2) > 2r_1 > M(R)$. Since f is analytic in $\overline{B}$ we have $\partial f(B) \subset f(\partial B) = f(C(0, R)) \cup f(\gamma_1)$, where $f(C(0, R)) \subset \overline{D}(0, M(R))$ and $M(R) < r_1$.

The boundary of $f(B)$ contains the boundary γ_2 of the unbounded component of the complement of $f(B)$. Thus γ_2 is a part of $f(\gamma_1)$ and is a continuum which lies in $|z| > 2r_1$ and indeed outside B.

One may now repeat the argument with B replaced by B_2, the region between $C(0, R)$ and γ_2. Certainly B_2 contains the annulus $R \leq |z| \leq 2r_1$. Then $f(B_2)$ has an 'outer' boundary γ_3 which is a continuum lying in $\{|z| > 4r_1\}$, $\gamma_3 \subset f(\gamma_2) \subset f^2(\gamma_1)$. By repeating this argument we see that there is a continuum $\gamma_{n+1} \subset f^n(\gamma_1)$ which lies in $\{z : |z| > 2^n r_1\}$. Thus $\gamma_1 \cap f^{-n}(\gamma_{n+1}) = A_n$ is a decreasing family of compact sets whose intersection is in $\gamma_1 \cap I(e = \infty)$. Each γ_n lies outside the corresponding B_{n-1}.

(b) It remains to consider the cases when either e is isolated but not Picard exceptional at e, or when there is a sequence of different $e_n \in E$ such that $e_n \to e$. We assume that $k = 2$. The general case is similar. In the case when e is isolated let us set $e_n = e$. Without loss of generality, we may take e to be finite. Then in either case there is a point $z_1 \in D_1 = D(e_1, 1)$ such that $f(z_1) = e_2'$ where $e_2' = e_1$, e_2 or e_3. If $r_2 > 0$ is sufficiently small then there is a simply-connected neighbourhood V_1 of z_1 in $D_1 \setminus \{e_1\}$ which is mapped locally univalently, except perhaps at z_1, onto $D_2 = D(e_2', r_2)$. Repeating the argument we see that there is some $z_2 \in D_2 \setminus \{e_2'\}$ such that $f(z_2) = e_3'$ where $e_3' = e_n$ for some $n > 3$. Further, there is some simply-connected neighbourhood V_2 of z_2 in $D_2 \setminus \{e_3'\}$ which is mapped locally univalently by f (except perhaps at z_2) to $D_3 = D(e_3', r_3)$, for some sufficiently small positive r_3. By taking r_2 small enough we achieve that there is a branch of f^{-1} analytic in $\overline{V}_2$ which satisfies $U_1 = f^{-1}(\overline{V}_2) \subset \overline{V}_1$. We certainly make $r_2 < r_1/2$.

Now continuing inductively we obtain discs $D_n = D(e_n', r_n)$ where e_n' is some e_j with $j > n + 1$, $r_n > 0$, points $z_n \in D_n \setminus \{e_n'\}$ such that $f(z_n) = e_{n+1}'$ and simply-connected

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neighbourhoods V_n of z_n in $D_n \setminus \{e'_n\}$ which are mapped locally univalently (except at z_n) to D_{n+1}. We also have $r_{n+1} < r_n/2$. Further, there is an analytic branch of f^{-1} in $\overline{V}_{n+1}$ so that $f^{-1}(\overline{V}_{n+1}) \subset \overline{V}_n$. Then $U_n = f^{-n}(\overline{V}_{n+1}) \subset \overline{V}_1 \subset D_1$. Since U_n is compact and decreases with n the intersection $H = \cap_n U_n \neq \emptyset$ and for $\beta \in H$ we have $f^n(\beta) \in \overline{V}_{n+1} \subset \overline{D}_{n+1}$ for every $n \in \mathbb{N}$. Thus $\beta \in I(f, e)$.

(2) There is some $z \in I(f, e)$. All $f^n(z)$ are different and belong to $I(f, e)$. Thus $I(f, e)$ contains at least $k + 1$ different values β_i all in $\{f^n(z)\}$.

Take any point $\alpha \in J(f)$ and any neighbourhood V of α. By property (5) of §3 one of β_i, say β_1 is not in $X(f)$. Thus some pre-image β' of β_1 lies in V. But $\beta' \in I(f, e)$. We conclude that $J(f) \subset \overline{I(f, e)}$. The interior of $I(f, e)$ is an open set in which all f^n are defined and hence $\operatorname{Int} I(f, e) \subset F(f)$ provided E contains at least three points. If, however, E contains only one or two points it is known that periodic points are dense in $J(f)$ and since periodic points are not in $I(f, e)$ it again follows that $\operatorname{Int} I(f, e) \subset F(f)$. Thus in all cases $J(f) \subset \partial I(f, e)$.

To prove the opposite inclusion take $\alpha \in \partial I(f, e)$ and suppose that $\alpha \in F(f)$, so that in $F(f)$ all f^n are defined meromorphically and hence do not take the value e. In a compact neighbourhood V of α, $V \subset F$, suppose that we have some subsequence f^{n_k} uniformly convergent to a meromorphic function g. Since there is a point $\beta \in I(f, e) \cap V$ we have $g(\beta) = e$. By Hurwitz's theorem $g \equiv e$. Thus is true for every convergent subsequence f^{n_k}, so in fact $f^n \to e$ in V. But this contradicts the fact that $\alpha \in \partial I(f, e)$. The proof of (2) is complete.

(3) In the general case (b) in the proof of Theorem B(1) the r_n may be chosen so small that H reduces to a single point. Since $U_n = f^{-n}(\overline{V}_{n+1})$ contains a pre-image $f^{-n-1}(e'_{n+2})$ of a point in $E(f) \subset J(f)$ it follows that the point β constructed as $\{\beta\} = \cap_n U_n$ belongs to $I(f, e) \cap J(f)$.

We return to the case (a) of Theorem B(1) and resume the notation of that proof. The compact set $K = \cap_n A_n$ is in $I(e) \cap \gamma_1$. If K meets $J(f)$ we are finished, so assume that $K \subset F(f)$. K is covered by some finite set $\{D_1, \ldots, D_p\}$ of components of $F(f)$. The sets A_n are decreasing compact sets so for some n_0 we have $A_{n_0} \subset \cup_1^p D_p \subset F(f)$. It follows that $\gamma_{n_0} = f(A_{n_0}) \subset F(f)$ and indeed $\gamma_n \subset F(f)$ for all $n \geq n_0$. Recall also that each γ_n winds around zero and lies in $\{z : |z| > r_1\}$.

Let G_n be the component of $F(f)$ which contains γ_n for $n \geq n_0$. Thus $f(G_n) \subset G_{n+1}$ and $f^n(z) \to \infty$ as $m \to \infty$, $z \in G_n$. If all G_n are different, then each is bounded since $f(\partial G_n) = \partial(f(G_n))$ and we get $\partial G_n \in I(e) \cap J(f)$.

It remains to consider the case when not all G_n are different, which can only occur if some $G_n = G_{n+1} = G$ (say), where G is an unbounded component of $F(f)$ with $f(G) \subset G$, $f^n \to \infty$ in G. Applying Lemma A.13 from the appendix with $K = A_{n_0}$ we see that there are constants A, C, n and a curve Γ which runs to ∞ in G with $f(\Gamma) \subset \Gamma$ such that for all z on Γ we have $|f(z)| < |z|^A$ while for any z, z' in any γ_n, $n > n_1$, we have $|z|^{1/C} \leq |z'| \leq |z|^C$.

Let u_n be a point on γ_n with minimum $|z|$, v_n be a point on γ_{n+1} with maximum $|z|$ and t_n a point in $\Gamma \cap \gamma_n$. Then $M(|u_n|) \leq \operatorname{Max}_{\gamma_n} |f(z)| = |v_n| \leq |f(t_n)|^c \leq |t_n|^{AC} \leq |u_n|^{A^2C}$. Since this holds for a sequence $|u_n|$ which tends to ∞ as $n \to \infty$ we have a contradiction to the assumption that $e = \infty$ is an essential singularity of f. The proof is complete. □

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5. *Components of $F(f)$*

If U is a component of $F(f)$ then for each $k \in \mathbb{N}$, $f^k(U)$ is contained in a unique component U_k of $F(f)$. If $U_k \neq U_n$ for $k \neq n$ then U is called 'wandering'. Otherwise U is preperiodic: there exist k and $p \in \mathbb{N}$ so that $U_k = U_{p+k}$. To know the behaviour of iterates in the preperiodic components it is enough to know the behaviour in periodic components. Since $F(f) = F(f^p)$ it is enough to know the properties for an invariant component such that $f(U) \subset U$. With little change the discussion in [**35**, §3, 13] yields the following.

THEOREM C. *If $f \in \mathbf{M}$ and U is a component of $F(f)$ such that $F(U) \subset U$ then precisely one of the following is true.*

(i) *U contains an attracting fixed point z_0, that is $f(z_0) = z_0$, $|f'(z_0)| < 1$. For $z \in U$ we have $f^n(z) \to z_0$ as $n \to \infty$. U is called the immediate attractive basin of z_0.*

(ii) *U is a domain of attraction of a parabolic fixed point $z_0 \in \partial U$, that is $f(z_0) = z_0$, $f'(z_0) = 1$ and for $z \in U$ we have $f^n(z) \to z_0$ as $n \to \infty$.*

(iii) *There is an analytic homeomorphism $\psi\colon U \to D$, where D is the unit disc, such that $\psi(f(\psi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case U is called a Siegel disc.*

(iv) *There is an analytic homeomorphism $\psi\colon U \to A$ where A is an annulus $A = \{z : 1 < |z| < r\}, r > 1$, such that $\psi(f(\psi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. U is called a Herman ring.*

(v) *There exists $z_0 \in \partial U$ such that $f^n(z) \to z_0$, for $z \in U$ as $n \to \infty$, but f is not meromorphic at z_0. There exists a path $\gamma \subset U$ leading to z_0 such that $f(\gamma) \subset \gamma$ and the spherical distance between $f(z)$ and z tends to zero as $z \to z_0$ on γ. In this case U is called essentially parabolic at z_0 or alternatively a Baker domain.*

Remark. The only point which calls for comment is that when we are not in cases (i), (iii) or (iv) of the theorem we have that all limit functions of $(f^n(z))$ in U are constant and that $f^n(z) \to \partial U$ as $n \to \infty$. If we fix $z \in U$ and join it to $f(z)$ by γ_0 in U, setting $\gamma_n = f^n(\gamma_0)$ we have that the curve $\gamma = \cup_n \gamma_n$ approaches ∂U and diam $\gamma_n \to 0$ (since γ_n is bounded in the Poincaré metric). If lim sup γ_n is not a point, it is a continuum Γ which contains uncountably many points of $E(f)^c$. For any such point e we have $f(e) = e$ since diam $\gamma_n \to 0$. But this is impossible. Hence γ approaches a unique point $z_0 \in \partial U$. If $z_0 \in E^c$ we have case (ii) by the usual arguments while if $z_0 \in E$ we have case (v).

We note here a result on the connectivity of invariant and periodic components in the case when $E(f)$ is compact and countable, that is $f \in \mathbf{K}$.

THEOREM D. [**14**] *If $f \in \mathbf{K}$ and U is a periodic component of $F(f)$, then the connectivity of U is 1, 2 or ∞.*

The proof depends on an application of Ahlfors' theory of covering surfaces. [**14**] also contains the interesting result that if U and U_1 are components of $F(f)$ such that $f(U) \subset U_1$, then for $f \in K$ the set $U_1 \setminus f(U)$ contains at most two points, while the more general condition cap $E(f) = 0$ implies that $U_1 \setminus f(U)$ has capacity zero.

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6. *A no wandering domains theorem*

If $f \in \mathbf{M}$, $f(z_0) = w_0$ and g is an analytic branch of f^{-1} at w_0, $g(w_0) = z_0$, and if we continue g analytically along some a path $\gamma = \gamma(t)$, $t \in [0, 1]$, $\gamma(0) = w_0$, $\gamma(1) = w_1$ until we meet a first singularity at w_1, then either $g(\gamma(t))$ has a limit point z_1 in $\widehat{\mathbb{C}} \setminus E(f)$ as $t \to 1$, in which case $f(z_1) = w_1$ and w_1 is an algebraic singularity of g, or $g(\gamma(t))$ has no limit point in $\widehat{\mathbb{C}} \setminus E(f)$ as $t \to 1$. In this case $g(\gamma(t))$ approaches a unique point $e \in E(f)$, since $E(f)$ is totally disconnected and any two distinct points e, e' of $E(f)$ are separated by a closed curve δ whose distance from $E(f)$ is positive. In this second case we say that w_1 is an asymptotic value of f at e corresponding to the path $g(\gamma)$.

We say that $f \in \mathbf{M}$ belongs to the class **MS** if the set of singular values of f^{-1} is finite, $C(f) = \{a_1, \ldots, a_p\}$. This is equivalent to saying that

$$f : \widehat{\mathbb{C}} \setminus (E(f) \cup f^{-1}(C(f))) \to \widehat{\mathbb{C}} \setminus C(f)$$

is a regular cover. For example, $\tan z$ and $\sin z$ are in **MS** with $E(f) = \{\infty\}$ and $C(f) = \{\pm i\}, \{\pm 1\}$, respectively. Any rational function is in **MS**.

LEMMA 4. *If f and g belong to* **MS** *so does $f \circ g$.*

Proof. Lemma 2 shows that $f \circ g \in \mathbf{M}$ with $E(f \circ g) = E(g) \cup g^{-1}(E(f))$. We see at once that any critical value of $h = f \circ g$ is either a critical value of f or has the form $f(b)$ where b is a critical value of g.

Any other singularity of h^{-1} is an asymptotic value l of h along a path $\gamma(t)$, $0 \le t < 1$, which lies in $\widehat{\mathbb{C}} \setminus E(h)$, such that $\gamma(t)$ tends to some $e \in E(h)$ as $t \to 1$. Also $g(\gamma(t)) \in \widehat{\mathbb{C}} \setminus E(f)$. If $g(\gamma(t))$ has a limit point $\beta \in \widehat{\mathbb{C}} \setminus E(f)$ as $t \to 1$ then $l = f(\beta)$ where β is an asymptotic value of g. Otherwise $g(\gamma(t))$ approaches a unique point e' of $E(f)$ as $t \to 1$ and $f \to l$ on $g(\beta)$. Thus $C(h) \subset C(f) \cup f\{C(g)\}$ which is finite. □

Lemma 4 allows us to construct many examples of functions in **MS**. In particular, the examples $\exp(R(z))$, where R is rational, and $\exp(\tan z)$ are in **MS** and the latter is not conjugate to any meromorphic function in $\mathbb{C}$. We may also find examples of $f \in \mathbf{MS}$ with $E(f)$ compact countable which are not iterates of any meromorphic function in $\mathbb{C}$. One may ask whether, for any countable compact set E, there is some $f \in \mathbf{MS}$ with $E(f) = E$.

Sullivan's result [**42**] that for rational f the set $F(f)$ has no wandering components is of enormous help in studying dynamics of f, since the preperiodic cases have been classified as in §5. This has been extended to the functions in **MS** for which $E(f) = \{\infty\}$, that is critically finite entire and meromorphic functions. The proof given in [**8**] extends much more generally.

Before we state the next theorem we need a notion which originates in the classification theory by Riemann surfaces.

If W is a domain in the plane and F is a function analytic in W, the Dirichlet integral of F is defined by $D_W(F) = \iint_W |F'(z)|^2 \, dx \, dy$.

An analytic function with finite Dirichlet integral is said to be of the class AD. The domain W is said to be of class O_{AD} if the only functions of class AD on W are constants. A closed set E in the plane whose complement W is connected and of class O_{AD} is 'removable' in some ways described in Theorems A.3 and A.6 and in Definitions A.1 and A.5.

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THEOREM E. *Define* **MSR** *to be the subset of those functions* $f \in \mathbf{MS}$ *such that the complement* $E(f)$ *is of class* O_{AD}. *Then* $F(f)$ *has no wandering components.*

The condition that $\mathbb{C} \setminus E(f)$ is of class O_{AD} is somewhat abstract. A sufficient condition to ensure this is that the linear measure of $E(f)$ is zero, see Theorem A.4. We do not assert that **MSR** is closed under composition. A subset which is closed under composition is $\mathbf{MS}_0 = \{f : f \in \mathbf{MS}$ and $E(f)$ has capacity zero$\}$, since $f, g \in \mathbf{MS}_0$ implies $\mathrm{Cap}\{E(g) \cup g^{-1}(E(f))\} = 0$, see, for example, [**40**, p. 69]. The most immediate application, for which we have many examples is for f in the class **K**. Clearly $\mathbf{MS} \cap \mathbf{K} \subset \mathbf{MS}_0 \subset \mathbf{MSR}$. The first step in the proof of Theorem E is a lemma.

LEMMA 5. *If* $f \in \mathbf{MS}$ *and if* $F(f)$ *has a wandering component, then* $F(f)$ *has a wandering component* U *such that, for each* $k = 0, 1, \ldots, f : U_k = f^k(U) \to U_{k+1}$ *is a bijection and* U_k *is simply-connected.*

Proof. Let the set $W = C(f) = \{a_1, \ldots, a_p\}$ and suppose that U_0 is a wandering component of $F(f)$. Denote by U_n the component of $F(f)$ which contains $f^n(U_0)$. We may suppose that U_0 is chosen so that each U_n, $n \in \mathbb{N}$, contains no value a_i. Hence $f(z) : U_n \to U_{n+1}$ is a regular cover. It will suffice to show that U_0 is simply-connected (as then each U_n, $n > 1$, which satisfies the same assumptions is also simply-connected and $f : U_n \to U_{n+1}$ is a bijection).

Suppose that U_0 is not simply-connected, so that we may find a Jordan curve γ which is not null-homotopic in U_0. If any $\gamma_n = f^n(\gamma)$ were null-homotopic in U_n the homotopy would lift back by the regular cover $f^n : U_0 \to U_n$ and γ would be null-homotopic in U_0. Thus each γ_n is non-null-homotopic in U_n. If any subsequence of $\{f^n\}$ has a non-constant limit function in a Fatou component V, then it follows that V must be preperiodic. The proof given by Fatou [**22**, II, p. 55] for rational functions remains applicable. Thus in U_0 all limit functions are constant and are approached locally uniformly. Hence as $n \to \infty$ the spherical diameter of γ_n, d_n say, approaches zero.

Let ϵ be a positive number such that 10ϵ is less than the greater of one and the minimum spherical distance between any pair of a_i, and further $4\epsilon < \operatorname{diam} E(f)$ (we may suppose that $E(f)$ contains at least two points since the result is already known if $E(f) \neq \emptyset$ or a singleton).

For $n \geq n_0 \operatorname{diam} \gamma_n < \epsilon$. Thus either γ_{n+1} lies in some spherical disc $D = D(a, \epsilon)$ which contains no points of W or there is some a_j, a_1 say, such that $\gamma_{n+1} \subset D(a_1, 2\epsilon) \setminus \{a_1\} = D_*$.

Take $p \in \gamma_n$ and denote by g the branch of f^{-1} such that $g(f(p)) = p$. In the first case above when $D = D(a, \epsilon)$, g is analytic and univalent in D and $g(D) = A$ is a simply-connected domain bounded by a Jordan curve $\partial A = g(\partial D)$.

In the second case above g continues analytically throughout D^*. Now [**36**] g either has an algebraic branch point at a_1 (we have to include here the possibility that g is analytic at a_1) or g has a logarithmic branch point. If a_1 is an algebraic branch point, g returns to its initial value after some number d of circuits round a_1. There is a Jordan curve Γ such that $f(\Gamma)$ covers the circle $C(a_1, 2\epsilon)$ just d times. One of the components A of the complement of Γ contains a point α such that $f(\alpha) = a_1$ and $f : A \setminus \{\alpha\} \to D^*$ is a d to 1 cover.

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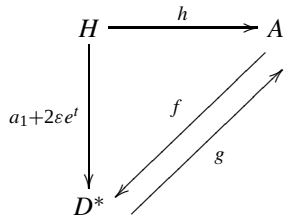


FIGURE 2. f is the universal covering map.

If, on the other hand, a_1 is a logarithmic branch point, the map h defined in $H = \{t : \operatorname{Re} t < 0\}$ by $h(t) = g(a_1 + 2\epsilon e^t)$ is univalent in H and indeed on ∂H. As $t = iy$, $y \in \mathbb{R}$, $y \to \infty$, so $f(h(t)) = a_1 + 2\epsilon e^{iy}$ shows that $h(t)$ must leave each compact set in $E(f)^c$. Thus (since E is totally disconnected) $h(iy)$ must tend to a unique $e \in E(f)$ as $y \to +\infty$. Since $h(H)$ omits an open set of values it follows that as $y \to -\infty$, $h(iy)$ must tend to the same limit e by Lindelöf's theorem (see Theorem A.10). We deduce that $g(D^*) = h(H)$ is a simply-connected domain A bounded by a Jordan curve Γ and that, except for e, f is analytic in $A \cup \Gamma$, and that $f : A \to D^*$ is the universal covering map, see Figure 2.

Thus in all the cases treated above there is a domain A bounded by a Jordan curve, such that $\gamma_n \in A$ and $E \subset A^c$. Thus the component Δ_n of γ_n^c which contains A^c must also contain E so that $\operatorname{diam} \Delta_n > 4\epsilon$. Since $\operatorname{diam} \gamma_n < \epsilon$ we see that for every component, say B, of γ_n^c which is different from Δ_n we have $\operatorname{diam} B < \epsilon$. Further, $\partial B \subset \gamma_n$ and f is meromorphic on $\overline{B}$ so $f(B)$ is a domain bounded by $\partial f(B) \subset f(\partial B) \subset \gamma_{n+1}$. Since $f(B) \subset f(A)$ we have $\operatorname{diam} f(B) < 4\epsilon$. Therefore, $f(B)$ does not meet Δ_{n+1}.

Thus all such components B and γ_n clearly map onto Δ_{n+1}^c. By induction f is analytic in all the open sets $V_n = (\overline{\Delta}_n)^c$, $n > n_0$, with values in $(\overline{\Delta}_{n+1})^c$ so $V_n \subset F(f)$. Thus $\gamma_n \cup V_n \subset F(f)$ and indeed is a subset of U_n. Then γ_n must be contractible to a point in U_n. But this contradicts what we have proved above. The proof is complete. □

For future use we note the following.

LEMMA 6. [**37**, p. 120] *If E is a totally disconnected compact plane set whose complement is of class O_{AD} and ϕ is a quasi-conformal map of $\mathbb{C}$, then $\phi(E)$ has a complement of class O_{AD}.*

LEMMA 7. [**32**, Ch. V, 3.6] *If G is a domain and E a compact set such that $G \setminus E$ is a domain and if E belongs to the class O_{AD}, in particular if E has linear measure zero, then any conformal map of $G \setminus E$ can be extended to a conformal map of G.*

Proof of Theorem E. This follows from the proof given in [**8**] with slight modifications. See also [**19**].

We suppose that $f \in \mathbf{MSR}$ with $C(f) = W = \{a_1, \ldots, a_p\}$ and that $F(f)$ has a wandering component U as in Lemma 5. Thus there is a conformal map $h : D = D(0, 1) \to U$ with inverse k. Let M be an integer exceeding $2p + 8$. Then we construct in [**8**] some $0 < \delta$, $T = \{\underline{t} \in \mathbb{R}^M : |t_i| \le \delta, 1 \le i \le M\}$ and explicit maps $\phi_{\underline{t}}(z) : T \times \overline{D} \to$

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$\overline{D}$ such that for fixed $\underline{t}$ the map $\phi = \phi_{\underline{t}}$ is a quasi-conformal homeomorphism of $\overline{D}$ to itself whose dilatation in D, $v_{\underline{t}}(z) = \phi_{\overline{z}}/\phi_z$, satisfies $\|v_{\underline{t}}(z)\|_\infty < \epsilon/3$. Further, $\phi_{\underline{t}}(z)$ is C^∞ in $\underline{t}$ and, for each $\underline{t}$, $\phi_{\underline{t}}(z)$ fixes $z = 0$ and also an open subset on ∂D which is independent of $\underline{t}$. If $\phi_{\underline{t}} \equiv \phi_{\underline{s}}$ on ∂D then $\underline{t} = \underline{s}$.

Just as in [**8**] we take $\chi_{\underline{t}} = h\phi_{\underline{t}}k : U \to U$ which is quasi-conformal with dilatation $\mu_{\underline{t}} = v_{\underline{t}}(k)\overline{k'}/k'$ which is measurable in U with $\|\mu_{\underline{t}}\|_\infty \leq \epsilon/3$.

With $B = O^-(E(f))$ we define the equivalence relation $\sim$ on $\widehat{\mathbb{C}} \setminus B$ by $x \sim y$ if and only if $f^m(x) = f^n(y)$ for some $m, n \in \mathbb{N}$. By Lemma 5 each iterate f^n is univalent in U so that a class $[x]$ meets U in at most one point. We can extend $\mu_{\underline{t}}(z)$ to $\widehat{\mathbb{C}}$ by push and pull back, that is:

$$\mu_{\underline{t}}(z)(f^n)'(z)/\overline{(f^n)'}(z) = \mu_{\underline{t}}(x)(f^m)'(x)/\overline{(f^m)'}(x) \tag{4}$$

if $z \in U$ and $f^n(z) = f^m(x)$, $m, n \in \mathbb{N}$, while $\mu_{\underline{t}} = 0$ if $[z] \cap U = \emptyset$ or $z \in B$.

Then $\mu_{\underline{t}}$ is measurable in $\widehat{\mathbb{C}}$ with $\|\mu_{\underline{t}}\|_\infty < \epsilon/3$, C^∞ in $\underline{t}$ and

$$\mu_{\underline{t}}(f(z)) = \mu_{\underline{t}} f'(z)/\overline{f'}(z) \tag{5}$$

for $z \notin E$, that is for almost all z.

Pick different finite b_1, b_2, $b_3 \notin W$ and a_{p+1}, a_{p+2}, a_{p+3} so that $f(a_{p+i}) = b_i$. By direct computation it is found as in [**8**], see also [**3**], that $\mu_t(z)$ satisfies the conditions of the measurable Riemann mapping theorem (Theorem A.12), so there is a unique quasi-conformal homeomorphism $\Phi_{\underline{t}}$ of $\widehat{\mathbb{C}}$ which fixes b_1, b_2, b_3 and has dilatation $\mu_{\underline{t}}$ almost everywhere. Since $\mu_{\underline{t}}(z)$ is in $C^1(T)$ it follows from [**1**] that $\Phi_{\underline{t}}(z)$ is in $C^1(T)$ for each fixed z.

From (4) and (5), the function $f_{\underline{t}} = \Phi_{\underline{t}} f \Phi_{\underline{t}}^{-1}$ is meromorphic in $R_{\underline{t}} = \Phi_{\underline{t}}(R)$, $R = \widehat{\mathbb{C}} \setminus E$ because $f_{\underline{t}}\Phi_{\underline{t}} = \Phi_{\underline{t}} f$ has the same dilatation as $\Phi_{\underline{t}}$. Also if $W_{\underline{t}} = \Phi_{\underline{t}}(W)$ then $f_{\underline{t}} : R_{\underline{t}} \setminus \Phi_{\underline{t}}(f^{-1}(W)) \to \widehat{\mathbb{C}} \setminus W_{\underline{t}}$ is an unbranched cover.

Now $\underline{t} \to \{\Phi_{\underline{t}}(a_1), \ldots, \Phi_{\underline{t}}(a_{p+3})\}$ is a C^1 map of T into $\mathbb{C}^{p+3} = \mathbb{R}^{2(p+3)}$. Since T is an M-dimensional cube and $M - 2(p+3) > 1$ there is a non-constant arc $\tau : \underline{t} = \tau(\sigma)$, $\sigma \in I = [0, \sigma_0]$, in T on which the map $\underline{t} \to \{\Phi_{\underline{t}}(a_1), \ldots, \Phi_{\underline{t}}(a_{p+3})\}$ is constant.

We shall simplify notation by writing Φ_σ for $\Phi_{\underline{t}(\sigma)}$, f_σ for $f_{\underline{t}(\sigma)}$, $a_i(\sigma) = \Phi_\sigma(a_i)$ and $R_\sigma = R_{\underline{t}(\sigma)}$.

LEMMA 8. (Generalized from [**19**]) *We have $f_\sigma = f_\tau$ and $\Phi_\sigma = \Phi_\tau$ for $\sigma, \tau \in I$.*

Proof. For fixed τ and σ in I

$$\begin{aligned} f \circ \Phi_\sigma^{-1} &: R_\sigma \setminus \Phi_\sigma(f^{-1}(W)) \to \widehat{\mathbb{C}} \setminus W, \\ f \circ \Phi_\tau^{-1} &: R_\tau \setminus \Phi_\tau(f^{-1}(W)) \to \widehat{\mathbb{C}} \setminus W, \end{aligned}$$

are unbranched coverings. For any $\lambda \in I$, $\Phi_\lambda^{-1}\Phi_\sigma$ is a quasi-conformal self-map of $\widehat{\mathbb{C}} \setminus W$ which fixes b_1, b_2, b_3 and $a_1, a_2, \ldots, a_{p+1}$. By the homotopy lifting theorem, $\Phi_\lambda^{-1}\Phi_\sigma$ lifts to a homeomorphism

$$\psi_{\sigma\lambda} : R_\sigma \setminus \Phi_\sigma(f^{-1}(W)) \to R_\tau \setminus \Phi_\tau(f^{-1}(W))$$

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which fixes $\Phi_\sigma(a_{p+1}) = \Phi_\tau(a_{p+1})$ in the commuting diagram

$$\begin{array}{ccc} R'_\sigma = R_\sigma \setminus \Phi_\sigma(f^{-1}(W)) & \xrightarrow{f\Phi_\sigma^{-1}} & \widehat{\mathbb{C}} \setminus W \\ \downarrow \psi_{\sigma\lambda} & & \downarrow \Phi_\lambda^{-1}\Phi_\sigma \\ R_\tau \setminus \Phi_\tau(f^{-1}(W)) & \xrightarrow{f\Phi_\tau^{-1}} & \widehat{\mathbb{C}} \setminus W \end{array} \tag{6}$$

(we note that both $f\Phi_\sigma^{-1}$ and $f\Phi_\tau^{-1}$ map $\alpha = \Phi_\sigma(a_{p+1})$ to b_1).

Now for any $\lambda, \sigma \in I$ $\Phi_\lambda(a_{p+2}) = \Phi_\sigma(a_{p+2})$, $\Phi_\lambda(b_2) = \Phi_\sigma(b_2)$ and $f(a_{p+2}) = b_2$. Thus from (6) $f\Phi_\tau^{-1}\psi_{\sigma\lambda}\Phi_\sigma(a_{p+2}) = \Phi_\lambda^{-1}\Phi_\sigma f(a_{p+2}) = b_2$, so that $\psi_{\sigma\lambda}\Phi_\sigma(a_{p+2}) \in \Phi_\tau f^{-1}(b_2)$, which is a discrete subset of R_τ. Since the left-hand side is continuous in λ it must be constant for $\lambda \in I$ and, in particular, setting $\lambda = \sigma$ and noting $\psi_{\sigma\sigma} = \Phi_\tau\Phi_\sigma^{-1}$, $\psi_{\sigma\lambda}\Phi_\sigma(a_{p+2}) = \psi_{\sigma\sigma}\Phi_\sigma(a_{p+2}) = \Phi_\tau(a_{p+2}) = \Phi_\sigma(a_{p+2})$. Similarly, $\psi_{\sigma\lambda}\Phi_\sigma(a_{p+3}) = \Phi_\sigma(a_{p+3})$.

Taking $\lambda = \tau$ in the commuting diagram (6) gives $f_\sigma = f_\tau\psi_{\sigma\tau}$. Thus $\psi_{\sigma\tau}$ is meromorphic and injective in $R_\sigma \setminus f_\sigma^{-1}\Phi_\tau(W) = R_\sigma \setminus \Phi_\sigma f^{-1}(W)$, that is conformal in $\widehat{\mathbb{C}} \setminus K$ where K is the compact set $\Phi_\sigma(E \cup f^{-1}(W))$. Recall that $f^{-1}(W)$ is countable with limit points in E. We claim that K has a complement of class O_{AD}. First K is totally disconnected. Suppose that the function F is univalent in the complement of $E \cup f^{-1}(W)$. Then each point $f^{-1}(W)$ is a removable singular point of F and the extension of F to the complement of E is univalent. By Theorem A.3, F is then linear. Thus $E \cup f^{-1}(W)$ has complement of class O_{AD} and Lemma 6 implies that the same is true for K. By Lemma 7 $\psi_{\sigma\tau}$ has an extension to a conformal map of $\widehat{\mathbb{C}}$, a Möbius transformation. Since $\psi_{\sigma\tau}$ fixes the three points $\Phi_\sigma(p+i)$, it is the identity. Thus $\Phi_\sigma = \Phi_\tau$ and Lemma 8 is proved. □

For each $\sigma \in I$ all images $\Phi_\sigma(U) = V$ are the same. Let H be a conformal map $H : D \to V$. Then both $H^{-1}\Phi_\sigma h$ and $\phi_{\underline{t}(\sigma)}$ map D to D with the same dilatation, since Φ_σ and $\chi_{\underline{t}(\sigma)} = h\phi_{\underline{t}(\sigma)}h^{-1}$ have the same dilatation in U. Hence there is a Möbius map $L_\sigma : D \to D$ such that $H^{-1}\Phi_\sigma h = L_\sigma\phi_{\underline{t}(\sigma)}$.

Since $\Phi_\sigma = \Phi_0$ for $\sigma \in I$ we have $L_\sigma\phi_{\underline{t}(\sigma)} = H^{-1}\Phi_\sigma h = H^{-1}\Phi_0 h = L_0\phi_{\underline{t}(0)}$ in D. Both sides are continuous in $\overline{D}$ so that the equality $L_\sigma\phi_{\underline{t}(\sigma)} = L_0\phi_{\underline{t}(0)}$ holds also on ∂D, in particular on the open subset of ∂D which is fixed by all $\phi_{\underline{t}(\sigma)}$. This implies that $L_\sigma = L_0$, $\sigma \in I$, and hence $\phi_{\underline{t}(\sigma)} = \phi_{\underline{t}(0)}$ on the whole of ∂D which contradicts the construction. The theorem is proved. □

7. *Baker domains*

The method of Eremenko and Lyubich [**20**, pp. 993–994] gives the following.

THEOREM F. *If $f \in \mathbf{M}$ and if $e \in E(f)$ is not a limit point of singularities of f^{-1} then there is no invariant component of $F(f)$ in which $f^n \to e$.*

Proof. Suppose that D is such a component. We may suppose that $e = \infty$ and take R so that in $A = \{z : R < |z| < \infty\}$ there is no singular point of f^{-1}. We may also suppose that $0 \notin E(f)$, $|f(0)| < R$.

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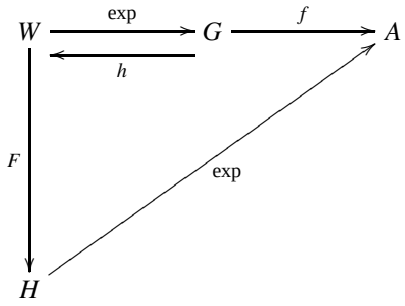


FIGURE 3. The analytic homeomorphism F.

Now $f(D) \subset D$ so $\infty \in \partial D$. By Theorem C there is a path $\gamma(t)$, $0 \leq t < 1$, in D such that $f(\gamma) \subset \gamma$, $\gamma(t) \to \infty$ and $f(\gamma(t)) \to \infty$ as $t \to 1$. For t near 1 we have $\gamma(t) \in A$, $f(\gamma(t)) \in \gamma \cap A$ and, taking g to be the branch of f^{-1} such that $g(f(\gamma(t))) = \gamma(t)$, we may continue g without restriction in A. Then g has either an algebraic branch point or a logarithmic branch point at ∞. But if ∞ is an algebraic branch point we must have $g(\infty) = \infty$ which conflicts with the assumption that $\infty \in E(f)$. Hence ∞ is a logarithmic branch point of g.

Thus if $H = \{z : \operatorname{Re} z > \ln R\}$ we have $g(e^t)$ univalent in H with values in a simply-connected unbounded region G such that $0, \infty \notin G$. We take a branch h of the logarithm to map G to a simply-connected domain W. Since $f \circ \exp : W \to A$ and $\exp : H \to A$ are both universal covers there is an analytic homeomorphism $F : W \to H$ such that $\exp \circ F = f \circ \exp$ in W. Note that W contains no disc of diameter 2π and the sets $W_n = W + 2n\pi i$ are disjoint. We may define F in $\cup_n W_n$ to be periodic with period $2\pi i$, see Figure 3.

Applying Koebe's $\frac{1}{4}$-theorem to $F^{-1} : H \to W$ shows that $|F'(z)| \geq (1/4\pi)(\operatorname{Re} F(z) - \ln R)$ in $\cup_n W_n$.

Now by assumption there is in D a disc $D_0 = D(z_0, r)$, $r > 0$, centred at some $z_0 \in \gamma(t)$, such that for all $n \in \mathbb{N}$, $D_n = f^n(D(z_0, r)) \subset A$. Since $z_n = f^n(z_0) \in D_n \cap \gamma$ and $f(z_n) = z_{n+1}$ we have $g(z_{n+1}) = z_n$, $g(D_{n+1}) = D_n$ and so all D_n, $n > 1$, in G. Next we put $C_n = hD_n$ in W. We have $\exp C_{n+1} = D_{n+1} = fD_n = f \exp C_n = \exp FC_n$. Hence C_{n+1} is $F(C_n) + 2\pi vi$ for some v.

Now let d_n be the radius of the largest disc centred at $t_n = hz_n$ which is contained in C_n. By the $\frac{1}{4}$-theorem $d_{n+1} \geq \frac{1}{4} d_n |F'(t_n))|$. Since $\operatorname{Re} t_n \to \infty$ as $n \to \infty$ we have $|F'(t_n)| \to \infty$ and we obtain a contradiction with the fact that C_n can have d_n at most π. □

COROLLARY. *If $f \in$ **MS**, then it has no Baker domains.*

For otherwise, there is some $p \in \mathbb{N}$, component G of $F(f)$ with $f^p(G) \subset G$, $f^{pn} \to e \in \partial G \cap E(f^p)$. But $f^p \in$ **MS** also and e cannot be a limit point of singularities of f^{-p}, so the result follows from Theorem F.

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COROLLARY. *If $f \in \mathbf{MSR}$, then every component of $F(f)$ is eventually periodic and the only periodic components are associated with attracting or parabolic periodic points or are part of a cycle of Siegel discs or Herman rings.*

8. *Relation between the singular orbits and the Fatou domains*

The results which hold for rational maps generalize with little change in the proofs.

LEMMA 9. *If $f \in \mathbf{M}$ and if $g_{n(k)}$ are branches of the inverses of $f^{n(k)}$ which are meromorphic in the domain U, $k \in \mathbb{N}$, then $\{g_{n(k)}\}$ is normal in U.*

Proof. If $E(f)$ contains fewer than three points, Theorem A(7) shows that repelling periodic points are dense in the perfect set $J(f)$. Since the solutions of $f^p(z) = z$ have limit points only at the countable set $E(f^p) = E \cup f^{-1}(E) \cup \cdots \cup f^{-(p-1)}(E)$ it follows that there are different integers p, q (≥ 3) and (distinct) p and q cycles: $A = \{\alpha_0, \alpha_1, \ldots, \alpha_p\}$ and $B = \{\beta_0, \beta_1, \ldots, \beta_q\}$. In $U \setminus A$ we have $g_{n(k)}(z) \neq \alpha_0, \alpha_1, \ldots, \alpha_p$ so that $\{g_{n(k)}\}$ is normal in $U \setminus A$. The same is true in $U \setminus B$ and hence in U.

If, however, E contains three points e_1, e_2, e_3 then $g_{n(k)} \neq e_1, e_2, e_3$ in U so the result follows.

LEMMA 10.

(i) *If $f \in \mathbf{M}$ and $G_1, \ldots, G_p$ is a periodic cycle of Fatou components in which the iterates converge to either an attracting or parabolic periodic cycle of points then $G_1 \cup \cdots \cup G_p$ contains the forward orbit of some singular point of f^{-1}.*

(ii) *If $G_1, \ldots, G_p$ is a cycle of Siegel discs or Herman rings (i.e. each G_i is a Siegel disc or Herman ring of f^p), then each point of $\cup_i(\partial G_i)$ is a limit of points in the orbit of singularities of f^{-1}.*

Proof. It suffices to consider the case $p = 1$; $G_1 = G$ is then an invariant component.

(i) Suppose that G contains no singular points of f^{-1}. If G contains an attracting fixed point p with $\lambda = f'(p)$ we have $|\lambda| < 1$ and may suppose that $\lambda \neq 0$, since otherwise p is the required singular point with $O(p) = \{p\} \subset G$.

Let g denote the branch of f^{-1} such that $g(p) = p$, $g'(p) = \lambda^{-1}$. Then if $D = D(p, R) \subset G$ we have g analytic in D. Further, for any $n \in \mathbb{N}$, $g^{(n)}$ is analytic at least in some neighbourhood of p and then by analytic continuation, noting the complete invariance of $F(f)$, in at least D. Thus by Lemma 9 $\{g^n\}$ is a normal family in D which contradicts the fact that $g^n(p) = p$, $(g^n)'(p) = \lambda^{-n} \to \infty$.

If G is a component in which f^n converges to a parabolic fixed point, which we may assume to be $0 \in \partial G$, then we have for some $p \in \mathbb{N}$ that the Taylor expansion of f near zero is

$$f(z) = z + a_{p+1}z^{p+1} + \cdots, \quad a_{p+1} \neq 0, z \in D(0, R)$$

and we denote by g the local inverse

$$g(z) = z - a_{p+1}z^{p+1} + \cdots, \quad z \in D(0, \rho), \rho > 0.$$

We may assume that $a_{p+1} < 0$ and that G contains

$$W = D(0, R) \cap \left\{ z : |\arg z| < \frac{\pi}{p} - \epsilon \right\}, \quad \text{where } 0 < \epsilon < \frac{\pi}{4p}.$$

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For small positive $x \in W$ we have $g(x) \in W$ and $g(g(x)) \in W$. By analytic continuation we find $g^2(z) = z - 2a_{p+1}z^{p+1} + \cdots$ and further $g^n(z) = z - na_{p+1}z^{p+1} + \cdots$ analytic and normal in W. By the local behaviour near parabolic points, g^n are defined and normal in $W'_\pm = D(0, s) \cap \{z : |\arg z \pm \pi/p| \leq 3\pi/4p\}$ for some $s > 0$ with $g^n \to 0$ in W' uniformly. Since $\{g^n\}$ is normal in $W \cup W'$ we have $g^n \to 0$ uniformly on the rays $\alpha_\pm : \{\arg z = 3\pi/2p$ or $-3\pi/2p, 0 \leq |z| \leq t\}$ and on $C_t = \{z : |z| = t, |\arg z| \leq 3\pi/2p\}$. Hence $g^n \to 0$ uniformly in the finite region V bounded by these rays and C_t. Now $z_n = f^n(t)$ is inside V for all n and $g^n(z_n) = t$ which contradicts the claim that $g^n \to 0$ in V. Thus the proof of (i) is complete.

(ii) If G is, for example, a Siegel disc and $z_0 \in \partial G$, suppose that some disc $D(z_0, R) = U$ contains no singular points of any f^{-n}. Denote by g the inverse of the univalent map $f|G$. Then g is defined at some points of U and has analytic continuation throughout U. The same is true for all iterates g^n of g. For some sequence $n_k \to \infty$ we have $f^{n_k} \to \mathrm{Id}$ in G and $g^{n_k} \to \mathrm{Id}$ in G. Since $\{g^{n_k}\}$ are normal in U we have $g^{n_k} \to \mathrm{Id}$ in U. Thus for all sufficiently large k, $g^{n_k}(U)$ covers a fixed neighbourhood V of z_0 and $f^{n_k}(V) \subset U$. But this contradicts the fact that $z_0 \in J(f) \cap V$. □

9. *Totally disconnected Julia sets*

In simple cases the results of §8 may allow us to determine the dynamics of a map in our class. As an example we have the following result.

THEOREM G. *Suppose that $f \in$ **MSR** and suppose that there is an attracting fixed point whose Fatou component G contains all the singular points of f^{-1}. Then $J(f)$ is totally disconnected.*

Remarks. (1) In dealing with a transcendental entire function we must remember that the singular point $\infty \in E(f) \subset J(f)$ so that f cannot possibly satisfy the conditions of the theorem and there can be no conflict with the known fact that $J(f)$ is never totally disconnected for this class of functions.

(2) The theorem is, however, interesting even in the case of transcendental meromorphic functions where the only known examples with totally disconnected $J(f)$ seems to be the function $\lambda \tan z$, $|\lambda| < 1$, or their conjugates.

Example 6. Let $R(z) = p(z)/q(z)$ where p and q are polynomials of degree (exactly) n without common zeros and such that:

(1) all zeros of p and q are simple;
(2) $q(0) \neq 0$; and
(3) $p(1) = 0$.

Then if the constant λ is sufficiently small, $f(z) = \lambda R(e^z)$, which is meromorphic in $\mathbb{C}$ with essential singularity ∞, has totally disconnected Julia set.

To justify the above claim note that the singularities of $f^{-1} \subseteq \lambda S$ where $S = \{\text{critical values of } R\} \cup \{R(0), R(\infty)\} = A \cup B$ (say).

By assumption (1) $0, \infty \notin A$. By assumption (2) $R(0) \neq \infty$. Since $\deg p = \deg q$ we have $R(\infty) \neq \infty, 0$. Thus S is a finite subset (of at most $2 \deg p$ points) of $\mathbb{C}$.

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Further, $f(0) = \lambda R(1) = 0$. Take $r > 0$ so that $R(e^z)$ is analytic in $D(0, 2r)$, and so bounded in $D = D(0, r)$. Then if λ_0 is sufficiently small we have for all $|\lambda| < \lambda_0$ that $f : D \to D$ so that D belongs to a component G of $F(f)$, zero is an attracting fixed point and $\lambda S \subset D$. Thus Theorem G applies to prove our claim.

Example 7. For sufficiently small constants λ, μ the function $f = \tan(\lambda \tan \mu z)$ has totally disconnected Julia set. For $f \in \mathbf{K} \cap \mathbf{MS}$ with an attracting fixed point at zero if $|\lambda\mu| < 1$. The singular points are $\pm i$ and $\pm \tan(\lambda i)$. For sufficiently small $|\lambda|$, $|\mu|$ we have $|\tan \lambda i| < 1$ and for all $|z| \leq 1$ that $|f(z)| \leq 2|\lambda\mu||z| \leq \frac{1}{2}|z|$. The function f therefore satisfies the assumptions of Theorem G.

Proof of Theorem G. We suppose that $f \in \mathbf{MSR}$ and that zero is an attracting fixed point of f which belongs to the component G of $F(f)$. The set of singular points is finite, $\{a_1, \ldots, a_p\} \subset G$. Clearly $f(G) \subset G$.

First we show that G is completely invariant. Take z in G such that $z_1 = f(z)$ is different from a_j, $1 \leq j \leq p$, and denote by g the branch of f^{-1} such that $g(z_1) = z$. If h is an arbitrary branch of f^{-1}, defined and hence analytic in a neighbourhood of z_1, there is some closed path γ in $\Omega = \widehat{\mathbb{C}} \setminus \{a_1, \ldots, a_p\}$ starting and ending at z_1, such that along γ the branch g continues analytically to h. But γ is homotopic in Ω to a path γ' which starts and ends at z_1 and has $\gamma' \subset G$. Continuation of g along γ' yields h but the continuation maps γ' into $F(f)$ and hence remains in G. Thus G is completely invariant.

By Theorems E and F there are neither Baker domains nor wandering components in $F(f)$. Further, each $O(a_j)$ lies in G and has only zero as a limit point. It follows from Lemma 10 that the only periodic component is G. This is indeed the only component of $F(f)$ since any component H of $F(f)$ must map by some f^n into G, which is completely invariant.

Now let Φ denote the analytic universal covering map from the unit disc D to G such that $\Phi(0) = 0$, and Ψ the local inverse of Φ which is analytic in a neighbourhood of zero with $\Phi(0) = 0$. Then $F = \Psi f \Phi$ is analytic in a neighbourhood of zero with $F(0) = 0$, $F'(0) = f'(0)$. Analytic continuation of F is possible throughout D and F is then analytic in D with $F : D \to D$.

We choose ρ such that $0 < \rho < 1$ holds and the circumference $C(0, \rho)$ contains none of the countable set of points $\Phi^{-1}(a_j)$, $1 \leq j \leq p$, while the disc $D(0, \rho)$ contains at least one of $\Phi^{-1}(a_j)$ for every $1 \leq j \leq p$.

Now $\Phi : D(0, \rho) \to K$ which contains $a_1, \ldots, a_p, 0$ and satisfies $f(K) \subset K$ since, by Schwarz's Lemma, F maps $D(0, \rho)$ into itself. The boundary ∂K is a subset of $\Gamma : \Phi(C(0, \rho))$. The curve Γ crosses itself only at points $\Phi(z)$ where $z \in C(0, \rho)$ is such that $Tz \in C(0, \rho)$ for some element T of the covering group which is different from the identity. We shall show that there are at most finitely many such points. If this is not the case there are sequences $z_n \in C(0, \rho)$ and $z'_n \in C(0, \rho)$, $T_n \neq \mathrm{Id}$ in the covering group such that $z_n \to \alpha$, $z'_n = T_n(z_n) \to \beta$. There is a neighbourhood $\Delta = D(\alpha, \eta)$ such that $T\Delta \cap \Delta = \emptyset$ for every covering transformation other than the identity. Let $\Delta' = D(\alpha, \eta/2)$ and take n so large that $z_n, z_{n+1} \in \Delta'$ and $|z'_n - z'_{n+1}| < \epsilon$ where ϵ is a number such that $0 < 4\epsilon < (1 - \rho)\eta$. Since T_{n+1}^{-1} maps z'_{n+1} to z_{n+1} and $D(z'_{n+1}, 1 - \rho)$ into the disc $D(z_{n+1}, 2)$, we have from Schwarz's lemma that $T_{n+1}^{-1} z'_n$ is in

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Δ, which implies that $T_{n+1}^{-1}T_n = \mathrm{Id}$. Thus for large n all T_n are the same, say T. But then T fixes both circles $C(0, \rho)$ and $C(0, 1)$ which is possible only if $T(0) = 0$, or since T is a covering transformation if $T = \mathrm{Id}$, which contradicts our assumption about T_n. Thus Γ is an analytic curve with at most a finite set of self-intersections.

We have shown that $\overline{K}^c$ has a finite set of components, each of which must be simply connected and bounded by a part of Γ. Denote these components by A_i, $1 \leq i \leq N$. We note that $f^{-1} : K^c \to K^c$ and indeed for any n the branches of f^{-n} are all analytic in each A_i and map A_i to disjoint domains $A_{i,s}^n$. Since $\Gamma \subset F(f)$, each component of $J(f)$ lies in the interior of some $A_{i,s}^n$ (for each n).

Suppose that some component μ of $J(f)$ contains more than one point. Now $f^n(\mu)$ is in the interior of some $A_{i(n)}$ and we may choose a sequence $n(k)$ such that $n(k) \to \infty$ as $k \to \infty$ and each $i(n(k))$ is the same, A_1 (say). Thus μ belongs to a component, which we call A^k (= some $A_{1,s}^{n(k)}$) of $f^{-n(k)}A_1$. The sequence of branches g_k of $f^{-n(k)}$, such that $g_k(A_1) = A^k$, is normal in A_1, and without loss of generality we may assume that g_k is locally uniformly convergent in A_1 with limit h.

Now $J_1 = J(f) \cap A_1$ is a compact subset of A_1. Thus if h is constant we must have $\operatorname{diam} g_k(J_1) \to 0$ as $k \to \infty$ which implies that $\operatorname{diam} \mu = 0$, against our assumption. Hence h is a non-constant function univalent in A_1. We take a Jordan curve δ in A_1 whose interior I contains J_1. Thus $h(J_1)$ is a compact subset of $h(I)$ which is bounded by $h(\delta)$. For sufficiently large k, $g_k(\delta)$ approximates $h(\delta)$ closely and there is an open set U such that for all large k we have $\mu \in g_k(J_1) \subset U \subset g_k(I)$. Thus $f^{n(k)}(U) \subset I \subset A_1$. In particular, all $f^{n(k)}$ and hence all f^n are analytic in U. Further, for any n there is a greater $n(k)$ so that $n(k) = n + j$ for some $j > 0$. Then $f^n(U) \subset f^{-j}f^{n(k)}(U) \subset f^{-j}(A_1)$, which is a set disjoint from K. It follows that $U \subset F(f)$ which contradicts the fact that $\mu \subset U$. We have now proved that all components of $J(f)$ are singletons. □

There exist cases where we can guarantee the existence of singleton components although the Julia set may not be totally disconnected. If f is a transcendental entire function then $J(f)$ is not totally disconnected (see, for example, [**9**, Theorem 9]), but it is shown in [**16**] that if there exists a multiply-connected component of $F(f)$, then singleton components are dense in $J(f)$.

In [**17**] it was shown that if f is a transcendental meromorphic function (where $E = \{\infty\}$) and if there exists a multiply-connected component of $F(f)$ of connectivity at least three, then singleton components are dense in $J(f)$. This result may be generalized with only slight changes in the proof.

THEOREM H. *Suppose that $f \in \mathbf{MA}_k$ and that $F(f)$ has a component H of connectivity at least k. Then singleton components are dense in $J(f)$.*

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A. *Appendix*

For the reader's convenience we include some less standard results of complex analysis relevant to the paper. The first results relate to the classifications of Riemann surfaces but are required only for plane domains W. See [**2**, Ch. 4]. In §6 we have already given the definition of the Dirichlet integral of a function analytic in W.

Definition A.1. A plane domain W is of class O_{AD} if the only analytic functions in W with finite Dirichlet integral are constants.

Definition A.2. A plane domain W is of class O_{AB} if the only bounded analytic functions in W are constants.

It is shown in [**2**] that O_{AB} is strictly contained in O_{AD}.

THEOREM A.3. [**2**] *A plane domain W is of class O_{AD} if and only if the only univalent functions in W are linear functions.*

THEOREM A.4. [**2**] *If the domain W is the complement of a closed bounded set E and E has linear measure zero, then $W \in O_{AB} \subset O_{AD}$.*

Remark. E has linear measure zero if it can be enclosed in a finite number of circular discs whose radii have an arbitrarily small sum.

Definition A.5. A closed set E is a removable singularity for any function f defined in a neighbourhood G of E with bounded Dirichlet integral if any such f has an analytic extension to G.

THEOREM A.6. [**2**] *A closed set E is a removable singularity in the sense of Definition A.5 if and only if its complement in the sphere is a domain of class O_{AD}.*

We remark that Lemma 6 states that plane sets whose complement is of class O_{AD} are quasiconformally invariant.

THEOREM A.7. [**37**, p.10] *If E is a closed totally disconnected set whose complement is of class O_{AB} and f is meromorphic in E^c with essential singularities at every point E, then $C(f, E^c, z_0) = \widehat{\mathbb{C}}$ for all $z_0 \in E$.*

Ahlfors' five island theorem has many uses in dynamics. It is also clearly related to the theory of normal families. See, e.g., the recent survey [**11**]. Some extensions to more general situations are mentioned in §2. One form of the theorem follows.

THEOREM A.8. [**44**] *Let f be a transcendental meromorphic function in $\mathbb{C}$, and let $D_1, D_2, \ldots, D_5$ be five simply connected domains in $\mathbb{C}$ with disjoint closures. Then there exists $j \in \{1, 2, \ldots, 5\}$ and, for any $R > 0$, a simply connected $G \subset \{z \in \mathbb{C} : |z| > R\}$ such that f is a conformal map G onto D_j. If f has only finitely many poles, then 'five' may be replaced by 'three'.*

THEOREM A.9. (Koebe's $\frac{1}{4}$-theorem [**44**]) *If f is a conformal map of $D(0, 1)$ onto the domain $G \subset \mathbb{C}$ such that $f(0) = 0$, $f'(0) = 1$, then G contains a disc $D(0, \rho)$ with $\rho \geq \frac{1}{4}$.*

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THEOREM A.10. (Lindelöf's theorem [**15**, p. 43]) *Suppose that f is analytic in $D(0,1)$ and continuous on $\overline{D}(0,1) \setminus \{1\}$.*

If $\lim f(e^{i\theta}) = e_+$ and e_- as $\theta \to 0$ through positive and negative values, respectively, where e_- and e_+ are finite, then $e_- \neq e_+$ implies that $C_D(f,1) = \widehat{\mathbb{C}}$. In particular, if $\widehat{\mathbb{C}} \setminus f(D)$ contains an open disc, the $e_- = e_+$.

Definition A.11. (Quasiconformal maps [**32**]) A function $u(x, y)$ is ACL (absolutely continuous on lines) in a plane domain D if, for every closed rectangle R in D with sides parallel to the x and y axes, $u(x, y)$ is absolutely continuous on almost every horizontal or vertical line in R.

A topological map ϕ of the plane domain D into $\widehat{\mathbb{C}}$ is called quasi-conformal if:

(i) ϕ is ACL in D; and

(ii) there is a constant k, $0 \leq k < 1$, such that and $|\phi_{\overline{z}}| \leq k|\phi_z|$ holds almost everywhere in D, where $\phi_z = \frac{1}{2}(\phi_x - i\phi_y)$, $\phi_{\overline{z}} = \frac{1}{2}(\phi_x + i\phi_y)$.

Further, $\mu(z) = \phi_{\overline{z}}/\phi_z$ is called the the complex dilatation of ϕ. If $\mu(z) = 0$ almost everywhere, then ϕ is conformal.

If f is a one-to-one conformal map from a domain D to a domain D_1 and ϕ is a quasi-conformal map defined on D and D_1 whose complex dilatation μ satisfies

$$\mu(f(z)) = \frac{\mu(z) f'(z)}{\overline{f'(z)}}, \quad \text{a.e. in } D,$$

then $\phi \circ f \circ \phi^{-1}$ is conformal in $\phi(D)$.

THEOREM A.12. (The measurable Riemann mapping theorem [**1**]) *Given any measurable function μ on $\mathbb{C}$ such that $\|\mu\|_\infty = \operatorname{ess\,sup}|\mu| < 1$, there exists a unique sense-preserving quasiconformal homeomorphism $\phi = \phi^\mu$ of $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ which has complex dilatation μ and fixes 0, 1, ∞.*

Ahlfors and Bers [**1**] proved that if μ depends continuously and differentially on parameters, the same is true of ϕ^μ. Their result includes the following.

Write $t = (t_1, \ldots, t_n)$ and $s = (s_1, \ldots, s_n)$. Suppose that for all t in some open set $\Delta \subset \mathbb{R}^n$ one has for the μ of A.12 that $\mu(t, z) \in L_\infty$ as a function of z with $\|\mu\|_\infty < 1$ and that (suppressing z)

$$\mu(t, s) = \mu(t) + \sum_{i=1}^{n} a_i(t) s_i + |s|\alpha(t, s)$$

with $\|\alpha(t, s)\|_\infty \leq c$, c constant and $\alpha(t, s) \to 0$ a.e. in z as $s \to 0$. Suppose that $\|a_i(t + s)\|_\infty$ are bounded and that $a_i(t + s) \to a_i(t)$ a.e. for $s \to 0$.

Then $\phi^{\mu(t)}$ is in $C^1(\Delta)$ as a function of t for fixed z.

LEMMA A.13. (The rate at which iterates can approach infinity in an unbounded invariant domain [**9**, Lemma 7]) *Let G be an unbounded invariant domain in $\mathbb{C}$ with at least two finite boundary points, and let g be analytic in G.*

Suppose that $g(G) \subset G$ and $g^n \to \infty$ in G. Then for any compact subset K of G there exist constants C and n_0 such that $|g^n(z')| \leq |g^n(z)|^C$ for all $z, z' \in K$, $n \geq n_0$. Further, there exist a constant $A > 1$ and a curve γ in G tending to ∞ which satisfies $g(\gamma) \subset \gamma$ such that $|z|^{1/A} \leq |g(z)| \leq |z|^A$ for $z \in \gamma$.

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