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# Betweenness relations and cycle-free partial orders

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## 1. *Introduction*

The intuition behind the notion of a *cycle-free partial order* (*CFPO*) is that it should be a partial ordering $(X, \leqslant)$ in which for any sequence of points $(x_0, x_1, \ldots, x_{n-1})$ with $n \geqslant 4$ such that $x_i$ is comparable with $x_{i+1}$ for each $i$ (indices taken modulo $n$) there are $i$ and $j$ with $j \neq i, i+1$ such that $x_j$ lies between $x_i$ and $x_{i+1}$. As its turn out however this fails to capture the intended class, and a more involved definition, in terms of the 'Dedekind–MacNeille completion' of $X$ was given by Warren [**5**]. An alternative definition involving the idea of a betweenness relation was proposed by P. M. Neumann [**1**]. It is the purpose of this paper to clarify the connections between these definitions, and indeed between the ideas of semi-linear order (or 'tree'), CFPO, and the betweenness relations described in [**1**]. In addition I shall tackle the issue of the axiomatizability of the class of CFPOs.

The study of CFPOs was originally proposed by Rubin as a way of generalizing trees, whereby the partial ordering is allowed to branch either upwards or downwards, and repeatedly, but never to 'return' to its starting point. His hope was that CFPOs would provide a class naturally extending that of trees for which reconstruction results similar to his in [**4**] could be obtained, and by using similar methods. Certainly the fact that 'piecewise patching' is possible for CFPOs is a major feature of the techniques used in [**5**] (which in turn are derived from those of [**2**]).

A typical problem which arises if we attempt to define 'cycle-free partial order' as above is that it rules out the 5-element partial ordering $\{a, b, c, d, e\}$ for which $a, b \leqslant c \leqslant d, e$ and with $a, b$ incomparable and $d, e$ incomparable. Warren [**5**] considered what was necessary to capture the intuitive notion, and proposed the definition mentioned above. This worked well in that it conformed to the geometric idea about absence of circuits, and on the basis of the definition he was able to develop a rather beautiful theory of CFPOs which are sufficiently homogeneous. I shall give a resume of the notions of 'Dedekind–MacNeille complete' and 'Dedekind–MacNeille completion' in Section 2, but just mention here that the idea is that $(X, \leqslant)$ is complete in this sense if all the maximal chains are Dedekind-complete as linear orderings, and if any two points which have an upper bound (lower bound) in $X$ have a least upper bound (greatest lower bound respectively). The Dedekind–MacNeille completion of $X$ is denoted by $X^D$.

Now in [**5**] all CFPOs were automatically supposed connected, and essentially appealing as connectedness does to quantification over the positive integers, it is hardly surprising that this cannot be captured by a set of first order formulae. The assumption was only made for convenience however, and I shall not require a CFPO to be connected. By a *cycle-free partial order* I shall therefore understand a partial

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ordering $(X, \leqslant)$ in which for any $a, b \in X$ there is *at most one* path from $a$ to $b$ in $X^D$ (rather than exactly one as in [**5**]). The 5-element partial ordering mentioned above then *is* a CFPO, because for example there is a unique path from $a$ to $b$, namely $a$, $c, b$. (The definition of 'path' in this context precludes the possibility that it pass through the same point more than once, as $a, c, d, c, b$ does.) A cycle-free partial order will thus be a disjoint union of connected components, each of which is a CFPO in Warren's sense. I therefore refer to a CFPO as in [**5**] as a *connected CFPO*.

The main results are then as follows. In the first place, the class of all CFPOs is first order axiomatizable, but not finitely axiomatizable. The existence of an axiomatization follows from the fact that not all the Dedekind–MacNeille completion is required in the definition; only $X$ together with the ramification points (written $X^+$ in [**5**]). In fact by modifying the list of axioms given it is shown that the theory of CFPOs is universal, so that a substructure of a CFPO is necessarily a CFPO. The non-finite-axiomatizability is rather clear, since for each $n \geqslant 3$ there is a partial ordering (a 'crown') containing a cycle of length $2n$ but none of shorter length. Using Neumann's idea, it is also shown that CFPOs form a pseudo-elementary class, meaning that in an expanded language they are axiomatized by a single sentence. The extra symbol needed is a ternary relation symbol for 'betweenness', though unfortunately the betweenness relation needs to be taken on $X^+$ rather than $X$. The reasons for this are explained in Section 3. In a similar way, there is a close connection between the class of semi-linear orders and B-relations, which has been explored in [**1**], and which I make some remarks about in Section 2.

We conclude Section 3 with a result due to J. Schmerl (which I am grateful to him for allowing me to reproduce) generalizing the ideas from this section and which can be used to give alternative derivations of several of them. I would also like to thank M. Droste, P. M. Neumann, M. Rubin, and the referee for helpful remarks.

The axioms for a (general) betweenness relation as given in [**1**] are as follows:

(B 1) $$B(x; y, z) \Rightarrow B(x; z, y),$$

(B 2) $$B(x; y, z) \,\&\, B(y; x, z) \Leftrightarrow x = y,$$

(B 3) $$B(x; y, z) \Rightarrow B(x; y, t) \quad \textit{or} \quad B(x; z, t),$$

(B 4) $$\neg B(x; y, z) \Rightarrow (\exists t \neq x)(B(t; x, y) \,\&\, B(t; x, z)).$$

If a ternary relation $B$ on a set $X$ satisfies (B 1)–(B 3), it is called a *B-relation*. For any ternary relation $B$ on $X$, and $x, y \in X$, we let the *closed interval* $[x, y]$ be $\{z \in X : B(z; x, y)\}$. Since we shall also be studying partial orderings, it is important to remark that interval notation will just be used here for B-relations. Any intervals in a partial ordering we need will be written out in full as $\{x : a \leqslant x \leqslant b\}$ for instance (except where they coincide with the corresponding $B$-interval). The first two lemmas are given in [**1**].

LEMMA 1·1. *If $(X, B)$ is a B-relation then*

(i) $B(x; z, t) \,\&\, B(y; z, t) \Rightarrow B(x; z, y)$ *or* $B(y; z, x)$,

(ii) $B(y; z, x) \,\&\, B(x; z, t) \Rightarrow B(y; z, t)$,

(iii) $B(y; x, z) \,\&\, B(z; x, t) \Rightarrow B(z; y, t)$,

(iv) $B(y; x, z) \,\&\, B(z; y, t) \,\&\, y \neq z \Rightarrow B(y; x, t) \,\&\, B(z; x, t)$,

(v) $B(x; y, y) \Rightarrow x = y$.

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LEMMA 1·2. *A B-relation $(X, B)$ is a betweenness relation if and only if for every $x, y, z \in X$, $[x, y] \cap [x, z] = \{x\} \Rightarrow B(x; y, z)$.*

Let us now consider extensions of B-relations. If $(X, B)$ is a B-relation we may extend it by increasing $B$ or $X$ or both. Here we shall just consider the former, with the object of obtaining another attractive characterization of betweenness relations, though there are natural ways to consider the latter as well (particularly when forming 'completions').

THEOREM 1·3. (i) *A B-relation $B$ on the set $X$ is maximal among all B-relations on $X$ if and only if it is a betweenness relation.*

(ii) *Any B-relation on $X$ can be extended to a betweenness relation on $X$.*

*Proof.* (i) Let $B$ be a betweenness relation on $X$, and suppose that $B'$ is a B-relation on $X$ properly containing $B$. Suppose $B'(x; y, z) \& \neg B(x; y, z)$. By (B 4) there is $t \neq x$ such that $B(t; x, y) \& B(t; x, z)$. Hence $B'(t; x, y) \& B'(t; x, z)$. From $B'(t; y, x) \& B'(x; y, z)$ we find by Lemma 1·1 (iii) that $B'(x; t, z)$. By (B 2), $x = t$, a contradiction.

Conversely, suppose that $B$ is a maximal B-relation on $X$. We have to show that it satisfies (B 4). Suppose that $[a, b] \cap [a, c] = \{a\}$ where $\neg B(a; b, c)$. Then by (B 1) and (B 2) and Lemma 1·1 (v), $a, b, c$ must be distinct. Let $B'$ extending $B$ be given by

$$B'(x; y, z) \Leftrightarrow B(x; y, z) \quad \text{or} \quad (x = a \,\&\, [x, y] \cap [x, z] = \{x\}).$$

Then $B'(a; b, c)$ since the second clause applies.

We verify that $B'$ is a B-relation. (B 1) is immediate, and so is half ($\Leftarrow$) of (B 2) from the corresponding property of $B$. If $B'(x; y, z) \& B'(y; x, z)$ where $x \neq y$ then we cannot have both $B(x; y, z)$ and $B(y; x, z)$. Suppose $\neg B(x; y, z)$. Then the second clause of the definition of $B'(x; y, z)$ must apply, and so $x = a$. Since we now have $[x, y] \cap [x, z] = \{x\}$, we deduce that $y \notin [x, z]$, and so $\neg B(y; x, z)$. But this implies $y = a$, contrary to $x \neq y$.

For (B 3), suppose for a contradiction that $B'(x; y, z) \& \neg B'(x; y, t) \& \neg B'(x; z, t)$. Then as $B$ satisfies (B 3), $x = a$ and $[x, y] \cap [x, z] = \{x\}$. Since $\neg B'(x; y, t) \& \neg B'(x; z, t)$, $[x, y] \cap [x, t] \neq \{x\}$ and $[x, z] \cap [x, t] \neq \{x\}$, so we may choose $u \in [x, y] \cap [x, t] - \{x\}$ and $v \in [x, z] \cap [x, t] - \{x\}$. By Lemma 1·1 (i), as $B(u; x, t) \& B(v; x, t)$ we have either $B(u; x, v)$ or $B(v; x, u)$. Without loss of generality assume the former. By Lemma 1·1 (ii), from $B(u; x, v)$ and $B(v; x, z)$ we deduce $B(u; x, z)$. Therefore $u \in [x, y] \cap [x, z]$ from which it follows that $u = x$, which is a contradiction.

This establishes (B 3) for $B'$, and so $B'$ is a B-relation extending $B$. Since $\neg B(a; b, c)$ and $B'(a; b, c)$, it is a proper extension, contrary to maximality. We deduce that $B$ satisfies (B 4), and so is a betweenness relation.

(ii) This follows by (i) and a straightforward application of Zorn's Lemma.

## 2. *Semi-linear orders and betweenness relations*

The notion of a semi-linear order was studied in [2] (though there they were referred to as 'trees'). We say that the partial ordering $(X, \leqslant)$ is a *semi-linear order* if for each $x \in X$, $\{y \in X : y \leqslant x\}$ is linearly ordered by $\leqslant$, and for each $x, y \in X$ there is $z \in X$ such that $z \leqslant x \,\&\, z \leqslant y$. This latter property makes $(X, \leqslant)$ a 'tree' rather than a 'forest'. If a semi-linear order has a minimal element (which will then be unique, and below

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every other element) we say it is a *rooted tree* (or *rooted semi-linear order*). Note that semi-linear orders as defined here are what are referred to as 'lower semi-linear orders' in [1], and there the authors concentrate on the dual notion, that of 'upper semi-linear order'.

If $x, y, z$ are members of a partially ordered set $(X, \leqslant)$, then $z$ is said to be the *least upper bound* of $x$ and $y$, written $z = x \vee y$, if $x, y \leqslant z \,\&\, \forall t(x, y \leqslant t \Rightarrow z \leqslant t)$. Similarly we may define the *greatest lower bound* (where it exists) of $x$ and $y$, written $x \wedge y$. If $\neg x \leqslant y \,\&\, \neg y \leqslant x$, $x$ and $y$ are *incomparable*, written $x \parallel y$. Points of the form $x \vee y, x \wedge y$ where $x \parallel y$ are called *ramification points*.

For any partially ordered set $(X, \leqslant)$ there is a natural notion of completion, called the *Dedekind–MacNeille completion*, $X^D$, in which certain extra points are added as needed. The construction is given in detail in [5], and we shall use the version described there, where two points have a common upper or lower bound in the original set if and only if they do in the completion. I now give a brief resume of the method.

If $I \subseteq X$ we let $\bigvee I = \{x \in X : I \leqslant x\}$ and $\bigwedge I = \{x \in X : x \leqslant I\}$ (the sets of upper and lower bounds of $I$). A *Dedekind ideal* is a subset $I$ of $X$ such that $I \neq \varnothing$, $\bigvee I \neq \varnothing$ and $\bigwedge \bigvee I = I$. A Dedekind ideal is *principal* if it has the form $X^{\leqslant x} = \{y \in X : y \leqslant x\}$ for some $x \in X$ (which clearly *is* a Dedekind ideal). We say that $(X, \leqslant)$ is *Dedekind complete* if every Dedekind ideal is principal. Then the Dedekind–MacNeille completion $X^D$ of $X$ is the least Dedekind complete extension of $X$, and it may be formed as the family of all Dedekind ideals of $X$, ordered by inclusion, and with $X$ embedded in $X^D$ via the mapping $x \mapsto X^{\leqslant x}$. A dual construction, using 'Dedekind filters' is also possible; but the use of ideals is equivalent, and slightly eases the proof of Theorem 3·12.

For our purposes here we only require $X^+$ which is defined to be the subset of $X^D$ containing all points of $X$ together with any ramification points of $X^D$. In $X^D$, any $x$, $y \in X$ which have a common upper (lower) bound have a least upper bound (greatest lower bound). These may or may not lie in the original structure, and $X^+$ is the smallest extension of $X$ in which all 2-element subsets of $X$ have least upper bounds or greatest lower bounds (provided they are correspondingly bounded in $X$).

If $(X, \leqslant)$ is a semi-linear order then we only need to add upward ramification points to form $X^+$ (since if $x, y \leqslant z$ then $x, y$ are already comparable). The most natural way (as described in [1]) to associate a ternary relation $B = B_{\leqslant}$ with a semi-linear $(X, \leqslant)$ is to let $B(x; y, z)$ if $y \leqslant x \leqslant z$ or $z \leqslant x \leqslant y$ or $y \parallel x \leqslant z$ or $z \parallel x \leqslant y$ or $y \parallel z \,\&\, x = y \wedge z$. Note that this is defined in terms of $(X, \leqslant)$ alone, though the *intuition* is derived in a very natural way from the geometrical visualization of $(X, \leqslant)$ and $X^+$: namely that $B(x; y, z)$ should hold if and only if the natural path in $X^+$ from $y$ to $z$ via $y \wedge z$ contains $x$.

LEMMA 2·1. *If $(X, \leqslant)$ is a semi-linear order then $B_{\leqslant}$ is a B-relation. It is a betweenness relation if and only if for every $x, y \in X$, $(\forall z)(z < x \Rightarrow z < y) \Rightarrow x \wedge y \in X$.*

*Proof.* The condition given is equivalent to one which was independently formulated in [1], there called 'normality'. The proof may now be read off from that of Theorem 33·1 in [1].

*Example* 2·2. A simple example where $B_{\leqslant}$ is not a betweenness relation is given by

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taking $X = \omega \cup \{a, b\}$ where $\omega = \{0, 1, 2, \dots\}$ ordered as usual with $n < a, n < b$ for all $n$ (and no other relationships). Here $\neg B(a; b, 0)$ but for every $t \neq a$, either $t \in \omega$ in which case $\neg B(t; a, b)$ or $t = b$ in which case $\neg B(t; a, 0)$. Other more complicated examples are given in [1], in Theorem 5·8 and Example 21·3.

There is a slightly different and perhaps less intuitive way to derive a B-relation $B = B'_{\leqslant}$ from a semi-linear order, wherein one just deletes the last case. Here $B(x; y, z)$ if $y \leqslant x \leqslant z$ or $z \leqslant x \leqslant y$ or $y \parallel x \leqslant z$ or $z \parallel x \leqslant y$. (This definition is given in [1] just before Theorem 33·1.) One way of seeing that this is also a B-relation is to pass to $(X^{++}, \leqslant)$ extending $(X, \leqslant)$, formed thus. Let $X^+$ be obtained from $X$ by adjoining all ramification points to $X$. Then $X^{++}$ is the same as $X^+$ except that in cases where $y \parallel z$ and $y \wedge z \in X$, we add an additional point immediately above $y \wedge z$ (and related to the other points as $y \wedge z$ is). Clearly $(X^{++}, \leqslant)$ is still a semi-linear order, and $B'_{\leqslant}$ is just the ternary relation on $(X, \leqslant)$ induced by $B_{(X^{++}, \leqslant)}$. Since B-relations are $\forall$-axiomatized, $B'_{\leqslant}$ is still a B-relation. It may be equal to $B_{\leqslant}$ or not, and in one respect at least has better properties than $B_{\leqslant}$.

In fact there is a whole collection of B-relations lying between $B_{\leqslant}$ and $B'_{\leqslant}$. If $R$ is any set of ramification points of $X$ (that is, which lie *in* $X$), then we let $B_{(\leqslant, R)}$ be given by $B_{(\leqslant, R)}(x; y, z)$ if $y \leqslant x \leqslant z$ or $z \leqslant x \leqslant y$ or $y \parallel x \leqslant z$ or $z \parallel x \leqslant y$ or $y \parallel z \,\&\, x = y \wedge z \,\&\, x \in R$. Thus the two cases $B_{\leqslant}$ and $B'_{\leqslant}$ correspond to $R$ the set of *all* ramification points of $X$, and $R = \varnothing$ respectively.

THEOREM 2·3. *Let $(X, \leqslant)$ be a semi-linear order.*

(i) *For any set $R$ of ramification points of $X$, $B_{(\leqslant, R)}$ is a B-relation restricting $B_{(X, \leqslant)}$.*

(ii) *For two sets $R$ and $S$ of ramification points, $B_{(\leqslant, R)} = B_{(\leqslant, S)}$ if and only if $R = S$.*

(iii) *$B_{\leqslant} = B'_{\leqslant}$ if and only if for every $y \parallel z$ in $X$, $y \wedge z \notin X$.*

(iv) *If $Y \subseteq X$ is semi-linear then $B'_{(Y, \leqslant)} = B'_{(X, \leqslant)} | Y$, but the corresponding assertion is not necessarily true for $B_{\leqslant}$.*

*Proof.* (i) We sketched the proof that $B'$ is a B-relation above, though a direct verification is also possible, and for $B_{(\leqslant, R)}$ the proof is similar.

(iv) The fact that $B'_{(Y, \leqslant)} = B'_{(X, \leqslant)} | Y$ follows since for $y, z \in Y, y \leqslant z, y \parallel z$ hold in $(X, \leqslant)$ if and only if they hold in $Y$. By definition of $B'$, $B'_{(X, \leqslant)} = B_{(X^{++}, \leqslant)} | X$. Hence if there are $y \parallel z \in X$ such that $y \wedge z \in X, B_{(X, \leqslant)} \neq B'_{(X, \leqslant)} | X$, providing the desired counter-example.

We remark that $B'$ need not be a betweenness relation. This is immediate if $B' \neq B$ by Theorem 1·3(i). Example 2·2 shows that this can happen even when $B' = B$.

We would like to say that any B-relation $B$ such that $B'_{\leqslant} \subseteq B \subseteq B_{\leqslant}$ has the form $B_{(\leqslant, R)}$ for some set of ramification points $R$, but a counter-example to this is provided by letting $X = \{a, b, c, d\}$ with $a < \{b, c, d\}, B(a; b, c), B(a; c, b), B(a; b, d), B(a; d, b)$, and no other non-trivial relationships. For this $B$ is equal to neither $B'_{\leqslant} (= B_{(\leqslant, \varnothing)})$ or $B_{\leqslant} (= B_{(\leqslant, \{a\})})$.

*Compatibility of partial orders and betweenness relations*

Following an idea of P. M. Neumann, we say that a partial ordering $\leqslant$ on a set $X$ is *compatible* with a B-relation $B$ on $X$ if (i) $x \leqslant y \leqslant z \Rightarrow B(y; x, z)$, and (ii) $x \leqslant z \,\&\, B(y; x, z) \Rightarrow x \leqslant y \leqslant z$.

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THEOREM 2·4. *If the semi-linear order $(X, \leqslant)$ is compatible with the B-relation $(X, B)$, then $B'_{\leqslant} \subseteq B \subseteq B_{\leqslant}$. Hence $B_{\leqslant}$ is the unique B-relation on $X$ which is maximal subject to being compatible with $\leqslant$.*

*Proof.* By definition of 'compatible', if $x \leqslant y \leqslant z$ then $B(y; x, z)$. Suppose that $x \parallel y \leqslant z$. By semi-linearity there is $t \leqslant x, y$. Since $t \leqslant y \leqslant z, B(y; t, z)$, so by (B 3), $B(y; t, x)$ or $B(y; z, x)$. If $B(y; t, x)$ then by compatibility, $t \leqslant y \leqslant x$, contrary to $x \parallel y$. Hence $B(y; x, z)$ as required.

Conversely, suppose that $B(x; y, z)$. If $y$ and $z$ are comparable, then by compatibility, either $y \leqslant x \leqslant z$ or $z \leqslant x \leqslant y$, each giving $B_{\leqslant}(x; y, z)$. If $y \parallel z$, then by semi-linearity we may take $t \leqslant y, z$, and by (B 3), $B(x; y, t)$ or $B(x; z, t)$. By compatibility, $t \leqslant x \leqslant y$ or $t \leqslant x \leqslant z$. Since this applies to any $t$ we may choose below both $y$ and $z$, $y \wedge z \leqslant x \leqslant y$ or $y \wedge z \leqslant x \leqslant z$. The three possibilities are $z \parallel x \leqslant y, y \parallel x \leqslant z$, and $x = y \wedge z$, each of which gives $B_{\leqslant}(x; y, z)$.

We now consider recovery of a semi-linear order from a B-relation.

THEOREM 2·5. *If $B$ is a B-relation on $X$, and $a \in X$, then $\leqslant = \{(x, y) : x, y \in X \,\&\, B(x; a, y)\}$ is a rooted semi-linear order on $X$, and $B'_{\leqslant} \subseteq B \subseteq B_{\leqslant}$.*

*Proof.* This is given in [1] Theorem 17·2.

It would be more satisfactory if we could say that for $\leqslant$ as defined in the theorem, $B$ was equal to one or other of $B_{\leqslant}$ or $B'_{\leqslant}$, but that is not possible.

*Example* 2·6. Let $X = \{a, x, y, z, t, u, v\}$ where $a < x < z < t < u, x < y, t < v, y \parallel z$, and $u \parallel v$, and $Y = \{a, x, y, z, u, v\}$ with the induced ordering. If $B = B_{(X, \leqslant)} \mid Y$ then $B(x; y, z)$ but not $B(z; u, v)$, so $B$ is equal to neither $B_{\leqslant}$ nor $B'_{\leqslant}$, though $\leqslant$ is the semi-linear order defined from $B$.

We know from Theorem 1·3 that any B-relation on $X$ can be extended to a maximal B-relation on $X$ (a betweenness relation). Example 2·2 shows however that this cannot necessarily be done while retaining compatibility with a semi-linear order. For by Theorem 2·4, the maximal B-relation on $X$ compatible with $\leqslant$ is $B_{\leqslant}$, which is not a betweenness relation in that example.

For betweenness relations, the match with semi-linear orders works out rather better.

THEOREM 2·7. *If $B$ is a betweenness relation on $X$, and $\leqslant$ is the semi-linear order given in Theorem 2·5 (for some $a$), then $B = B_{\leqslant}$.*

*Proof.* By Theorem 2·5, $B \subseteq B_{\leqslant}$. Since $B$ is a betweenness relation on $X$, it is maximal by Theorem 1·3, and therefore $B = B_{\leqslant}$.

The following result, on patching a family of pairwise disjoint betweenness relations, is required in Section 3. We say that a subset $Y$ of a partially ordered set $(X, \leqslant)$ is *convex* if $y_1 \leqslant x \leqslant y_2 \,\&\, y_1, y_2 \in Y \Rightarrow x \in Y$, and the subset $Y$ of $(X, B)$ where $B$ is a B-relation is called *convex* if $y_1, y_2 \in Y \,\&\, B(x; y_1, y_2) \Rightarrow x \in Y$.

LEMMA 2·8. *Let $\mathscr{X}$ be a family of pairwise disjoint sets and for each $X \in \mathscr{X}$ let $B_X$ be a betweenness relation on $X$. Then there is a betweenness relation $B$ on $\bigcup \mathscr{X}$ extending each $B_X$, and for which each $X \in \mathscr{X}$ is convex.*

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*Proof.* For each $X$ pick $a_X \in X$ and let $C_X$ be a maximal chain of the semi-linear order $\leqslant_X$ on $X$ given by Theorem 2·5. Let $\leqslant$ be a linear ordering of $\mathscr{X}$ and partially order $\bigcup \mathscr{X}$ by $\preccurlyeq$:

$$x \preccurlyeq y \text{ if } x \leqslant_X y \text{ for some } X \in \mathscr{X} \text{ or } x \in X, y \in Y \text{ where } X < Y, \text{ and } x \in C_X.$$

It is easy to see that $\preccurlyeq$ is a semi-linear ordering of $\bigcup \mathscr{X}$ with each $X \in \mathscr{X}$ convex. Let $B = B_{\preccurlyeq}$.

$B$ extends $B_X$: if $B_X(x;y,z)$ then by Theorem 2·7, $y \leqslant x \leqslant z$ or $z \leqslant x \leqslant y$ or $y \parallel x \leqslant z$ or $z \parallel x \leqslant y$ or $y \parallel z \,\&\, x = y \wedge z$ in $\leqslant_X$. But since $\leqslant_X = \preccurlyeq | X$ and $X$ is convex in $(X, \preccurlyeq)$, the corresponding condition holds in $\preccurlyeq$, so that $B_{\preccurlyeq}(x;y,z)$.

$X$ is convex: suppose $B(x;y,z)$ with $y,z \in X$. If $y \preccurlyeq x \preccurlyeq z$ then as $X$ is a convex subset of $\bigcup \mathscr{X}$ as a partial ordering, $x \in X$ and similarly if $z \preccurlyeq x \preccurlyeq y$. If $y \parallel x \preccurlyeq z$ and $x \notin X$ then $x \in C_Y$ for some $Y < X$. But then also $x \preccurlyeq y$, contrary to assumption. Similarly if $z \parallel x \preccurlyeq y$. If $y \parallel z \,\&\, x = y \wedge z$ then $x \in X$ follows from $\leqslant_X$ semi-linear and $X$ convex as a partial ordering.

$B$ a betweenness relation: we appeal to Lemma 2·1. Suppose $(\forall z)(z \prec x \Rightarrow z \prec y)$ with the object of showing that $x \wedge y \in \bigcup \mathscr{X}$. If $x, y \in X \in \mathscr{X}$ then $(\forall z)(z < x \Rightarrow z < y)$ holds in $X$, and as $B_X$ is a betweenness relation and $B_X = B_{\leqslant_X}$, $x \wedge y \in X$. Otherwise $x \in X, y \in Y$ with $X \neq Y$. If $Y < X$ then for every $z \in C_Y$, $z \prec x$ and hence $z \prec y$. But this means that $z <_Y y$, contrary to the maximality of $C_Y$. Thus $X < Y$. If $x \in C_X$ then $x \preccurlyeq y$ so $x \wedge y = x \in X$. Otherwise as $C_X$ is a maximal chain in $(X, \leqslant_X)$ there is $t \in C_X$ with $x \not\leqslant t$. Let $z <_X x$ be arbitrary. Then $z \prec x$ so by assumption $z \prec y$. As $t \prec y$ and $\preccurlyeq$ is semi-linear, $t \preccurlyeq z$ or $z \prec t$. As $z, t \in X$, $t \leqslant_X z$ or $z <_X t$. But $t \leqslant_X z$ and $z \prec x$ give $t \leqslant_X x$, contrary to its choice. Hence $z <_X t$. As $z$ was arbitrary, and $B_X$ is a betweenness relation, $x \wedge t \in X$. But clearly $x \wedge t = x \wedge y$ so also $x \wedge y \in X$.

## 3. *Cycle-free partial orders*

Let us call a 4-element partial ordering $\{a,b,c,d\}$ with $a \leqslant b \leqslant d \,\&\, a \leqslant c \leqslant d \,\&\, b \parallel c$ a *diamond* and a $2n$-element partial ordering $\{a_i : 0 \leqslant i < 2n\}$ $(n \geqslant 3)$ with $a_{2i} \leqslant a_{2i \pm 1}$ the only non-trivial relationships, and where the indices are taken modulo $2n$, a *$2n$-crown*. By an *embedding* of partial orderings $(X, \leqslant) \to (Y, \leqslant)$ I understand a 1-1 map $f: X \to Y$ such that $x_1 \leqslant x_2 \Leftrightarrow f(x_1) \leqslant f(x_2)$. A *strong embedding* is an embedding which also preserves (finite) suprema and infima, that is in addition we require $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ whenever $x_1 \vee x_2$ exists in $X$, and similarly for $\wedge$.

THEOREM 3·1. *The following are equivalent:*

(i) $(X, \leqslant)$ *is a cycle-free partial order,*

(ii) $X^+$ *embeds no diamond or crown,*

(iii) $X^+$ *strongly embeds no diamond or crown.*

*Proof.* $\neg$ (ii) $\Rightarrow$ $\neg$ (iii): suppose $\{a,b,c,d\}$ is a diamond in $(X^+, \leqslant)$, ordered as above. Then $\{b \wedge c, b, c, b \vee c\}$ is a diamond strongly embedded in $(X^+, \leqslant)$. Similarly for crowns. (This technique is described in [5], Lemma 2·4·4.)

$\neg$ (iii) $\Rightarrow$ $\neg$ (i): suppose $\{a,b,c,d\}$ is a diamond in $(X^+, \leqslant)$. Then there are (at least) two maximal chains in $X^D$ from $a$ to $d$, one passing through $b$, and one through $c$, so $X^+$ is not cycle-free, and by [5] Lemma 2·3·9, nor is $X$. Similarly for crowns.

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$\neg$ (i) $\Rightarrow \neg$ (ii): if $X$ is not a CFPO there are $a, b$ such that there are two paths from $a$ to $b$ in $X^D$. Concatenating these paths and refining as necessary provides a diamond or crown in $X^+$.

THEOREM 3·2. (i) *The class of cycle-free partial orders is axiomatizable but not finitely axiomatizable.*

(ii) *The class of connected cycle-free partial orders is not axiomatizable.*

*Proof.* (i) According to Theorem 3·1 it suffices to show how to express the existence of a diamond or $2n$-crown in $X^+$. We have to show how to refer to the ramification points by pairs of members of $X$. The assertion that there is a diamond or $2n$-crown may be written

$$\exists x_0 \exists x_1 \ldots \exists x_{2n-1} \Phi,$$

where $x_i$ range over $X^+$, and $\Phi$ is a conjunction of inequalities and incomparabilities between the $x_i$, and for the diamond case, $n = 2$. Let us fix a particular combination of choices of $x_i$ in $X$, or as upward or downward ramification points. We show that the above formula is equivalent to one in which we quantify over $X$, and then take the conjunction over all such possibilities.

If $x_i \in X$ we leave it unchanged; if $x_i$ is an upward or downward ramification point we replace $\exists x_i$ by $\exists y_i \exists z_i$ and have to modify the formula to express the condition that $x_i = y_i \wedge z_i$ or $x_i = y_i \vee z_i$. It suffices to make the following translations:

$$a \leqslant b \vee c \Leftrightarrow (\forall x)(b \leqslant x \,\&\, c \leqslant x \rightarrow a \leqslant x),$$
$$a \geqslant b \vee c \Leftrightarrow b \leqslant a \,\&\, c \leqslant a,$$
$$a \leqslant b \wedge c \Leftrightarrow a \leqslant b \,\&\, a \leqslant c,$$
$$a \geqslant b \wedge c \Leftrightarrow (\forall x)(b \geqslant x \,\&\, c \geqslant x \rightarrow a \geqslant x),$$

and formulae expressing $a \vee b, a \wedge b \leqslant (\geqslant) c \vee d, c \wedge d$ are obtained by combining these. For instance,

$$a \vee b \leqslant c \vee d \Leftrightarrow (\forall x)(c \leqslant x \,\&\, d \leqslant x \rightarrow a \leqslant x) \,\&\, (\forall x)(c \leqslant x \,\&\, d \leqslant x \rightarrow b \leqslant x),$$
$$a \wedge b \leqslant c \vee d \Leftrightarrow (\forall x)(\forall y)(c \leqslant x \,\&\, d \leqslant x \,\&\, y \leqslant a \,\&\, y \leqslant b \rightarrow y \leqslant x).$$

Thus the class of cycle-free partial orders is axiomatized by the axioms $T_{PO}$ for partial orderings together with $\Gamma = \{\neg \phi_n : n \geqslant 2\}$ where $\phi_2$ says there is a diamond, and for $n \geqslant 3$, $\phi_n$ says that there is a $2n$-crown in $X^+$. It is not finitely axiomatizable since for any $N$ there is a partial ordering satisfying $\{\neg \phi_n : 2 \leqslant n < N\}$, namely a $2N$-crown, which is *not* a CFPO.

(ii) This is an easy consequence of compactness. Suppose that $T$ is a set of axioms for connected CFPOs. Adjoin new constant symbols $\underline{c}_1, \underline{c}_2$ and sentences $\psi_n$ asserting that $\underline{c}_1, \underline{c}_2$ are not joined by any path having less than $n$ alternations. Then any finite subset of $T' = T \cup \{\psi_n : n \in \omega\}$ has a model, and so $T'$ has too. But this cannot be connected.

J. Schmerl has observed that the class of connected CFPOs is not even closed under elementary equivalence, since the 'alternating' CFPO 'ALT' with domain $\mathbf{Z}$ and with $2n \leqslant 2n \pm 1$ and no other non-trivial relations is a connected CFPO, but two disjoint copies of ALT, which is elementarily equivalent to ALT, is a CFPO which is not connected.

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*Remark* 3·3. The trick used above is actually only needed for diamonds, though using it in general perhaps makes things a bit clearer. It is needed for diamonds in view of the following example of Neumann's. Let $X = \{a, b, c, d, e, f\}$ with $a < d, e$; $b < d, e, f$; and $c < e, f$. Then in the completion there are two new points $x$ and $y$ say, and $a, b < x < d, e$ and $b, c < y < e, f$. Hence $\{b, x, y, e\}$ is a diamond in $X^+$, but $X$ contains no diamond since all its maximal chains have length 2.

THEOREM 3·4. *For any partial ordering $(X, \leqslant)$, if $X^+$ embeds a $2n$-crown, then either $X$ embeds a $2n$-crown, or $X^+$ embeds a diamond or a $2m$-crown for some $m < n$. Hence $(X, \leqslant)$ is a CFPO if and only if $X^+$ embeds no diamond, and $X$ embeds no crown.*

*Proof.* Suppose that $\{a_i : 0 \leqslant i < 2n\}$ is a $2n$-crown strongly embedded in $X^+$, with $a_{2i} \leqslant a_{2i \pm 1}$ for each $i$ (with indices taken modulo $2n$). Pick $b_{2i} \leqslant a_{2i}$ and $b_{2i+1} \geqslant a_{2i+1}$ for each $i$, with $b_i \in X$. If distinct $b_{2i}$ are comparable, or distinct $b_{2i+1}$ are, then $X^+$ contains either a diamond, or a smaller crown. Otherwise we have a $2n$-crown in $X$.

Seeking a further improvement of this result, we next show that the class of CFPOs is axiomatizable by a set of universal sentences. This then implies that a substructure of a CFPO is a CFPO. By standard model theory it is actually *equivalent* to this statement, so if we proved preservation under substructures, this would give the result automatically. It seems easier however to give the axiomatization by universal sentences explicitly.

LEMMA 3·5. *The statement 'no diamond is embeddable in $(X^+, \leqslant)$' is expressible as a universal sentence in the language of partial orderings (that is with variables ranging over $X$).*

*Proof.* We observe that there are finitely many configurations (six in fact) such that $(X^+, \leqslant)$ embeds a diamond if and only if $(X, \leqslant)$ embeds at least one of these configurations. The result then follows since for each configuration, the assertion that $(X, \leqslant)$ does not embed it can be expressed as a universal sentence.

First note that if $(X^+, \leqslant)$ embeds a diamond $x \leqslant y, z \leqslant w, y \parallel z$ then it embeds a diamond in which $x, w \in X$. The six configurations then correspond to these cases: (1) $y, z \in X$; (2) $y \in X, z \in \uparrow \mathrm{Ram}\,(X)$ (an upward ramification point); (3) $y \in X, z \in \downarrow \mathrm{Ram}\,(X)$ (downward); (4) $y, z \in \uparrow \mathrm{Ram}\,(X)$; (5) $y, z \in \downarrow \mathrm{Ram}\,(X)$; and (6) $y \in \uparrow \mathrm{Ram}\,(X), z \in \downarrow \mathrm{Ram}\,(X)$. We do (1) and (2) as typical cases.

(1) This is just the original diamond. The required universal sentence here is

$$\forall x \forall y \forall z \forall w \neg (x \leqslant y \,\&\, x \leqslant z \,\&\, y \leqslant w \,\&\, z \leqslant w \,\&\, \neg (y \leqslant z) \,\&\, \neg (z \leqslant y)).$$

(2) Suppose this happens and (1) does not. Let $z = r \wedge s, r, s \in X$. Then $w \wedge r \wedge s = z$, so if $w \wedge r, w \wedge s \geqslant y$ then $z = (w \wedge r) \wedge (w \wedge s) \geqslant y$, contrary to hypothesis. Thus we assume that $w \wedge r \not\geqslant y$. But $w \wedge r \geqslant z$, so also $w \wedge r \not\leqslant y$. So we may replace $z$ by $w \wedge r$. By [5] Lemma 2·4·7, $(y \wedge z, z]$ contains a member $t$ of $X \cup \downarrow \mathrm{Ram}\,(X)$. Since we are assuming that (1) does not occur, $t \in \downarrow \mathrm{Ram}\,(X)$. Since $t \in (y \wedge z, z], t \not\leqslant y$ so if $t = u \vee v$, either $x \vee u \not\leqslant y$ or $x \vee v \not\leqslant y$. So we assume the former and replace $t$ by $x \vee u$. So we have

$$x \leqslant y \,\&\, x \leqslant r \,\&\, y \leqslant w \,\&\, u \leqslant r \,\&\, u \leqslant w \,\&\, x \parallel u \,\&\, y \parallel u \,\&\, y \parallel r \,\&\, w \parallel r.$$

The desired universal sentence is the universal closure of the negation of this.

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THEOREM 3·6. *The class of CFPOs can be axiomatized by a set of universal sentences. Hence any substructure of a CFPO is a CFPO, and a partial ordering is a CFPO if and only if every finite subset is.*

*Proof.* This follows from Theorem 3·4 and Lemma 3·5 since the assertion that $X$ embeds no $2n$-crown is clearly universal for each $n$.

We now move on to consider compatibility with regard to CFPOs rather than semi-linear orders. Ideally as suggested in [1] we would like to say that $(X, \leqslant)$ is a CFPO if and only if it is compatible with a B-relation on $X$, or alternatively, the same with 'betweenness relation' in place of 'B-relation'. Neither of these works as it stands. The former fails since in a partial ordering where all chains have length at most 2, $\leqslant$ is automatically (and vacuously) compatible with the trivial B-relation. Although as described in [5] many interesting CFPOs do only have chains of length at most 2, there are many such partial orderings which we would certainly *not* want to regard as CFPOs under any definition (such as the $2n$-crowns for a start). The latter fails since although we wish to regard all semi-linear orders as 'cycle-free', the results in Section 2 show that there is not always a betweenness relation compatible with a given semi-linear order. The best we can do, so far as obtaining a characterization of CFPOs using this idea is once again to move to the structure with ramification points included. We first give two lemmas to aid the proof. Note that for a B-relation $B$ which is compatible with a partial ordering $(X, \leqslant)$ we may use interval notations unambiguously for comparable elements (as follows at once from the compatibility hypotheses). In addition, to avoid having to say which of $a$ and $b$ is the greater, we shall use $(a, b)$ to stand for $\{x: a < x < b\}$ if $a < b$ and $\{x: b < x < a\}$ if $b < a$.

LEMMA 3·7. *Suppose that $B$ is a B-relation compatible with the partial ordering $(X^+, \leqslant)$. Let $a, b \in X^+$, and $\langle c_0, c_1, \ldots, c_n \rangle$ be a connecting set from $a$ to $b$ in $X^+$ such that the members of $\{(c_i, c_{i+1}): 0 \leqslant i < n\} \cup \{\{c_i\}: 0 \leqslant i \leqslant n\}$ are pairwise disjoint. Then*

$$\bigcup_{0 \leqslant i < n} (c_i, c_{i+1}) \subseteq [a, b] \subseteq \bigcup_{0 \leqslant i < n} [c_i, c_{i+1}].$$

*Proof.* By (B 3), for any $x, y, z, [x, y] \subseteq [x, z] \cup [y, z]$, and the right-hand inclusion follows from this by induction.

We also use induction to establish the left-hand inclusion. If $n = 1$ it is immediate. Assuming the result for $n-1$ we have $\bigcup_{0 \leqslant i < n-1}(c_i, c_{i+1}) \subseteq [a, c_{n-1}]$. The disjointness hypothesis ensures that $[a, c_{n-1}] \cap [c_{n-1}, b] = \{c_{n-1}\}$. Now using (B 3) again,

$$[a, c_{n-1}] \subseteq [a, b] \cup [c_{n-1}, b]$$

and hence $[a, c_{n-1}] \subseteq [a, b] \cup \{c_{n-1}\}$. Similarly $[c_{n-1}, b] \subseteq [a, b] \cup \{c_{n-1}\}$. Therefore $\bigcup_{0 \leqslant i < n}(c_i, c_{i+1}) \subseteq [a, c_{n-1}] \cup [c_{n-1}, b] \subseteq [a, b] \cup \{c_{n-1}\}$, and since by disjointness again, $c_{n-1} \notin \bigcup_{0 \leqslant i < n}(c_i, c_{i+1})$, the result follows.

We remark that this lemma is a generalization of (part of) Theorem 2·4.

LEMMA 3·8. *Under the same hypotheses as Lemma 3·7, except that $B$ is now assumed to be a betweenness relation,*

$$[a, b] = \bigcup_{0 \leqslant i < n} [c_i, c_{i+1}].$$

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*Proof.* We just have to show that $c_i \in [a, b]$ for $0 < i < n$. But this is immediate from the fact that $[a, c_i] \cap [c_i, b] = \{c_i\}$.

THEOREM 3·9. *A partially ordered set $(X, \leqslant)$ is a cycle-free partial order if and only if there is a betweenness relation $B$ on $(X^+, \leqslant)$ compatible with $\leqslant$.*

*Proof.* If $(X, \leqslant)$ is a connected CFPO, then we consider the natural ternary relation $B$ on $X^+$ given by $B(x; y, z)$ if $x$ lies on the path joining $y$ and $z$. First we sketch in outline why this is a B-relation. That (B 1) holds is immediate since as sets, the paths from $y$ to $z$ and from $z$ to $y$ are equal. For (B 2) note that if $B(x; y, z)$ then the path from $x$ to $z$ is a subset of that from $y$ to $z$, and it is a proper subset unless $x = y$. To verify (B 3) one considers the point $u$ at which the path from $t$ to $x$ joins that from $y$ to $z$ (which contains $x$). If $u$ lies between $y$ and $x$ then $B(x; z, t)$, and if it lies between $x$ and $z$, then $B(x; y, t)$. Looking at (B 4), suppose that $[x, y] \cap [x, z] = \{x\}$. Thus the unique paths in $X^+$ from $x$ to $y$ and from $x$ to $z$ only intersect at $x$. Let the segments of the paths next to $x$ be $[a, x]$ and $[b, x]$ respectively. If $a \leqslant x \leqslant b$ or $b \leqslant x \leqslant a$ then the path joining $y$ to $z$ in $X^+$ is equal to $[x, y] \cup [x, z]$, which contains $x$, giving $B(x; y, z)$. If $x \leqslant a$ and $x \leqslant b$, then $x \leqslant a \wedge b \leqslant a, b$ and as $[x, y] \cap [x, z] = \{x\}$, and $a \wedge b \in X^+$, $x = a \wedge b$. Hence the path from $y$ to $z$ is again equal to $[x, y] \cup [x, z]$ and $B(x; y, z)$ holds.

Finally to see that $B$ is compatible with $\leqslant$. If $x \leqslant y \leqslant z$ then the path from $x$ to $z$ is just the interval $\{t : x \leqslant t \leqslant z\}$, and $y$ lies in this. If $B(t; x, z)$ then $t$ must lie in this interval, and hence $x \leqslant t \leqslant z$.

Now consider a general, not necessarily connected, CFPO $(X, \leqslant)$, with set $\mathscr{C}$ of connected components. For each $C \in \mathscr{C}$ let $B_C$ be the betweenness relation given as above. Then by Lemma 2·8 there is a betweenness relation $B$ on $X^+$ extending each $B_C$ and for which each $C^+$ is convex. Since $B_C$ is compatible with $\leqslant$ on $C^+$, convexity readily implies that $B$ is compatible with $\leqslant$ on $X^+$.

Conversely, suppose that $B$ is a betweenness relation on $(X^+, \leqslant)$ compatible with $\leqslant$. It is immediate from compatibility that $X^+$ embeds no diamonds. We have to show that it embeds no crowns. Suppose that $\{a_i : 0 \leqslant i < 2n\}$ forms a $2n$-crown for $n \geqslant 3$ with $a_{2i} < a_{2i \pm 1}$ the only non-trivial relations. By Theorem 3·1 we suppose that the crown is strongly embedded in $X^+$. This means that $a_{2i} = a_{2i-1} \wedge a_{2i+1}$ and $a_{2i+1} = a_{2i} \vee a_{2i+2}$. Then $\langle a_0, a_1, \ldots, a_n \rangle$ and $\langle a_0, a_{2n-1}, a_{2n-2}, \ldots, a_n \rangle$ form two connecting sets from $a_0$ to $a_n$ fulfilling the hypotheses of Lemma 3·8. Hence $[a_0, a_n] = \bigcup_{0 \leqslant i < n} [a_i, a_{i+1}] = \bigcup_{n \leqslant i < 2n} [a_i, a_{i+1}]$. Thus $a_1 \in [a_i, a_{i+1}]$ for some $i$ with $n \leqslant i < 2n$, contradiction.

THEOREM 3·10. *The class of cycle-free partial orders is a pseudo-elementary class.*

*Proof.* We expand the language of partial orderings by a 6-place predicate $A$. The intention is that $A(u, v, w, x, y, z)$ will hold if and only if $B(r; s, t)$, where $r, s, t$ are the points determined by the pairs $(u, v), (w, x), (y, z)$ respectively. The pair $(u, v)$ (for instance) determines $r$ according to one of three cases: (i) $u = v = r$; (ii) $u \parallel v, u, v$ have a common lower bound, and $r = u \wedge v$; and (iii) $u \parallel v, u, v$ have a common upper bound, and $r = u \vee v$. (It will be automatic that (ii) and (iii) cannot both hold, though we could explicitly stipulate this if we wished by a first order formula.) We take as our single sentence $\theta$ axiomatizing the class of CFPOs in the expanded language one which is equivalent to the conjunction of the universal closures of formulae about $A$

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expressing (B 1)–(B 4), together with the compatibility conditions. Let us write $L(w,x,y,z)$ as an abbreviation for the following formula, which stands for 'the point of $X^+$ coded by $(w,x)$ is $\leqslant$ the point of $X^+$ coded by $(y,z)$':

$$[w = x \,\&\, y = z \,\&\, w \leqslant y]$$

or

$$[w = x \,\&\, (y \parallel z \,\&\, (w \leqslant y \wedge z \text{ or } w \leqslant y \vee z))]$$

or

$$[y = z \,\&\, (w \parallel x \,\&\, (w \wedge x \leqslant y \text{ or } w \vee x \leqslant y))]$$

or

$$[w \parallel x \,\&\, y \parallel z \,\&\, (w \wedge x \leqslant y \wedge z \text{ or } w \wedge x \leqslant y \vee z \text{ or } w \vee x \leqslant y \wedge z \text{ or } w \vee x \leqslant y \vee z)].$$

Here it is understood that $w \wedge x \leqslant y \vee z$ really stands for '$w$ and $x$ have a common lower bound and $y$ and $z$ have a common upper bound', in addition to what was written, and then this part of the formula is re-expressed without reference to $\wedge$ and $\vee$ by the method described in the proof of Theorem 3·2(i).

The formulae expressing what is required are then as follows:

(B 1) $$A(u,v,w,x,y,z) \to A(u,v,y,z,w,x),$$

(B 2) $$A(u,v,w,x,y,z) \,\&\, A(w,x,u,v,p,q) \leftrightarrow L(u,v,w,x) \,\&\, L(w,x,u,v),$$

(B 3) $$A(u,v,w,x,y,z) \to A(u,v,x,w,p,q) \text{ or } A(u,v,p,q,w,x),$$

(B 4) $$\neg A(u,v,w,x,y,z) \to (\exists p)(\exists q)(A(p,q,u,v,w,x) \,\&\, A(p,q,u,v,y,z) \,\&\, \neg L(p,q,u,v) \,\&\, \neg L(u,v,p,q)),$$

(where we have used $L$ to help express equality in (B 2) and (B 4)),

$$L(w,x,u,v) \,\&\, L(u,v,y,z) \to A(u,v,w,x,y,z),$$

$$L(w,x,y,z) \,\&\, A(u,v,w,x,y,z) \to L(w,x,u,v) \,\&\, L(u,v,y,z).$$

If $(X, \leqslant)$ is a CFPO then by Theorem 3·9 there is a betweenness relation $B$ on $X^+$ compatible with $\leqslant$, and if $A$ is determined from $B$ by the above recipe, then $\theta$ holds in the resulting structure. Conversely if $(X, \leqslant, A)$ is a model for $\theta$ then $A$ determines a betweenness relation on $X^+$ compatible with $\leqslant$, so by Theorem 3·9, $(X, \leqslant)$ is a CFPO.

Given appropriate assumptions on $X$ (unfortunately rather strong at present), Theorem 3·9 can be improved as follows.

THEOREM 3·11. *Suppose $(X, \leqslant)$ is a partial ordering for which $X^+$ is dense and $X$ is dense in $X^+$. Then $(X, \leqslant)$ is a CFPO if and only if there is a B-relation compatible with $\leqslant$.*

*Proof.* If $X$ is a CFPO then the restriction of the natural betweenness relation on $X^+$ to $X$ is a B-relation compatible with $\leqslant$.

Conversely suppose $B$ is compatible with $\leqslant$, and that $\{a_i : 0 \leqslant i < 2n\}$ forms a $2n$-crown strongly embedded in $X^+$. As $X$ is dense in $X^+$, which is itself dense, we may choose $x_i \in (a_i, a_{i+1}) \cap X$. We show that for each $i, B(x_i; x_{i-1}, x_{i+1})$. Take $i = 1$ for ease, and let $y_0 \leqslant a_0, y_1 \geqslant a_1, y_2 \leqslant a_2, y_3 \geqslant a_3, y_i \in X$. Then $x_1 \in [y_2, y_1] \subseteq [y_2, x_2] \cup [x_2, x_0] \cup [x_0, y_1]$ by (B 3). But by compatibility, $x_1 \notin [y_2, x_2] \cup [x_0, y_1]$, so $x_1 \in [x_2, x_0]$,

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giving $B(x_1; x_0, x_2)$ as desired. Now using Lemma 1·1(iv) repeatedly and the fact that the $x_i$ are clearly distinct, we find that $B(x_j; x_i, x_k)$ whenever $i < j < k$. Hence $B(x_1; x_0, x_{2n})$. But $x_0 = x_{2n}$ so $B(x_1; x_0, x_0)$ and by Lemma 1·1(v), $x_0 = x_1$, a contradiction. We argue similarly if $X^+$ embeds a diamond.

J. Schmerl has observed that the ideas of this section can be generalized in the following way.

THEOREM 3·12. *Let $(Q, \leqslant)$ be a finite partially ordered set with $|Q| = n$. Then there is a set $\mathcal{Q}$ of partially ordered sets with at most $2n^2$ points, such that for any partially ordered set $(X, \leqslant)$ the following are equivalent:*

(i) *$Q$ is embeddable in $X^D$,*

(ii) *there is $Y \in \mathcal{Q}$ which is embeddable in $X$.*

*Proof.* Let $Q = \{q_1, q_2, \ldots, q_n\}$, and let $\mathcal{Q}$ be the set of partially ordered sets which may be written as the disjoint union of four sets $A$, $B$, $C$, and $D$ where $A = \{a_{ij} : q_i \not\leqslant q_j\}, B = \{b_{ij} : q_i \not\leqslant q_j\}, C = \{c_i : 1 \leqslant i \leqslant n\}, D = \{d_i : 1 \leqslant i \leqslant n\}$, and such that $q_i \not\leqslant q_j \Rightarrow a_{ij} \not\leqslant b_{ij}$, and $q_i \leqslant q_s \Rightarrow (a_{ij} \leqslant b_{rs} \,\&\, a_{ij} \leqslant d_s \,\&\, c_i \leqslant b_{rs} \,\&\, c_i \leqslant d_s)$. Thus the relations between (certain) elements are specified, but not all, which is why we have a *set* of partial orders, rather than just one.

Now consider an arbitrary partially ordered set $(X, \leqslant)$.

(i) $\Rightarrow$ (ii): suppose that $f$ is an embedding of $Q$ into $X^D$, and as above, view $X^D$ as the set of Dedekind ideals of $X$. Pick $c_i \in f(q_i)$ and $d_i \in \bigvee f(q_i)$ (where $\bigvee$ is calculated in $X$); and if $q_i \not\leqslant q_j$, which means that $f(q_i) \not\subseteq f(q_j)$, let $a_{ij} \in f(q_i) - f(q_j)$, and choose $b_{ij} \in \bigvee f(q_j)$ with $a_{ij} \not\leqslant b_{ij}$. This determines a partially ordered set $Y \subseteq X$ with domain $A \cup B \cup C \cup D$ according to the above notation, and it satisfies the required conditions, so lies in $\mathcal{Q}$.

(ii) $\Rightarrow$ (i): suppose that $Y \subseteq X$ with $Y \in \mathcal{Q}$. We embed $Q$ into $X^D$ by means of the map $f$ given by $f(q_i) = \bigwedge X_i$ where $X_i = \{b_{rs} : q_i \leqslant q_s\} \cup \{d_s : q_i \leqslant q_s\}$. Observe that $X_i \neq \varnothing$, (because $d_i \in X_i$), and $\bigwedge X_i \neq \varnothing$, (because $c_i \in \bigwedge X_i$). To see that $f$ is an embedding we suppose first that $q_i \leqslant q_j$. Then clearly $X_i \supseteq X_j$ and so $f(q_i) = \bigwedge X_i \leqslant \bigwedge X_j = f(q_j)$. If $q_i \not\leqslant q_j$ then $a_{ij} \in \bigwedge X_i$, but $a_{ij} \notin \bigwedge X_j$ since $a_{ij} \not\leqslant b_{ij} \in X_j$, which shows that $f(q_i) \not\leqslant f(q_j)$.

We conclude by remarking how some of the results of Section 3 may be derived from Theorem 3·12. Theorem 3·2(i) follows immediately, since the existence of a diamond or $2n$-crown in $X^D$ is equivalent to the embeddability into $X$ of one of a finite list of finite partial orderings. Moreover the axiomatization is actually already universal (since saying that a particular partial ordering *can* be embedded is clearly existential), so that Theorem 3·6 also follows. Finally, since the proof of Theorem 3·12 is sufficiently effective, it is also possible to derive Theorem 3·10.

## REFERENCES

[1] S. A. ADELEKE and P. M. NEUMANN. Relations related to betweenness: their structure and automorphisms, in preparation.

[2] M. DROSTE. *Structure of partially ordered sets with transitive automorphism groups*. Memoirs of the American Math. Soc. **57** (1985), No. 334.

[3] H. M. MACNEILLE. Partially ordered sets. *Trans. Amer. Math. Soc.* **42** (1937), 416–460.

[4] M. RUBIN. *The reconstruction of trees from their automorphism groups*. Contemporary Mathematics **151** (American Math. Soc., 1993).

[5] R. WARREN. *The structure of k-CS-transitive cycle-free partial orders*, to appear in the Memoirs of the American Math. Soc.

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