Math. Proc. Camb. Phil. Soc. (1986), **99**, 179
Printed in Great Britain

Fronsdal *-quantization and Fell inducing

By M. A. HENNINGS
St John's College, Oxford, OX1 3JP

(*Received* 6 *March* 1985)

Abstract

The process of Fronsdal inducing is discussed as a method of quantizing physical systems, and is seen to reduce to a technique for obtaining representations of *-algebras.

It is asked whether this above process is at all successful, and whether it is at all related to the representation inducing methods of Fell. The two inducing techniques are shown to be equivalent for algebras satisfying three specified properties, and it is also shown that all the strictly irreducible representations of such algebras can be found by these methods.

Examples are produced whereby it is shown that algebras of the type required by the theory are of physical interest.

1. *Introduction*

Quantization theory can be viewed as the procedure of identifying a collection $\mathscr{A}$ of classical observables of a physical system, represented by C^∞ functions on some symplectic manifold (phase space) M, and then obtaining a physically meaningful bijection from $\mathscr{A}$ to a space of (unbounded) linear maps on some pre-Hilbert space $\mathscr{H}_0$.

It is well known that attempting to satisfy the Dirac quantum condition, namely identifying $\mathscr{A}$ as a Lie subalgebra of $C^\infty(M)$ under the natural Poisson bracket $\{\,,\}$ and then looking for a Lie algebra homomorphism from $\mathscr{A}$ to End $(\mathscr{H}_0)$, is inadequate. We cannot make classical and quantum mechanics correspond completely to each other, or at least not in their present forms. Moyal [9] showed how we should attempt to alleviate this problem. By considering quantum mechanics as a statistical system, he found that, when $M = \mathbb{R}^{2n}$, the collection of classical observables $\mathscr{A}$ should be regarded as a Lie algebra with respect to the *Moyal bracket*

$$\{f*g\} = \frac{2}{\hbar}\left(\sin\frac{\hbar}{2}\{\,,\}\right)(f,g) \quad (f,g\in\mathscr{A}), \tag{1}$$

in which case a physically meaningful Lie algebra homomorphism between $\mathscr{A}$ and End $(\mathscr{H}_0)$ is obtainable. The Moyal bracket is a deformation of the Poisson bracket in the sense that

$$\{f*g\} = \{f,g\} + O(\hbar^2) \quad (f,g\in\mathscr{A}), \tag{2}$$

and so it is tempting to state (and we do) that traditional classical mechanics using the Poisson bracket is merely a (macroscopically very good) approximation for the true state of affairs.

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

Not only this, but it is also possible to regard $\mathscr{A}$ as an algebra with the *Moyal product* for multiplication, where the map $(f,g)\mapsto f*g$ satisfies

$$f*g = f\cdot g + O(\hbar) \quad (f,g\in\mathscr{A}), \tag{3}$$

$$f*g - g*f = i\hbar\{f*g\} \quad (f,g\in\mathscr{A}), \tag{4}$$

so the algebra structure of $\mathscr{A}$ is a deformation of the natural algebra structure of $C^\infty(M)$ given by pointwise multiplication, and it also defines the correct Lie algebra structure on $\mathscr{A}$. Also the natural involution of $\mathscr{A}$ given by complex conjugation turns $\mathscr{A}$ into a *-algebra, and the remainder of the process of quantization can be reduced to the problem of finding a pre-*-representation of the *-algebra $\mathscr{A}$.

Bayen *et al.*[1] and Fronsdal[3] have considered the problem of extending this theory. They discuss methods whereby a Lie subalgebra $\mathscr{A}_0$ of $C^\infty(M)$ contained in $\mathscr{A}$ can be identified, called the algebra of distinguished observables, and how an associative multiplication operation $*$ can be defined on $\mathscr{A}$ such that

$$a*f - f*a = i\hbar\{a,f\} \quad (a\in\mathscr{A}_0; f\in\mathscr{A}). \tag{5}$$

$\mathscr{A}_0$ is chosen to denote the natural physical invariants and geometrical symmetries of the system – $\mathscr{A}_0$ is generated by the polynomials $1, x_1, \ldots, x_{2n}$ in the case $M = \mathbb{R}^{2n}$. Any such multiplication on $\mathscr{A}$ is called an *$\mathscr{A}_0$-invariant Fronsdal *-product.*

We shall not be concerned in this paper with the question of whether such *-products can be defined; we shall assume from henceforth that we have an algebra of physical observables – the algebra structure being given by an invariant Fronsdal *-product of some sort – and that a meaningful involution can be defined on $\mathscr{A}$ turning it into a *-algebra. Hence the problem of quantizing this system degenerates into that of obtaining a pre-*-representation of the *-algebra $\mathscr{A}$.

Fronsdal and co-workers have also considered this problem, and have postulated, at least formally, a general method of approach so that the observable $f\in\mathscr{A}$ is represented by the restriction of the operator $f*$ to some invariant subspace of $\mathscr{A}$. The main aim of this paper will be to consider under what circumstances this approach is effective, and also how well it relates to the more standard technique of obtaining *-representations of $\mathscr{A}$ by Fell inducing [2]. Before we start the main body of the theory, the next two sections will be devoted to establishing the notations and procedures to be used in this work. All linear spaces and algebras will be assumed complex.

The contents of this paper form a part of the author's D.Phil. thesis, and many thanks are due to Dr K. C. Hannabuss for his support and advice over the years.

2. *Fronsdal *-quantization and Fronsdal inducing*

We assume now that we have an associative *-algebra $\mathscr{A}$ of observables with respect to an invariant *-product and some involution *. For simplicity we shall omit the star when writing products, so the algebra structure of $\mathscr{A}$ will be denoted by $(a,b)\mapsto ab$.

The process of Fronsdal and coworkers is very simple in essence. We define a *-*polarization* X to be a subspace of $\mathscr{A}$ which is stable and irreducible under the algebra of endomorphisms given by $f\mapsto af$ for all $a\in\mathscr{A}$. In other words, a *-polarization

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

is a minimal left ideal of $\mathscr{A}$. If a *-polarization X can be found, then we have found a representation of the algebra $\mathscr{A}$ given by R^0: $\mathscr{A} \to \mathrm{End}(X)$, where

$$(R^0 a)f = af \quad (a \in \mathscr{A}; f \in X). \tag{6}$$

This process is known as *Fronsdal *-quantization*, if it can be done. Of course, comparatively few algebras possess minimal ideals, and this at first seems to pose a problem. However, it is shown in the author's thesis [4] how physical considerations frequently generate algebras $\mathscr{A}$ that do possess minimal ideals, and so in the sequel we shall always assume that *-polarizations can be found for $\mathscr{A}$.

Given a *-polarization X of $\mathscr{A}$, can we define an inner product structure $\langle\ ,\ \rangle_X$ on X such that the map R^0: $\mathscr{A} \to \mathrm{End}(X)$ becomes a pre-*-representation of the *-algebra $\mathscr{A}$? If we can, we call R^0 the pre-*-representation of $\mathscr{A}$ obtained from X by *Fronsdal inducing*.

Finally, we notice one simple method that can be used to try to find *-polarizations of $\mathscr{A}$. If we pick any subalgebra $\mathscr{A}_p$ of $\mathscr{A}$ and a character K of $\mathscr{A}_p$, then the space

$$X_K = \{f \in \mathscr{A} \mid fa = K(a)f \quad \forall a \in \mathscr{A}_p\} \tag{7}$$

is an $\mathscr{A}$-invariant subspace of $\mathscr{A}$, since the algebra $\mathscr{A}$ is associative. We might well attempt now to choose $\mathscr{A}_p$ and K so that X_K is also irreducible. It is not certain, of course, whether such combinations of $\mathscr{A}_p$ and K can be found and, even if they can, whether all *-polarizations of $\mathscr{A}$ can be obtained in this form.

3. *Fell inducing*

The following section gives a brief summary of the notation and basic constructions used by Fell [2] in his inducing procedure.

Definition 1. Let L be a linear space and $\mathscr{H}$ a Hilbert space. An *operator inner product* on L, acting in $\mathscr{H}$, is a map V: $L \times L \to \mathscr{B}(\mathscr{H})$, the space of bounded linear operators on $\mathscr{H}$, such that:

(*a*) $V_{s,t}$ is linear in t and conjugate-linear in s;

(*b*) for any $n \in \mathbb{N}$ and $t_j \in L$ and $\xi_j \in \mathscr{H}$,

$$\sum_{i,j=1}^{n} \langle \xi_i, V_{t_i,t_j} \xi_j \rangle \geqslant 0.$$

If we consider the algebraic tensor product $M = L \otimes \mathscr{H}$, then the formula

$$\langle s \otimes \xi, t \otimes \eta \rangle_0 = \langle \xi, V_{s,t} \eta \rangle \quad (s, t \in L; \xi, \eta \in \mathscr{H}) \tag{8}$$

defines a pre-inner product $\langle\ ,\ \rangle_0$ on M, so if we factor out the radical

$$\mathscr{R} = \{\zeta \in M \mid \langle \omega, \zeta \rangle_0 = 0; \forall \omega \in M\} \tag{9}$$

and complete the factor space $\mathscr{H}_0(L, V) = M/\mathscr{R}$, then we obtain a new Hilbert space $\mathscr{H}(L, V)$ called the Hilbert space deduced from V. The image of $s \otimes \xi$ in $\mathscr{H}(L, V)$ will be written $s \tilde{\otimes} \xi = s \otimes \xi + \mathscr{R}$, and the inner product in $\mathscr{H}(L, V)$ will also be written as $\langle\ ,\ \rangle_0$, so that

$$\langle s \tilde{\otimes} \xi, t \tilde{\otimes} \eta \rangle_0 = \langle \xi, V_{s,t} \eta \rangle \quad (s, t \in L; \xi, \eta \in \mathscr{H}). \tag{10}$$

Now let us let $\mathscr{A}$ be our *-algebra, and let L be a left-$\mathscr{A}$-module with an operator inner product V defined on it.

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

Definition 2. The $\mathscr{A}$-module L is called *Hermitian* with respect to V if

$$V_{as,t} = V_{s,a^*t}$$

for all $a \in \mathscr{A}$ and all $s, t \in L$.

If L is Hermitian with respect to V, then for any $a \in \mathscr{A}$ the equation

$$F_a^0(s \tilde{\otimes} \xi) = as \tilde{\otimes} \xi \quad (s \in L; \xi \in \mathscr{H}) \tag{11}$$

defines a linear endomorphism F_a^0 of $\mathscr{H}_0(L, V)$, which yields a pre-*-representation F^0 of $\mathscr{A}$ on $\mathscr{H}_0(L, V)$, called the pre-*-representation of $\mathscr{A}$ deduced from L and V.

L is said to be V-bounded if each map F_a^0 is continuous on $\mathscr{H}_0(L, V)$, in which case each map F_a^0 extends to a continuous linear map F_a on $\mathscr{H}(L, V)$, and F is a *-representation of $\mathscr{A}$, called the *-representation of $\mathscr{A}$ deduced from L and V.

The procedure outlined above is known as Fell inducing. We now observe that various simplifications occur when $\mathscr{H} = \mathbb{C}$, in which case the operator inner products on L are precisely the pre-inner products on L. Such operator inner products are called *scalar-valued*. In this case $M = L \otimes \mathbb{C} \equiv L$, so we shall write $M = L$, and define the pre-inner product $\langle\ ,\ \rangle_0$ on L by

$$\langle s, t \rangle_0 = V_{s,t} \quad (s, t \in L). \tag{12}$$

We construct the radical $\mathscr{R}$ and the Hilbert space $\mathscr{H}(L, V)$ as usual, and the image of $s \in L$ in $\mathscr{H}(L, V)$ will now be denoted $\tilde{s} = s + \mathscr{R}$.

4. *Statement of the problem*

Given a *-algebra $\mathscr{A}$, we shall be concerned with the following questions:

(*a*) Can pre-*-representations of $\mathscr{A}$ be obtained by Fronsdal inducing?

(*b*) Can pre-*-representations of $\mathscr{A}$ be obtained by Fell inducing?

(*c*) Can any Fronsdal-induced pre-*-representation of $\mathscr{A}$ be obtained by Fell inducing, and vice versa?

(*d*) Can all the irreducible pre-*-representations of $\mathscr{A}$ be found by these methods?

We shall answer these questions for a particular type of *-algebra $\mathscr{A}$. For simplicity we shall restrict ourselves to topological *-algebras $\mathscr{A}$ and to continuous strictly irreducible pre-*-representations of $\mathscr{A}$. This extra requirement is not strictly necessary, and in §7 we will discuss examples of non-topological *-algebras for which our results hold.

5. *Statement and proof of results*

Throughout this section we shall make free use of the results of Rickart[**11**] concerning annihilators.

If we have a *-algebra $\mathscr{A}$, consider the following list of properties of the algebra:

(P 1) $\mathscr{A}$ is a *-algebra in which $x^*x = 0$ implies $x = 0$.

(P 2) $\mathscr{A}$ is a topological *-algebra such that if $\mathscr{L}$ is a closed proper maximal left ideal of $\mathscr{A}$, then the right annihilator $\mathscr{A}_r(\mathscr{L})$ is a non-zero minimal right ideal.

(P 3) Any division subalgebra of $\mathscr{A}$ is isomorphic to $\mathbb{C}$.

The results that we shall prove will be true for algebras that satisfy various combinations of the properties (P 1), (P 2) and (P 3).

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

We first summarize the results of Rickart concerning *-algebras $\mathscr{A}$ which satisfy (P1).

PROPOSITION 1. *If $\mathscr{A}$ is a *-algebra satisfying* (P1), *then* $\mathscr{A}_l(\mathscr{A}) = \mathscr{A}_r(\mathscr{A}) = \{0\}$.

Proof. If $x \in \mathscr{A}_r(\mathscr{A})$ then $yx = 0$ for all y in $\mathscr{A}$, so that in particular $x^*x = 0$ and thus $x = 0$. Hence $\mathscr{A}_l(\mathscr{A}) = \mathscr{A}_r(\mathscr{A})^* = \{0\}$.

PROPOSITION 2. *If $\mathscr{A}$ is a *-algebra satisfying* (P1), *then for any non-zero minimal right ideal $\mathscr{R}$ in $\mathscr{A}$ there exists a unique non-zero minimal Hermitian idempotent p such that $\mathscr{R} = p\mathscr{A}$. Also $p\mathscr{A}p$ is a division algebra with identity p, and $\mathscr{L} = \mathscr{A}p$ is a non-zero minimal left ideal.*

Proof. See Rickart [**10**].

We can now obtain our first result.

THEOREM 1. *If $\mathscr{A}$ is a *-algebra satisfying* (P1) *and* (P2), *then for any strictly irreducible continuous pre-*-representation T^0 of $\mathscr{A}$ on some pre-Hilbert space $\mathscr{H}^0$ we can find a non-zero minimal left ideal $\mathscr{L}$ of $\mathscr{A}$ and define an inner product structure $\langle\,,\rangle_1$ on $\mathscr{L}$ such that the natural module structure R^0 of $\mathscr{A}$ on $\mathscr{L}$ defines a continuous pre-*-representation R^0 of $\mathscr{A}$ on $(\mathscr{L}, \langle\,,\rangle_1)$ which is unitarily equivalent to T^0.*

Hence, for any *-algebra $\mathscr{A}$ satisfying (P1) and (P2) the process of Fronsdal inducing works, and can be used to obtain all the strictly irreducible continuous pre-*-representations of $\mathscr{A}$. We observe in passing that we can also obtain the continuous *-representation R of $\mathscr{A}$ acting on the Hilbert space completion of $(\mathscr{L}, \langle\,,\rangle_1)$ which is unitarily equivalent to the continuous *-representation T of $\mathscr{A}$ on $\mathscr{H}$, the Hilbert space completion of $\mathscr{H}^0$, obtained by completing T^0.

Proof of Theorem 1. For any non-zero $\xi \in \mathscr{H}^0$ we observe that $\mathscr{A}\xi := \{T^0_a\xi | a \in \mathscr{A}\}$ is equal to $\mathscr{H}^0$. If we define $\operatorname{ann}(\xi) = \{a \in \mathscr{A} | T^0_a\xi = 0\}$, then $\operatorname{ann}(\xi)$ is a proper closed left ideal in $\mathscr{A}$. Suppose that $\mathscr{J}$ is a left ideal in $\mathscr{A}$ such that $\operatorname{ann}(\xi) \subsetneq \mathscr{J} \subseteq \mathscr{A}$. Pick $j \in \mathscr{J} - \operatorname{ann}(\xi)$. Then $T^0_j\xi \neq 0$, so that $\mathscr{A}(T^0_j\xi) = \mathscr{H}^0$, and we can find $\alpha \in \mathscr{A}$ with $T^0_{\alpha j}\xi = T^0_\alpha(T^0_j\xi) = \xi$. Thus $T^0_{(a-a\alpha j)}\xi = T^0_a\xi - T^0_aT^0_{\alpha j}\xi = T^0_a\xi - T^0_a\xi = 0$ for all $a \in \mathscr{A}$, so that $a - a\alpha j \in \operatorname{ann}(\xi)$ for all $a \in \mathscr{A}$, and so $a = (a - a\alpha j) + a\alpha j \in \mathscr{J}$ for all $a \in \mathscr{A}$, so that $\mathscr{J} = \mathscr{A}$. Hence $\operatorname{ann}(\xi)$ is a closed proper maximal left ideal in $\mathscr{A}$, and so $\mathscr{A}_r(\operatorname{ann}(\xi))$ is a non-zero minimal right ideal.

Let p be the unique non-zero Hermitian idempotent specified by Proposition 2 with $\mathscr{R} = \mathscr{A}_r(\operatorname{ann}(\xi))$. By Proposition 1, Rickart([**11**], lemma 2·8·2), and Wolfson([**14**], lemma 2·1), we deduce that

$$\operatorname{ann}(\xi) = \mathscr{A}_l(\mathscr{A}_r(\operatorname{ann}(\xi))) = \{z \in \mathscr{A} | zp = 0\} \quad \text{and} \quad \mathscr{L} := \mathscr{A}p = \{z \in \mathscr{A} | zp = z\}.$$

Consequently, for any $b \in \mathscr{L}$ we have $T^0_b\xi = 0$ if and only if $b = bp = 0$, so we can define an inner product structure on $\mathscr{L}$ by setting

$$\langle b, c\rangle_1 = \langle T^0_b\xi, T^0_c\xi\rangle \quad (b, c \in \mathscr{L}). \tag{13}$$

Since $(ap - a)p = 0$ for all $a \in \mathscr{A}$, we deduce that $T^0_{ap}\xi = T^0_a\xi$ for all $a \in \mathscr{A}$, so that the map $X: (\mathscr{L}, \langle\,,\rangle_1) \to (\mathscr{H}^0, \langle\,,\rangle)$ given by

$$Xb = T^0_b\xi \quad (b \in \mathscr{L}) \tag{14}$$

is bijective and unitary. The natural module action R^0 of $\mathscr{A}$ on $\mathscr{L}$ given by $R^0_a b = ab$ for $a \in \mathscr{A}$ and $b \in \mathscr{L}$ is now seen by (13) and (14) to define a pre-*-representation of $\mathscr{A}$ on $(\mathscr{L}, \langle\,,\rangle_1)$ such that

$$XR^a_0 = T^0_a X \quad (a \in \mathscr{A}). \tag{15}$$

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

Thus R^0 is indeed a continuous strictly irreducible pre-*-representation of $\mathscr{A}$ on $(\mathscr{L}, \langle\,,\,\rangle_1)$ which is unitarily equivalent to T^0.

The details of the proof tell us another thing: not only can we always find the *-polarization $\mathscr{L}$, but we can always find it by the simple method indicated by (7). This is because $\mathscr{L} = \{z \in \mathscr{A} | zp = z\}$, and so $\mathscr{L}$ is of the form X_K where $\mathscr{A}_P$ is the one-dimensional subalgebra of $\mathscr{A}$ generated by p, and K is the character of $\mathscr{A}_P$ defined by $K(p) = 1$.

We can also go a fair way towards relating this result to the problem of Fell inducing.

THEOREM 2. *Let $\mathscr{A}$ and T^0 be as in Theorem* 1, *and let T be the continuous *-representation of $\mathscr{A}$ on $\mathscr{H}$, the Hilbert space completion of $\mathscr{H}^0$, obtained by completing T^0. Then the *-representation T may be obtained by Fell-inducing methods.*

Proof. If we consider the map $V: \mathscr{A} \times \mathscr{A} \to \mathscr{B}(\mathscr{H})$ given by $V_{a,b} = T_{pa^*bp}$ for $a, b \in \mathscr{A}$, it is easy to show that V is an operator inner product on $\mathscr{A}$ with respect to which the $\mathscr{A}$-module $\mathscr{A}$ in Hermitian. Thus we can form the pre-Hilbert space $(\mathscr{H}_0(\mathscr{A}, V), \langle\,,\,\rangle_0)$ and the Fell-induced pre-*-representation F^0 of $\mathscr{A}$ on $\mathscr{H}_0(\mathscr{A}, V)$. Simple calculations show

$$\left\langle \sum_{i=1}^{n} a_j \otimes \xi_j, \sum_{j=1}^{m} b_j \otimes \eta_j \right\rangle_0 = \left\langle \sum_{i=1}^{n} T_{a_i p} \xi_i, \sum_{j=1}^{m} T_{b_j p} \eta_j \right\rangle,$$

and since the space spanned by the vectors $\{T_{ap}\xi | a \in \mathscr{A}; \xi \in \mathscr{H}\}$ contains $\mathscr{H}^0$, it is dense in $\mathscr{H}$, so that the radical of the operator inner product V is given by

$$\mathscr{R} = \left\{ \sum_{i=1}^{n} a_i \otimes \xi_i \in \mathscr{A} \otimes \mathscr{H} \,\middle|\, \sum_{i=1}^{n} T_{a_i p} \xi_i = 0 \right\}.$$

So we can define an isometric map $Y^0: \mathscr{H}_0(\mathscr{A}, V) \to \mathscr{H}$ by setting

$$Y^0(a \tilde{\otimes} \xi) = T_{ap}\xi \quad (a \in \mathscr{A}; \xi \in \mathscr{H}), \tag{16}$$

and since the range of Y^0 is dense in $\mathscr{H}$, Y^0 extends to a unitary map $Y: \mathscr{H}(\mathscr{A}, V) \to \mathscr{H}$. It is easy to check that $Y^0 F^0_a = T_a Y^0$ for all $a \in \mathscr{A}$, so that $\mathscr{A}$ is V-bounded, and we can extend the continuous pre-*-representation F^0 to a continuous *-representation F of $\mathscr{A}$ on $\mathscr{H}(\mathscr{A}, V)$ such that $YF_a = T_a Y$ for all $a \in \mathscr{A}$. In other words, Fell-inducing methods produce the *-representation F which is unitarily equivalent to T.

Thus both Fronsdal- and Fell-inducing methods are almost equally effective. However, we cannot guarantee to obtain precisely T^0 by Fell-inducing methods, only its completion T, so we do not yet have full control of the problem. Also the constructions used above are undesirable because they do not enable us to use them to calculate *-representations of $\mathscr{A}$. For example, we cannot define the inner product structure on a *-polarization $\mathscr{L}$ without already knowing the representation T^0 that we are trying to obtain.

It is to remove this obstruction to our procedure that we now include (P3), the simple spectral property of Gelfand and Mazur.

THEOREM 3. *Let $\mathscr{A}$ be a *-algebra satisfying* (P1), (P2), *and* (P3), *and let T^0 be a continuous strictly irreducible pre-*-representation of $\mathscr{A}$ on $\mathscr{H}^0$. Then T^0 can be directly obtained by either Fronsdal- or Fell-inducing methods.*

Proof. For a non-zero $\xi \in \mathscr{H}^0$ we can define the non-zero Hermitian idempotent p

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

and the associated *-polarization $\mathscr{L} = \mathscr{A}p$ as in Theorem 1. In this case, however, $p\mathscr{A}p = \mathbb{C}p$, and so we can define a sesquilinear form $\langle\ ,\ \rangle_2$ on $\mathscr{L}$ by setting

$$\langle a, b\rangle_2 p = a^*b \quad (a, b \in \mathscr{L}). \tag{17}$$

Simple calculations show that $\|T^0_p\xi\|^2\langle a, b\rangle_2 = \langle a, b\rangle_1$ for any $a, b \in \mathscr{L}$, and since $T^0_p\xi \neq 0$ we deduce that $\langle\ ,\ \rangle_2$ is an inner product on $\mathscr{L}$, and that the natural module structure R^0 of $\mathscr{A}$ on $\mathscr{L}$ becomes a continuous pre-*-representation of $\mathscr{A}$ on $(\mathscr{L}, \langle\ ,\ \rangle_2)$ which is unitarily equivalent to T^0, the intertwining operator $\tilde{X}$: $\mathscr{L} \to \mathscr{H}^0$ being given by

$$\|T^0_p\xi\|\,\tilde{X}a = T^0_a\xi \quad (a \in \mathscr{L}).$$

Similarly we can now define a scalar-valued operator inner product $\tilde{V}$ on $\mathscr{A}$ by setting

$$p\tilde{V}_{a,b} = pa^*bp \quad (a, b \in \mathscr{A}), \tag{18}$$

so that $\tilde{V}_{a,b} = \langle ap, bp\rangle_2$. The radical of V is thus given by $\mathscr{R} = \{b \in \mathscr{A} \mid bp = 0\}$, and so the map Z: $\mathscr{H}_0(\mathscr{A}, \tilde{V}) \to \mathscr{L}$ given by

$$Z\tilde{a} = ap \quad (a \in \mathscr{A}) \tag{19}$$

is bijective and unitary, and since $ZF^0_a = R^0_a Z$ for all $a \in \mathscr{A}$, we deduce that the Fell-induced continuous pre-*-representation F^0 of $\mathscr{A}$ on $\mathscr{H}_0(\mathscr{A}, V)$ is unitarily equivalent to R^0, and hence to T^0.

Thus, for a *-algebra $\mathscr{A}$ which satisfies properties (P1), (P2), and (P3), both Fronsdal- and Fell-inducing techniques are practicable and exhaustive. We have also seen that every non-zero minimal Hermitian idempotent p generates a continuous strictly irreducible pre-*-representation of $\mathscr{A}$ by either inducing method.

6. *Examples of such topological algebras*

We shall now discuss a collection of types of topological *-algebras, and show that a wide variety of algebras of interest satisfy the properties (P1), (P2), and (P3), so that the results of §5 enable us to use either Fronsdal or Fell inducing techniques when quantizing these algebras. The condition that is most difficult to satisfy is (P2), and so we shall consider this first.

PROPOSITION 3. *Any semisimple dual *-algebra satisfies* (P2).

Proof. See Kaplansky [7].

By semisimplicity we shall always understand the algebraic notion of semi-simplicity used by Jacobson [6], and we remember that a dual algebra is a topological algebra for which $\mathscr{A}_l(\mathscr{A}_r(\mathscr{L})) = \mathscr{L}$ and $\mathscr{A}_r(\mathscr{A}_l(\mathscr{R})) = \mathscr{R}$ for any closed left ideal $\mathscr{L}$ and any closed right ideal $\mathscr{R}$ in $\mathscr{A}$. It is possible to extend this result to cover the case of annihilator algebras, those algebras for which $\mathscr{A}_r(\mathscr{L}) = \{0\}$ if and only if $\mathscr{L} = \mathscr{A}$ and $\mathscr{A}_l(\mathscr{R}) = \{0\}$ if and only if $\mathscr{R} = \mathscr{A}$ for any closed left ideal $\mathscr{L}$ and any closed right ideal $\mathscr{R}$ in $\mathscr{A}$, at least if the algebra is also assumed to be a Q-algebra.

PROPOSITION 4. *Any semisimple annihilator Q-*-algebra satisfies* (P2).

Proof. See Hennings [4].

It is shown in Rickart [11] that the H^*-algebras first introduced by Ambrose are semisimple dual *-algebras, and so satisfy (P2). It is possible to prove an extension of

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

this result to cover the case of a full Hilbert algebra. Let $\mathscr{A}$ be a full Hilbert algebra with involution * and associated Hilbert space $\mathscr{H}$. An analogous argument to that found in Rickart [11] for the case of H^*-algebras proves the following result.

LEMMA 1. *If $\mathscr{L}$ is a closed left ideal in $\mathscr{A}$, then $\mathscr{A}_r(\mathscr{L})^* = \mathscr{M}^\perp \cap \mathscr{A}$, where $\mathscr{M}$ is the closure of $\mathscr{L}$ in $\mathscr{H}$. An analogous result holds for closed right ideals.*

Proof. See Hennings [4].

PROPOSITION 5. *Any full Hilbert algebra $\mathscr{A}$ satisfies* (P 2).

Proof. If $\mathscr{L}$ is a closed left ideal in $\mathscr{A}$, then $\mathscr{A}_r(\mathscr{L}) = (\mathscr{M}^\perp \cap \mathscr{A})^* = (\mathscr{M}^*)^\perp \cap \mathscr{A}$. Now $(\mathscr{M}^*)^\perp$ is a right-invariant subspace of $\mathscr{H}$, and so Rieffel ([12], proposition 2·7) shows that $\mathscr{A}_r(\mathscr{L}) = (\mathscr{M}^*)^\perp \cap \mathscr{A}$ is a closed right ideal in $\mathscr{A}$ which is dense in $(\mathscr{M}^*)^\perp$. Thus the closure of $\mathscr{A}_r(\mathscr{L})$ in $\mathscr{H}$ is equal to $(\mathscr{M}^*)^\perp$, and so

$$[\mathscr{A}_l(\mathscr{A}_r(\mathscr{L}))]^* = [(\mathscr{M}^*)^\perp]^\perp \cap \mathscr{A} = \mathscr{M}^* \cap \mathscr{A}.$$

Thus $\mathscr{A}_l(\mathscr{A}_r(\mathscr{L})) = \mathscr{M} \cap \mathscr{A} = \mathscr{L}$. A similar result holds for closed right ideals. Since Rieffel [12] has observed that $\mathscr{A}$ may be regarded as an A^*-algebra, it is semisimple. Thus $\mathscr{A}$ is a semisimple dual *-algebra, and hence it satisfies (P 2).

If we move on to consider the condition (P 3), then the Gelfand–Mazur theorem states that any Banach *-algebra satisfies (P 3). More generally, we have the following result.

PROPOSITION 6. *Any locally multiplicatively convex *-algebra $\mathscr{A}$ satisfies* (P 3).

Proof. See Michael [8].

Hence we obtain the following results.

COROLLARY 1. *Any locally multiplicatively convex semisimple dual *-algebra in which $x^*x = 0$ implies $x = 0$ satisfies* (P 1), (P 2), *and* (P 3).

COROLLARY 2. *Any annihilator C^*-algebra satisfies* (P 1), (P 2), *and* (P 3).

COROLLARY 3. *Any full Hilbert algebra satisfies* (P 1), (P 2), *and* (P 3).

Proof. It is evident that $x^*x = 0$ implies $x = 0$ in the full Hilbert algebra $\mathscr{A}$. Although the natural topology of $\mathscr{A}$ may not be locally multiplicatively convex in general, it is shown in Rieffel [12] that $\mathscr{A}$ may be regarded as a Banach *-algebra with respect to a new norm, and hence we see that $\mathscr{A}$ satisfies (P 3) as well.

Consequently we can throw the full power of our results at all annihilator C^*-algebras and full Hilbert algebras. In Hennings [4] it is shown how, for any integer n, the collection of spherical harmonics up to order n may be treated as an annihilator C^*-algebra, and so now we know all about quantizing spin observables for a single particle. Also it is known (Hudson *et al.* [5]) how to find a full Hilbert algebra $\mathscr{A}_\sigma$ related to the twisted convolution algebra of a locally compact abelian group G with multiplier σ, and the full knowledge that we now have about quantizing this algebra $\mathscr{A}_\sigma$ is very useful when attempting to quantize some physical system with the group G as phase space.

Finally, for any Hilbert space $\mathscr{H}$, the space $\mathscr{K}(\mathscr{H})$ of all compact linear operators on $\mathscr{H}$ is an annihilator C^*-algebra, and the spaces $\mathscr{T}_1(\mathscr{H})$ and $\mathscr{T}_2(\mathscr{H})$ of trace-class and Hilbert–Schmidt operators on $\mathscr{H}$, with their appropriate norms, form semisimple annihilator Banach *-algebras in which $T^*T = 0$ implies $T = 0$, and so we know all about finding *-representations of all three algebras $\mathscr{K}(\mathscr{H})$, $\mathscr{T}_1(\mathscr{H})$, and $\mathscr{T}_2(\mathscr{H})$.

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

7. *Non-topological algebras*

We shall end our discussion of this problem by considering two cases where the equivalence and exhaustiveness of Fronsdal- and Fell-inducing methods can be shown for non-topological algebras.

(*a*) *Generalized Hilbert algebras*

If we consider a full right Hilbert algebra $\mathscr{A}$ as defined in Stratila and Zsido [**13**], then we do not have a topological algebra, because only right multiplication is continuous. However, Phillips [**10**] has shown that, with respect to a new norm, $\mathscr{A}$ becomes a Banach *-algebra, and hence $\mathscr{A}$ satisfies (P 3). It is also evident that $\mathscr{A}$ satisfies (P 1).

Let us also require that $\mathscr{A}$ be strongly orthocomplemented in the sense that $\mathscr{A} = \mathscr{L} \oplus \mathscr{L}^{\textcircled{\perp}}$ for any closed left ideal $\mathscr{L}$ in $\mathscr{A}$, where $\mathscr{L}^{\textcircled{\perp}} = \mathscr{L}^{\perp} \cap \mathscr{A}$ is the relative orthogonal complement of $\mathscr{L}$. It has been shown [**4**] that if $\mathscr{L}$ is a proper closed maximal left ideal in $\mathscr{A}$, then $\mathscr{A}_r(\mathscr{L})$ is a non-zero minimal right ideal in $\mathscr{A}$. Hence $\mathscr{A}$ satisfies (P 2), except for being non-topological. A quick check of the proofs of Theorems 1, 2 and 3 shows that left multiplication need not be continuous for their results to be true, and so we deduce that any continuous strictly irreducible pre-*-representation of a strongly orthocomplemented full right Hilbert algebra can be obtained by either Fell- or Fronsdal-inducing methods.

The property of strong orthocomplementarity was introduced by Phillips [**10**] for generalized Hilbert algebras, and Yood [**15**] first showed that all full Hilbert algebras are strongly orthocomplemented, so that imposing this extra condition is sensible when first extending the results for Hilbert algebras to generalized Hilbert algebras.

(*b*) *Dense algebras of linear transformations*

Finally, we shall briefly discuss one more case of interest. Let A be a vector space equipped with an inner product and an involution. We let $\mathscr{E}(A)$ be the algebra of all linear transformations on A which possess an adjoint with respect to the inner product. In [**4**] it has been shown how using the involution turns $\mathscr{E}(A)$ into a dense algebra of linear transformations of the type studied by Wolfson [**14**]. If we let $T_0(A)$ denote the subalgebra of $\mathscr{E}(A)$ consisting of all such linear transformations of finite rank, then the following results have been established. We say that a pre-*-representation T of $\mathscr{E}(A)$ on $\mathscr{H}$ is null if $T_R = 0$ for all $R \in T_0(A)$. (P 1) certainly holds for the algebra $\mathscr{E}(A)$.

LEMMA 2. *If T is a non-null strictly irreducible pre-*-representation of $\mathscr{E}(A)$ on $\mathscr{H}$, then $\mathscr{L} = \operatorname{ann}(\xi)$ is a proper maximal left ideal for any non-zero $\xi \in \mathscr{H}$, and $\mathscr{A}_r(\mathscr{L})$ is a non-zero minimal right ideal.*

LEMMA 3. *$P \cdot \mathscr{E}(A) \cdot P = \mathbb{C}P$ whenever P is a non-zero minimal Hermitian idempotent in $\mathscr{E}(A)$.*

THEOREM 4. *Any strictly irreducible non-null pre-*-representation of $\mathscr{E}(A)$ can be obtained by either Fell- or Fronsdal-inducing methods. Moreover, there is, to within equivalence, only one such pre-*-representation of $\mathscr{E}(A)$, and that is the trivial one given by its action on A.*

This example is of physical interest, because it is shown in [**4**] how an algebra

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press

$\mathcal{O}(\mathbb{R}^{2n})$ can be found – very closely related to the algebra $\mathcal{O}_m(\mathbb{R}^{2n})$ of distributions of weak decrease – which is isomorphic to just such an algebra $\mathscr{E}(\mathscr{S}(\mathbb{R}^n))$, where $\mathscr{S}(\mathbb{R}^n)$ is the space of Schwartz functions on $\mathbb{R}^n$. Thus we deduce that there is essentially only one way of quantizing this algebra, and so the Weyl transformation for the free particle is uniquely defined.

REFERENCES

[1] F. BAYEN, M. FLATO, C. FRONSDAL, A. LICHNEROWICZ and D. STERNHEIMER. Deformation theory and quantisation. I. Deformations of symplectic structures. *Ann. Physics* **111** (1978), 61–110.

[2] J. M. G. FELL. *Induced Representations and Banach *-algebraic Bundles*. Springer Lecture Notes in Mathematics 582 (Springer-Verlag, 1977).

[3] C. FRONSDAL. Some ideas about quantisation. *Rep. Math. Phys.* **15·1** (1979), 111–145.

[4] M. A. HENNINGS. Fronsdal *-quantisation and the abstract inducing procedures of Fell and Rieffel. D.Phil. thesis (Oxford, 1984).

[5] R. L. HUDSON, M. D. WILKINSON and S. N. PECK. Fourier analysis on multiplier representations of locally compact abelian groups. *Rep. Math. Phys.* **19·1** (1984), 91–100.

[6] N. JACOBSON. The radical and semisimplicity for arbitrary rings. *Amer. J. Math.* **67** (1945), 300–320.

[7] I. KAPLANSKY. Dual rings. *Ann. of Math.* **49·3** (1948), 689–701.

[8] E. A. MICHAEL. Locally-multiplicatively-convex topological algebras. *Mem. Amer. Math. Soc.* **11** (1952).

[9] J. E. MOYAL. Quantum mechanics as a statistical theory. *Math. Proc. Cambridge Philos. Soc.* **45** (1949), 99–124.

[10] J. PHILLIPS. Complementation for right ideals in generalised Hilbert algebras. *Trans. Amer. Math. Soc.* **197** (1974), 409–417.

[11] C. E. RICKART. *General Theory of Banach Algebras* (Van Nostrand, 1960).

[12] M. A. RIEFFEL. Square-integrable representations of Hilbert algebras. *J. Funct. Anal.* **3** (1969), 265–300.

[13] S. STRATILA and L. ZSIDO. *Lectures on von Neumann Algebras* (Abacus Press, 1979).

[14] K. G. WOLFSON. Annihilator rings. *J. Lond. Math. Soc.* **31** (1956), 94–104.

[15] B. YOOD. Hilbert algebras as topological algebras. *Ark. Mat.* **12** (1974), 131–151.

https://doi.org/10.1017/S0305004100064070 Published online by Cambridge University Press