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Stability of a Leap-Frog Discontinuous Galerkin Method for Time-Domain Maxwell's Equations in Anisotropic Materials

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Abstract. In this work we discuss the numerical discretization of the time-dependent Maxwell's equations using a fully explicit leap-frog type discontinuous Galerkin method. We present a sufficient condition for the stability and error estimates, for cases of typical boundary conditions, either perfect electric, perfect magnetic or first order Silver-Müller. The bounds of the stability region point out the influence of not only the mesh size but also the dependence on the choice of the numerical flux and the degree of the polynomials used in the construction of the finite element space, making possible to balance accuracy and computational efficiency. In the model we consider heterogeneous anisotropic permittivity tensors which arise naturally in many applications of interest. Numerical results supporting the analysis are provided.

AMS subject classifications: 35Q61, 65M60

Key words: Maxwell's equations, fully explicit leap-frog discontinuous Galerkin method, stability.

1 Introduction

Maxwell's equations are a fundamental set of partial differential equations which describe electromagnetic wave interactions with materials. The advantages of using discontinuous Galerkin time domain (DGTD) methods on the simulation of electromagnetic waves propagation, when compared with classical finite-difference time-domain methods, finite volume time domain methods or finite element time domain methods, have been reported by several authors (see *e.g.* [9] and references therein cited for an overview). DGTD methods gather many desirable features such as being able to achieve

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high-order accuracy and easily handle complex geometries. Moreover, they are suitable for parallel implementation on modern multi-graphics processing units. Local refinement strategies can be incorporated due to the possibility of considering irregular meshes with hanging nodes and local spaces of different orders.

The staggered leapfrog time-stepping algorithm is a popular choice for time domain Maxwell's equations (*e.g.* [1, 9, 21]) due to its simplicity, as it does not require to save in memory previous states, accuracy and robustness.

Despite the relevance of the anisotropic case in applications (*e.g.* [4,14,24]), most of the formulation of the DGTD methods presented in the literature are restricted to isotropic materials [11, 12, 16]. Motivated by our application of interest described in [2, 20], in the present paper we consider a model with a heterogeneous anisotropic permittivity tensor. The treatment of anisotropic materials within a DGTD framework was discussed for instance in [9] (with central fluxes) and in [13] (with upwind fluxes). The stability analysis of DGTD methods for Maxwell's equations was considered in [9], where the scheme that is defined with the central fluxes leads to a locally implicit time method in the case of Silver-Müller absorbing boundary conditions, and [15], where the scheme is defined with the upwind fluxes leading to an implicit method. Our derivation extends the results in [9] and [15] to a fully explicit in time method for both cases, central fluxes and upwind fluxes.

We consider the formulation in two dimensions as well as an extension to a three dimensional problem and we combine the nodal DG method [11] for the integration in space, considering both central and upwind fluxes, with an explicit leap-frog type method for the time integration. We present a rigorous proof of stability showing the influence of the mesh size, the choice of the numerical flux and choice of the degree of the polynomials used in the construction of the finite element space and the boundary conditions, which can be either perfect electric, perfect magnetic or first order Silver-Müller.

This paper consists in six sections after this introduction. In Section 2, we state the problem and in Section 3 we describe the formulation of the numerical method for the two-dimensional problem. In Section 4 we derive stability and convergence results for the method described in the previous section. We illustrate the theoretical results with numerical examples in Section 5. In the last section we extend the stability results to the three dimensional case.

2 The governing equations

The electromagnetic field consists of coupled electric and magnetic fields, known as electric field intensity, E, and magnetic induction, B. The effects of these two fundamental fields on matter can be characterized by the electric displacement and the magnetic field intensity vectors, frequently denoted by D and H, respectively. The knowledge of the material properties can be used to derive a useful relation between D and E and between B and H. Here we will consider the constitutive relations of the form $D=\epsilon E$ and $B=\mu H$,

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where ϵ is the medium's electric permittivity and μ is the medium's magnetic permeability.

In three-dimensional spaces for heterogeneous anisotropic linear media with no source, these equations can be written in the form [9]

$$\epsilon\frac{\partial E}{\partial t}=\text{curl } H, \qquad \mu\frac{\partial H}{\partial t}=-\text{curl } E. \tag{2.1}$$

In a similar fashion to [23], we decompose the electromagnetic wave in a transverse electric (TE) mode and a transverse magnetic (TM) mode, this way reducing the number of equations implemented in our model. This assumption is appropriate when studying *e.g.* truly 2D photonic crystals [6] or the electrodynamic properties of 2D materials like graphene [17].

In what follows we shall analyse the time domain Maxwell's equations in the transverse electric (TE) mode, as in [13], where the only non-vanishing components of the electromagnetic fields are E_x, E_y and H_z. For this case, and assuming no conductivity effects, the equations in the non-dimensional form are

$$\epsilon\frac{\partial E}{\partial t}=\nabla\times H \quad \text{in } \Omega\times(0,T_f], \tag{2.2}$$

$$\mu\frac{\partial H}{\partial t}=-\text{curl } E \quad \text{in } \Omega\times(0,T_f], \tag{2.3}$$

where $E=(E_x,E_y)$ and $H=(H_z)$. This equations are set and solved on the bounded polygonal domain $\Omega\subset\mathbb{R}^2$. Note that we use the following notation for the vector and scalar curl operators

$$\nabla\times H=\left(\frac{\partial H_z}{\partial y},-\frac{\partial H_z}{\partial x}\right)^T, \quad \text{curl } E=\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}.$$

The electric permittivity of the medium ϵ and the magnetic permeability of the medium μ are varying in space, being ϵ an anisotropic tensor

$$\epsilon=\begin{pmatrix}\epsilon_{xx} & \epsilon_{xy}\\ \epsilon_{yx} & \epsilon_{yy}\end{pmatrix}, \tag{2.4}$$

while we consider isotropic permeability μ. We assume that electric permittivity tensor ϵ is symmetric and uniformly positive definite for almost every $(x,y)\in\Omega$, and it is uniformly bounded with a strictly positive lower bound, *i.e.*, there are constants $\underline{\epsilon}>0$ and $\overline{\epsilon}>0$ such that, for almost every $(x,y)\in\Omega$,

$$\underline{\epsilon}|\xi|^2\le\xi^T\epsilon(x,y)\xi\le\overline{\epsilon}|\xi|^2, \qquad \forall\xi\in\mathbb{R}^2.$$

We also assume that there are constants $\underline{\mu}>0$ and $\overline{\mu}>0$ such that, for almost every $(x,y)\in\Omega$,

$$\underline{\mu}\le\mu(x,y)\le\overline{\mu}.$$

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Let the unit outward normal vector to the boundary be denoted by n. We can define an effective permittivity [13] by

$$\epsilon_{eff} = \frac{\det(\epsilon)}{n^T \epsilon n},$$

that is used to characterize the speed with which a wave travels along the direction of the unit normal,

$$c = \sqrt{\frac{n^T \epsilon n}{\mu \det(\epsilon)}}.$$

The model equations (2.2)-(2.3) must be complemented by proper boundary conditions. Here we consider the most common, either the perfect electric conductor boundary condition (PEC)

$$n \times E = 0 \quad \text{on } \partial\Omega, \tag{2.5}$$

the perfect magnetic conductor boundary condition (PMC),

$$n \times H = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

or the first order Silver-Müller absorbing boundary condition

$$n \times E = c\mu n \times (H \times n) \quad \text{on } \partial\Omega. \tag{2.7}$$

Initial conditions

$$E(x,y,0) = E_0(x,y) \quad \text{and} \quad H(x,y,0) = H_0(x,y) \quad \text{in } \Omega,$$

must also be provided.

We can write Maxwell's equations (2.2)-(2.3) in a conservation form

$$Q\frac{\partial q}{\partial t} + \nabla \cdot F(q) = 0 \quad \text{in } \Omega \times (0, T_f], \tag{2.8}$$

with

$$Q = \begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}, \quad q = \begin{pmatrix} E_x \\ E_y \\ H_z \end{pmatrix} \quad \text{and} \quad F(q) = \begin{pmatrix} 0 & -H_z \\ H_z & 0 \\ E_y & -E_x \end{pmatrix},$$

where $\nabla\cdot$ denotes the divergence operator.

3 A leap-frog discontinuous Galerkin method

The aim of this section is to derive our computational method. We will consider a nodal discontinuous Galerkin method for the space discretization and a leap-frog method for the time integration.

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3.1 The discontinuous Galerkin method

Assume that the computational domain Ω is partitioned into K triangular elements T_k such that $\overline{\Omega}=\cup_k T_k$. For simplicity, we consider that the resulting mesh $\mathcal{T}_h$ is conforming, that is, the intersection of two elements is either empty or an edge.

Let h_k be the diameter of the triangle $T_k\in\mathcal{T}_h$, and h be the maximum element diameter,

$$h_k=\sup_{P_1,P_2\in T_K}\|P_1-P_2\|,\quad h=\max_{T_k\in\mathcal{T}_h}\{h_k\}.$$

We assume that the mesh is regular in the sense that there is a constant $\tau>0$ such that

$$\forall T_k\in\mathcal{T}_h,\quad \frac{h_k}{\tau_k}\leq\tau, \tag{3.1}$$

where τ_k denotes the maximum diameter of a ball inscribed in T_k.

On each element T_k, the solution fields are approximated by polynomials of degree less or equal to N. The global solution $q(x,y,t)$ is then assumed to be approximated by the piecewise N order polynomials

$$q(x,y,t)\simeq\tilde{q}(x,y,t)=\bigoplus_{k=1}^{K}\tilde{q}_k(x,y,t),$$

defined as the direct sum of the K local polynomial solutions $\tilde{q}_k(x,t)=(\tilde{E}_{x_k},\tilde{E}_{y_k},\tilde{H}_{z_k})$. We use the notation $\tilde{E}_x(x,y,t)=\bigoplus_{k=1}^{K}\tilde{E}_{x_k}(x,y,t)$, $\tilde{E}_y(x,y,t)=\bigoplus_{k=1}^{K}\tilde{E}_{y_k}(x,y,t)$, $\tilde{H}_z(x,y,t)=\bigoplus_{k=1}^{K}\tilde{H}_{z_k}(x,y,t)$. The finite element space is then taken to be

$$V_N=\{v\in L^2(\Omega)^3:v|_{T_k}\in P_N(T_k)^3\},$$

where $P_N(T_k)$ denotes the space of polynomials of degree less or equal to N on T_k. The fields are expanded in terms of interpolating Lagrange polynomials $L_i(x,y)$,

$$\tilde{q}_k(x,y,t)=\sum_{i=1}^{N_p}\tilde{q}_k(x_i,y_i,t)L_i(x,y)=\sum_{i=1}^{N_p}\tilde{q}_{ki}(t)L_i(x,y).$$

Here N_p denotes the number of coefficients that are utilized, which is related with the polynomial order N via $N_p=(N+1)(N+2)/2$.

In order to deduce the method, we start by multiplying Eq. (2.8) by test functions $v\in V_N$, usually the Lagrange polynomials, and integrate over each element T_k. The next step is to employ one integration by parts and to substitute in the resulting contour integral the flux F by a numerical flux F^*. Reversing the integration by parts yields

$$\int_{T_k}\left(Q\frac{\partial\tilde{q}}{\partial t}+\nabla\cdot F(\tilde{q})\right)\cdot v(x,y)\,dxdy=\int_{\partial T_k}n\cdot(F(\tilde{q})-F^*(\tilde{q}))\cdot v(x,y)\,ds,$$

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where $n=(n_x,n_y)$ is the outward pointing unit normal vector of the contour.

The approximate fields are allowed to be discontinuous across element boundaries. In this way, we introduce the notation for the jumps of the field values across the interfaces of the elements, $[\tilde{E}]=\tilde{E}^- - \tilde{E}^+$ and $[\tilde{H}]=\tilde{H}^- - \tilde{H}^+$, where the superscript "$+$" denotes the neighboring element and the superscript "$-$" refers to the local cell. Furthermore we introduce, respectively, the cell-impedances and cell-conductances $Z^\pm=\mu^\pm c^\pm$ and $Y^\pm=(Z^\pm)^{-1}$ where

$$c^\pm=\sqrt{\frac{n^T\epsilon^\pm n}{\mu^\pm \det(\epsilon^\pm)}}.$$

At the outer cell boundaries we set $Z^+=Z^-$.

The coupling between elements is introduced via numerical flux obtained using the Rankine-Hugoniot condition [12, 13], defined by

$$n\cdot(F(\tilde{q})-F^*(\tilde{q}))=\begin{pmatrix} \frac{-n_y}{Z^++Z^-}\left(Z^+[\tilde{H}_z]-\alpha\left(n_x[\tilde{E}_y]-n_y[\tilde{E}_x]\right)\right) \\ \frac{n_x}{Z^++Z^-}\left(Z^+[\tilde{H}_z]-\alpha\left(n_x[\tilde{E}_y]-n_y[\tilde{E}_x]\right)\right) \\ \frac{1}{Y^++Y^-}\left(Y^+\left(n_x[\tilde{E}_y]-n_y[\tilde{E}_x]\right)-\alpha[\tilde{H}_z]\right) \end{pmatrix}.$$

The parameter $\alpha\in[0,1]$ in the numerical flux can be used to control dissipation. Taking $\alpha=0$ yields a non dissipative central flux while $\alpha=1$ corresponds to the classic upwind flux.

In order to discretize the boundary conditions we set $[\tilde{E}_x]=2\tilde{E}_x^-$, $[\tilde{E}_y]=2\tilde{E}_y^-$, $[\tilde{H}_z]=0$ and $[\tilde{E}_x]=0$, $[\tilde{E}_y]=0$, $[\tilde{H}_z]=2\tilde{H}_z^-$, for PEC and PMC boundary conditions, respectively. For Silver-Müller absorbing boundary conditions, using the same kind of approach as in [1], we consider, for upwind fluxes $Z^-\tilde{H}_z^+=n_x\tilde{E}_y^+-n_y\tilde{E}_x^+$ or equivalently $\tilde{H}_z^+=Y^-(n_x\tilde{E}_y^+-n_y\tilde{E}_x^+)$ and, for central fluxes $Z^-\tilde{H}_z^+=(n_x\tilde{E}_y^--n_y\tilde{E}_x^-)$ and $Y^-(n_x\tilde{E}_y^+-n_y\tilde{E}_x^+)=\tilde{H}_z^-$. This is equivalent to consider, for both upwind and central fluxes, $\alpha=1$ for numerical flux at the outer boundary and $[\tilde{E}_x]=\tilde{E}_x^-$, $[\tilde{E}_y]=\tilde{E}_y^-$ and $[\tilde{H}_z]=\tilde{H}_z^-$.

In what follows, $(\cdot,\cdot)_{T_k}$ and $(\cdot,\cdot)_{\partial T_k}$ denote the classical $L^2(T_k)$ and $L^2(\partial T_k)$ inner-products, respectively. The same notation is also used for the classical inner-products in $(L^2(T_k))^2$ and $(L^2(\partial T_k))^2$.

3.2 Time discretization

To define a fully discrete scheme, we divide the time interval $[0,T]$ into M subintervals by points $0=t^0<t^1<\cdots<t^M=T$, where $t^m=m\Delta t$, Δt is the time step size and T is such that $T+\Delta t/2\leq T_f$. The unknowns related to the electric field are approximated at integer time-stations t^m and are denoted by $\tilde{E}_k^m=\tilde{E}_k(.,t^m)$. The unknowns related to the magnetic field are approximated at half-integer time-stations $t^{m+1/2}=(m+\frac{1}{2})\Delta t$ and are denoted by $\tilde{H}_k^{m+1/2}=\tilde{H}_k(.,t^{m+1/2})$. With the above setting, we can now formulate the leap-frog DG method: given an initial approximation $(\tilde{E}_{x_k}^0,\tilde{E}_{y_k}^0,\tilde{H}_{z_k}^{1/2})\in V_N$, for each $m=0,1,\cdots,M-1$,

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find $(\tilde{E}_{x_k}^{m+1},\tilde{E}_{y_k}^{m+1},\tilde{H}_{z_k}^{m+1/2})\in V_N$ such that, $\forall(u_k,v_k,w_k)\in V_N$,

$$\left(\epsilon_{xx}\frac{\tilde{E}_{x_k}^{m+1}-\tilde{E}_{x_k}^{m}}{\Delta t}+\epsilon_{xy}\frac{\tilde{E}_{y_k}^{m+1}-\tilde{E}_{y_k}^{m}}{\Delta t},u_k\right)_{T_k}=\left(\partial_y\tilde{H}_{z_k}^{m+1/2},u_k\right)_{T_k}$$
$$+\left(\frac{-n_y}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x[\tilde{E}_y^m]-n_y[\tilde{E}_x^m]\right)\right),u_k\right)_{\partial T_k}, \tag{3.2}$$

$$\left(\epsilon_{yx}\frac{\tilde{E}_{x_k}^{m+1}-\tilde{E}_{x_k}^{m}}{\Delta t}+\epsilon_{yy}\frac{\tilde{E}_{y_k}^{m+1}-\tilde{E}_{y_k}^{m}}{\Delta t},v_k\right)_{T_k}=-\left(\partial_x\tilde{H}_{z_k}^{m+1/2},v_k\right)_{T_k}$$
$$+\left(\frac{n_x}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x[\tilde{E}_y^m]-n_y[\tilde{E}_x^m]\right)\right),v_k\right)_{\partial T_k}, \tag{3.3}$$

$$\left(\mu\frac{\tilde{H}_{z_k}^{m+3/2}-\tilde{H}_{z_k}^{m+1/2}}{\Delta t},w_k\right)_{T_k}=\left(\partial_y\tilde{E}_{x_k}^{m+1}-\partial_x\tilde{E}_{y_k}^{m+1},w_k\right)_{T_k}$$
$$+\left(\frac{1}{Y^++Y^-}\left(Y^+(n_x[\tilde{E}_y^{m+1}]-n_y[\tilde{E}_x^{m+1}])-\alpha[\tilde{H}_z^{m+1/2}]\right),w_k\right)_{\partial T_k}. \tag{3.4}$$

The boundary conditions are considered as described in the previous section.

We want to emphasize that the scheme (3.2)-(3.4) is fully explicit in time, in opposition to [15], where the scheme is defined with the upwind fluxes involving the unknowns E_k^{m+1} and $H_k^{m+3/2}$ and to [9], where the scheme that is defined with the central fluxes leads to a locally implicit time method in the case of Silver-Müller absorbing boundary conditions.

4 Stability and convergence analysis

The aim of this section is to provide a sufficient condition for the L^2-stability of the leap-frog DG method (3.2)-(3.4).

Choosing $u_k=\Delta t\tilde{E}_{x_k}^{[m+1/2]}$, $v_k=\Delta t\tilde{E}_{y_k}^{[m+1/2]}$ and $w_k=\Delta t\tilde{H}_{z_k}^{[m+1]}$, where $\tilde{E}^{[m+1/2]}=\left(\tilde{E}^m+\tilde{E}^{m+1}\right)/2$ and $\tilde{H}^{[m+1]}=\left(\tilde{H}^{m+1/2}+\tilde{H}^{m+3/2}\right)/2$, we have

$$\begin{aligned}&\left(\epsilon\tilde{E}_k^{m+1},\tilde{E}_k^{m+1}\right)_{T_k}-\left(\epsilon\tilde{E}_k^{m},\tilde{E}_k^{m}\right)_{T_k}\\ =&2\Delta t\left(\nabla\times\tilde{H}_{z_k}^{m+1/2},\tilde{E}_k^{[m+1/2]}\right)_{T_k}\\ &+2\Delta t\left(\frac{-n_y}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x[\tilde{E}_y^m]-n_y[\tilde{E}_x^m]\right)\right),\tilde{E}_{x_k}^{[m+1/2]}\right)_{\partial T_k}\\ &+2\Delta t\left(\frac{n_x}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x[\tilde{E}_y^m]-n_y[\tilde{E}_x^m]\right)\right),\tilde{E}_{y_k}^{[m+1/2]}\right)_{\partial T_k}\end{aligned} \tag{4.1}$$

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and

$$\begin{aligned}
&\left(\mu\tilde{H}_{z_k}^{m+3/2},\tilde{H}_{z_k}^{m+3/2}\right)_{T_k}-\left(\mu\tilde{H}_{z_k}^{m+1/2},\tilde{H}_{z_k}^{m+1/2}\right)_{T_k}\\
=&-2\Delta t\left(\text{curl }\tilde{E}_k^{m+1},\tilde{H}_{z_k}^{[m+1]}\right)_{T_k}\\
&+2\Delta t\left(\frac{1}{Y^++Y^-}\left(Y^+\left(n_x[\tilde{E}_y^{m+1}]-n_y[\tilde{E}_x^{m+1}]\right)-\alpha[\tilde{H}_z^{m+1/2}]\right),\tilde{H}_{z_k}^{[m+1]}\right)_{\partial T_k}. \qquad (4.2)
\end{aligned}$$

Using the identity,

$$\left(\text{curl }\tilde{E}_k^{m+1},\tilde{H}_{z_k}^{[m+1]}\right)_{T_k}=\left(\nabla\times\tilde{H}_{z_k}^{[m+1]},\tilde{E}_k^{m+1}\right)_{T_k}+\left(n_x\tilde{E}_{y_k}^{m+1}-n_y\tilde{E}_{x_k}^{m+1},\tilde{H}_{z_k}^{[m+1]}\right)_{\partial T_k},$$

summing (4.1) and (4.2) from $m=0$ to $m=M-1$, and integrating by parts, we get

$$\begin{aligned}
&\left(\epsilon\tilde{E}_k^M,\tilde{E}_k^M\right)_{T_k}+\left(\mu\tilde{H}_{z_k}^{M+1/2},\tilde{H}_{z_k}^{M+1/2}\right)_{T_k}\\
=&\left(\epsilon\tilde{E}_k^0,\tilde{E}_k^0\right)_{T_k}+\left(\mu\tilde{H}_{z_k}^{1/2},\tilde{H}_{z_k}^{1/2}\right)_{T_k}\\
&+\Delta t\left(\nabla\times\tilde{H}_{z_k}^{1/2},\tilde{E}_k^0\right)_{T_k}-\Delta t\left(\nabla\times\tilde{H}_{z_k}^{M+1/2},\tilde{E}_k^M\right)_{T_k}+2\Delta t\sum_{m=0}^{M-1}A_k^m, \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
A_k^m=&\left(\frac{-n_y}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x[\tilde{E}_y^m]-n_y[\tilde{E}_x^m]\right)\right),\tilde{E}_{x_k}^{[m+1/2]}\right)_{\partial T_k}\\
&+\left(\frac{n_x}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x[\tilde{E}_y^m]-n_y[\tilde{E}_x^m]\right)\right),\tilde{E}_{y_k}^{[m+1/2]}\right)_{\partial T_k}\\
&+\left(\frac{1}{Y^++Y^-}\left(Y^+\left(n_x[\tilde{E}_y^{m+1}]-n_y[\tilde{E}_x^{m+1}]\right)-\alpha[\tilde{H}_z^{m+1/2}]\right),\tilde{H}_{z_k}^{[m+1]}\right)_{\partial T_k}\\
&-\left(n_x\tilde{E}_{y_k}^{m+1}-n_y\tilde{E}_{x_k}^{m+1},\tilde{H}_{z_k}^{[m+1]}\right)_{\partial T_k}.
\end{aligned}$$

Let us denote by F^{int} the set of internal edges and F^{ext} the set of edges that belong to the boundary $\partial\Omega$. Let ν_k be the set of indices of the neighboring elements of T_k. For each $i\in\nu_k$, we consider the internal edge $f_{ik}=T_i\cap T_k$, and we denote by n_{ik} the unit normal oriented from T_i towards T_k. For each boundary edge $f_k=T_k\cap\partial\Omega$, n_k is taken to be the unitary outer normal vector to f_k. Summing over all elements $T_k\in\mathcal{T}_h$ we obtain

$$\sum_{T_k\in\mathcal{T}_h}A_k^m=B_1^m+B_2^m,$$

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where $B_1^m = B_{11}^m + B_{12}^m + B_{13}^m$ with

$$\begin{aligned} B_{11}^m = \sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \Bigg(& \frac{-(n_y)_{ki}}{Z_i+Z_k}\left(Z_i[\tilde{H}_{z_k}^{m+1/2}] - \alpha\left((n_x)_{ki}[\tilde{E}_{y_k}^m] - (n_y)_{ki}[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{x_k}^{[m+1/2]} \\ & + \frac{-(n_y)_{ik}}{Z_i+Z_k}\left(Z_k[\tilde{H}_{zi}^{m+1/2}] - \alpha\left((n_x)_{ik}[\tilde{E}_{yi}^m] - (n_y)_{ik}[\tilde{E}_{xi}^m]\right)\right)\tilde{E}_{xi}^{[m+1/2]} \\ & - \frac{Y_i(n_y)_{ki}}{Y_i+Y_k}[\tilde{E}_{x_k}^{m+1}]\tilde{H}_{z_k}^{[m+1]} - \frac{Y_k(n_y)_{ik}}{Y_i+Y_k}[\tilde{E}_{xi}^{m+1}]\tilde{H}_{zi}^{[m+1]} \\ & + (n_y)_{ki}\tilde{E}_{x_k}^{m+1}\tilde{H}_{z_k}^{[m+1]} + (n_y)_{ik}\tilde{E}_{xi}^{m+1}\tilde{H}_{zi}^{[m+1]}\Bigg) ds, \end{aligned} \tag{4.4}$$

$$\begin{aligned} B_{12}^m = \sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \Bigg(& \frac{(n_x)_{ki}}{Z_i+Z_k}\left(Z_i[\tilde{H}_{z_k}^{m+1/2}] - \alpha\left((n_x)_{ki}[\tilde{E}_{y_k}^m] - (n_y)_{ki}[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{y_k}^{[m+1/2]} \\ & + \frac{(n_x)_{ik}}{Z_i+Z_k}\left(Z_k[\tilde{H}_{zi}^{m+1/2}] - \alpha\left((n_x)_{ik}[\tilde{E}_{yi}^m] - (n_y)_{ik}[\tilde{E}_{xi}^m]\right)\right)\tilde{E}_{yi}^{[m+1/2]} \\ & + \frac{Y_i(n_x)_{ki}}{Y_i+Y_k}[\tilde{E}_{y_k}^{m+1}]\tilde{H}_{z_k}^{[m+1]} + \frac{Y_k(n_x)_{ik}}{Y_i+Y_k}[\tilde{E}_{yi}^{m+1}]\tilde{H}_{zi}^{[m+1]} \\ & - (n_x)_{ki}\tilde{E}_{y_k}^{m+1}\tilde{H}_{z_k}^{[m+1]} - (n_x)_{ik}\tilde{E}_{yi}^{m+1}\tilde{H}_{zi}^{[m+1]}\Bigg) ds, \end{aligned} \tag{4.5}$$

$$B_{13}^m = -\sum_{f_{ik}\in F^{int}} \int_{f_{ik}} \left(\frac{\alpha}{Y_i+Y_k}[\tilde{H}_{z_k}^{m+1/2}]\tilde{H}_{z_k}^{[m+1]} + \frac{\alpha}{Y_i+Y_k}[\tilde{H}_{zi}^{m+1/2}]\tilde{H}_{zi}^{[m+1]} \right) ds \tag{4.6}$$

and B_2^m has the terms related with the outer boundary

$$\begin{aligned} B_2^m = \sum_{f_k\in F^{ext}} \int_{f_k} \Bigg(& \frac{-(n_y)_k}{2Z_k}\left(Z_k[\tilde{H}_{z_k}^{m+1/2}] - \alpha\left((n_x)_k[\tilde{E}_{y_k}^m] - (n_y)_k[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{x_k}^{[m+1/2]} \\ & + \frac{(n_x)_k}{2Z_k}\left(Z_k[\tilde{H}_{z_k}^{m+1/2}] - \alpha\left((n_x)_k[\tilde{E}_{y_k}^m] - (n_y)_k[\tilde{E}_{x_k}^m]\right)\right)\tilde{E}_{y_k}^{[m+1/2]} \\ & + \frac{1}{2Y_k}\left(Y_k\left((n_x)_k[\tilde{E}_{y_k}^{m+1}] - (n_y)_k[\tilde{E}_{x_k}^{m+1}]\right) - \alpha[\tilde{H}_{z_k}^{m+1/2}]\right)\tilde{H}_{z_k}^{[m+1]} \\ & - \left((n_x)_k\tilde{E}_{y_k}^{m+1} - (n_y)_k\tilde{E}_{x_k}^{m+1}\right)\tilde{H}_{z_k}^{[m+1]}\Bigg) ds. \end{aligned} \tag{4.7}$$

Lemma 4.1. *Let B_{11}^m, B_{12}^m and B_{13}^m be defined by* (4.4), (4.5) *and* (4.6), *respectively, and* $B_1^m =$

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$B_{11}^m+B_{12}^m+B_{13}^m$. *Then*

$$\sum_{m=0}^{M-1}B_1^m\leq\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{1}{4(Z_i+Z_k)}\Bigg(-\alpha\Big((n_y)_{ki}[\tilde{E}_{x_k}^0]-(n_x)_{ki}[\tilde{E}_{y_k}^0]\Big)^2$$
$$+2\Big((n_x)_{ki}\Big(Z_i\tilde{E}_{y_k}^0+Z_k\tilde{E}_{yi}^0\Big)-(n_y)_{ki}\Big(Z_i\tilde{E}_{x_k}^0+Z_k\tilde{E}_{xi}^0\Big)\Big)[\tilde{H}_{z_k}^{1/2}]$$
$$+\alpha\Bigg(\Big((n_y)_{ki}[\tilde{E}_{x_k}^M]-(n_x)_{ki}[\tilde{E}_{y_k}^M]\Big)^2-\Big([\tilde{H}_{z_k}^{1/2}]^2-[\tilde{H}_{z_k}^{M+1/2}]^2\Big)\Bigg)$$
$$+2\Big((n_y)_{ki}\Big(Z_i\tilde{E}_{x_k}^M+Z_k\tilde{E}_{xi}^M\Big)-(n_x)_{ki}\Big(Z_i\tilde{E}_{y_k}^M+Z_k\tilde{E}_{yi}^M\Big)\Big)[\tilde{H}_{z_k}^{M+1/2}]\Bigg)ds.$$

Proof. Since $Z_i/(Z_i+Z_k)+Y_i/(Y_i+Y_k)=Z_k/(Z_i+Z_k)+Y_k/(Y_i+Y_k)=1$, $Z_i/(Z_i+Z_k)=Y_k(Y_i+Y_k)$ and $Z_k(Z_i+Z_k)=Y_i(Y_i+Y_k)$, summing from $m=0$ to $m=M-1$, we conclude

$$\sum_{m=0}^{M-1}B_{11}^m=\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{(n_y)_{ki}}{2(Z_i+Z_k)}\Bigg(-\Big(Z_i\tilde{E}_{x_k}^0+Z_k\tilde{E}_{xi}^0\Big)[\tilde{H}_{z_k}^{1/2}]+\alpha\Big((n_x)_{ki}[\tilde{E}_{y_k}^0]-(n_y)_{ki}[\tilde{E}_{x_k}^0]\Big)[\tilde{E}_{x_k}^0]$$
$$+\alpha\sum_{m=0}^{M-1}\Big((n_x)_{ki}[\tilde{E}_{y_k}^{m+1}]-(n_y)_{ki}[\tilde{E}_{x_k}^{m+1}]+(n_x)_{ki}[\tilde{E}_{y_k}^m]-(n_y)_{ki}[\tilde{E}_{x_k}^m]\Big)[\tilde{E}_{x_k}^{m+1}]$$
$$+\Big(Z_i\tilde{E}_{x_k}^M+Z_k\tilde{E}_{xi}^M\Big)[\tilde{H}_{z_k}^{M+1/2}]-\alpha\Big((n_x)_{ki}[\tilde{E}_{y_k}^M]-(n_y)_{ki}[\tilde{E}_{x_k}^M]\Big)[\tilde{E}_{x_k}^M]\Bigg)ds.$$

In the same way, for B_{12}^m we have

$$\sum_{m=0}^{M-1}B_{12}^m=\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{(n_x)_{ki}}{2(Z_i+Z_k)}\Bigg(\Big(Z_i\tilde{E}_{y_k}^0+Z_k\tilde{E}_{yi}^0\Big)[\tilde{H}_{z_k}^{1/2}]-\alpha\Big((n_x)_{ki}[\tilde{E}_{y_k}^0]-(n_y)_{ki}[\tilde{E}_{x_k}^0]\Big)[\tilde{E}_{y_k}^0]$$
$$-\alpha\sum_{m=0}^{M-1}\Big((n_x)_{ki}[\tilde{E}_{y_k}^{m+1}]-(n_y)_{ki}[\tilde{E}_{x_k}^{m+1}]+(n_x)_{ki}[\tilde{E}_{y_k}^m]-(n_y)_{ki}[\tilde{E}_{x_k}^m]\Big)[\tilde{E}_{y_k}^{m+1}]$$
$$-\Big(Z_i\tilde{E}_{y_k}^M+Z_k\tilde{E}_{yi}^M\Big)[\tilde{H}_{z_k}^{M+1/2}]+\alpha\Big((n_x)_{ki}[\tilde{E}_{y_k}^M]-(n_y)_{ki}[\tilde{E}_{x_k}^M]\Big)[\tilde{E}_{y_k}^M]\Bigg)ds,$$

and for B_{13}^m

$$\sum_{m=0}^{M-1}B_{13}^m=-\sum_{m=0}^{M-1}\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{\alpha}{2(Y_i+Y_k)}[\tilde{H}_{z_k}^{m+1/2}]\Big([\tilde{H}_{z_k}^{m+1/2}]+[\tilde{H}_{z_k}^{m+3/2}]\Big)ds.$$

Observing that, for general sequences $\{a^m\}$ and $\{b^m\}$, hold

$$\sum_{m=0}^{M-1}\Big(a^{m+1}+a^m\Big)a^{m+1}=\frac{1}{2}\Bigg(-(a^0)^2+(a^M)^2+\sum_{m=0}^{M-1}\Big(a^m+a^{m+1}\Big)^2\Bigg),$$

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$$\sum_{m=0}^{M-1}\left(a^{m+1}+a^m\right)b^{m+1}=\frac{1}{2}\left(-a^0b^0+a^Mb^M+\sum_{m=0}^{M-1}\left(a^mb^m+2a^mb^{m+1}+a^{m+1}b^{m+1}\right)\right),$$

we get

$$\begin{aligned}
&\sum_{m=0}^{M-1}(B_{11}^m+B_{12}^m)\\
&\leq\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{1}{4(Z_i+Z_k)}\Bigg(-\alpha(n_y)_{ki}^2\left(-[\tilde{E}_{x_k}^0]^2+[\tilde{E}_{x_k}^M]^2\right)\\
&\quad+\alpha(n_x)_{ki}(n_y)_{ki}\left(-[\tilde{E}_{x_k}^0][\tilde{E}_{y_k}^0]+[\tilde{E}_{x_k}^M][\tilde{E}_{y_k}^M]\right)-2(n_y)_{ki}\left(Z_i\tilde{E}_{x_k}^0+Z_k\tilde{E}_{xi}^0\right)[\tilde{H}_{z_k}^{1/2}]\\
&\quad+2\alpha(n_y)_{ki}\left((n_x)_{ki}[\tilde{E}_{y_k}^0]-(n_y)_{ki}[\tilde{E}_{x_k}^0]\right)[\tilde{E}_{x_k}^0]+2(n_y)_{ki}\left(Z_i\tilde{E}_{x_k}^M+Z_k\tilde{E}_{xi}^M\right)[\tilde{H}_{z_k}^{M+1/2}]\\
&\quad-2\alpha(n_y)_{ki}\left((n_x)_{ki}[\tilde{E}_{y_k}^M]-(n_y)_{ki}[\tilde{E}_{x_k}^M]\right)[\tilde{E}_{x_k}^M]-\alpha(n_x)_{ki}^2\left(-[\tilde{E}_{y_k}^0]^2+[\tilde{E}_{y_k}^M]^2\right)\\
&\quad+\alpha(n_x)_{ki}(n_y)_{ki}\left(-[\tilde{E}_{x_k}^0][\tilde{E}_{y_k}^0]+[\tilde{E}_{x_k}^M][\tilde{E}_{y_k}^M]\right)+2(n_x)_{ki}\left(Z_i\tilde{E}_{y_k}^0+Z_k\tilde{E}_{yi}^0\right)[\tilde{H}_{z_k}^{1/2}]\\
&\quad-2\alpha(n_x)_{ki}\left((n_x)_{ki}[\tilde{E}_{y_k}^0]-(n_y)_{ki}[\tilde{E}_{x_k}^0]\right)[\tilde{E}_{y_k}^0]-2(n_x)_{ki}\left(Z_i\tilde{E}_{y_k}^M+Z_k\tilde{E}_{yi}^M\right)[\tilde{H}_{z_k}^{M+1/2}]\\
&\quad+2\alpha(n_x)_{ki}\left((n_x)_{ki}[\tilde{E}_{y_k}^M]-(n_y)_{ki}[\tilde{E}_{x_k}^M]\right)[\tilde{E}_{y_k}^M]\Bigg)ds.
\end{aligned}$$

We also have

$$\begin{aligned}
\sum_{m=0}^{M-1}B_{13}^m&=-\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{\alpha}{4(Y_i+Y_k)}\left([\tilde{H}_{z_k}^{1/2}]^2-[\tilde{H}_{z_k}^{M+1/2}]^2+\sum_{m=0}^{M-1}\left([\tilde{H}_{z_k}^{m+1/2}]+[\tilde{H}_{z_k}^{m+3/2}]\right)^2\right)ds\\
&\leq-\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\frac{\alpha}{4(Y_i+Y_k)}\left([\tilde{H}_{z_k}^{1/2}]^2-[\tilde{H}_{z_k}^{M+1/2}]^2\right)ds,
\end{aligned}$$

which concludes the proof. □

Let us now analyze the term B_2^m for different kinds of boundary conditions.

Lemma 4.2. *Let B_2^m be defined by (4.7). Then*

$$\begin{aligned}
\sum_{m=0}^{M-1}B_2^m\leq\sum_{f_k\in F^{ext}}\int_{f_k}\frac{\beta_1}{4Z_k}&\left(-\left((n_y)_k\tilde{E}_{x_k}^0-(n_x)_k\tilde{E}_{y_k}^0\right)^2+\left((n_y)_k\tilde{E}_{x_k}^M-(n_x)_k\tilde{E}_{y_k}^M\right)^2\right)\\
&+\frac{\beta_2}{2}\left(\tilde{H}_{z_k}^{1/2}\left((n_x)_k\tilde{E}_{y_k}^0-(n_y)_k\tilde{E}_{x_k}^0-\frac{\beta_3}{2Y_k}\tilde{H}_{z_k}^{1/2}\right)\right.\\
&\left.-\tilde{H}_{z_k}^{M+1/2}\left((n_x)_k\tilde{E}_{y_k}^M-(n_y)_k\tilde{E}_{x_k}^M-\frac{\beta_3}{2Y_k}\tilde{H}_{z_k}^{M+1/2}\right)\right)ds,
\end{aligned}$$

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where $\beta_1=\alpha,\beta_2=0$ *for PEC,* $\beta_1=0,\beta_2=1$, $\beta_3=\alpha$ *for PMC, and* $\beta_1=\beta_2=\frac{1}{2}$, $\beta_3=1$ *for Silver-Müller boundary conditions.*

Proof. First we consider PEC boundary conditions. We have

$$B_2^m=\sum_{f_k\in F^{ext}}\int_{f_k}\frac{\alpha}{Z_k}\Bigg((n_y)_k\Big((n_x)_k\tilde{E}_{y_k}^m-(n_y)_k\tilde{E}_{x_k}^m\Big)\tilde{E}_{x_k}^{[m+1/2]}$$
$$-(n_x)_k\Big((n_x)_k\tilde{E}_{y_k}^m-(n_y)_k\tilde{E}_{x_k}^m\Big)\tilde{E}_{y_k}^{[m+1/2]}\Bigg)ds.$$

Summing from $m=0$ to $m=M-1$ we obtain

$$\sum_{m=0}^{M-1}B_2^m=\sum_{f_k\in F^{ext}}\int_{f_k}\frac{\alpha}{4Z_k}\Bigg(-\Big((n_x)_k\tilde{E}_{y_k}^0-(n_y)_k\tilde{E}_{x_k}^0\Big)^2+\Big((n_x)_k\tilde{E}_{y_k}^M-(n_y)_k\tilde{E}_{x_k}^M\Big)^2$$
$$-4\sum_{m=0}^{M-1}\Big((n_x)_k\tilde{E}_{y_k}^{[m+1/2]}-(n_y)_k\tilde{E}_{x_k}^{[m+1/2]}\Big)^2\Bigg)ds$$
$$\leq\sum_{f_k\in F^{ext}}\int_{f_k}\frac{\alpha}{4Z_k}\Bigg(-\Big((n_x)_k\tilde{E}_{y_k}^0-(n_y)_k\tilde{E}_{x_k}^0\Big)^2+\Big((n_x)_k\tilde{E}_{y_k}^M-(n_y)_k\tilde{E}_{x_k}^M\Big)^2\Bigg)ds.$$

For PMC boundary conditions we have

$$B_2^m=\sum_{f_k\in F^{ext}}\int_{f_k}\Bigg(\tilde{H}_{z_k}^{m+1/2}\Big((n_x)_k\tilde{E}_{y_k}^{[m+1/2]}-(n_y)_k\tilde{E}_{x_k}^{[m+1/2]}\Big)$$
$$-\Big(\frac{\alpha}{Y_k}\tilde{H}_{z_k}^{m+1/2}+(n_x)_k\tilde{E}_{y_k}^{m+1}-(n_y)_k\tilde{E}_{x_k}^{m+1}\Big)\tilde{H}_{z_k}^{[m+1]}\Bigg)ds.$$

Summing from $m=0$ to $m=M-1$ results

$$\sum_{m=0}^{M-1}B_2^m\leq\frac{1}{2}\sum_{f_k\in F^{ext}}\int_{f_k}\Bigg(\tilde{H}_{z_k}^{1/2}\Big((n_x)_k\tilde{E}_{y_k}^0-(n_y)_k\tilde{E}_{x_k}^0-\frac{\alpha}{2Y_k}\tilde{H}_{z_k}^{1/2}\Big)$$
$$-\tilde{H}_{z_k}^{M+1/2}\Big((n_x)_k\tilde{E}_{y_k}^M-(n_y)_k\tilde{E}_{x_k}^M-\frac{\alpha}{2Y_k}\tilde{H}_{z_k}^{M+1/2}\Big)\Bigg)ds.$$

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For Silver-Müller absorbing boundary conditions we have

$$B_2^m = \frac{1}{2}\sum_{f_k\in F^{ext}}\int_{f_k}\Bigg(\left(-(n_y)_k\tilde{H}_{z_k}^{m+1/2}+\frac{(n_y)_k}{Z_k}\left((n_x)_k\tilde{E}_{y_k}^m-(n_y)_k\tilde{E}_{x_k}^m\right)\right)\tilde{E}_{x_k}^{[m+1/2]}$$
$$+\left((n_x)_k\tilde{H}_{z_k}^{m+1/2}-\frac{(n_x)_k}{Z_k}\left((n_x)_k\tilde{E}_{y_k}^m-(n_y)_k\tilde{E}_{x_k}^m\right)\right)\tilde{E}_{y_k}^{[m+1/2]}$$
$$-\left(\frac{1}{Y_k}\tilde{H}_{z_k}^{m+1/2}+(n_x)_k\tilde{E}_{y_k}^{m+1}-(n_y)_k\tilde{E}_{x_k}^{m+1}\right)\tilde{H}_{z_k}^{[m+1]}\Bigg)ds.$$

Summing from $m=0$ to $m=M-1$, and taking into account the previous cases, we deduce

$$\sum_{m=0}^{M-1}B_2^m \le \sum_{f_k\in F^{ext}}\int_{f_k}\frac{1}{8Z_k}\Bigg(-\left((n_y)_k\tilde{E}_{x_k}^0-(n_x)_k\tilde{E}_{y_k}^0\right)^2+\left((n_y)_k\tilde{E}_{x_k}^M-(n_x)_k\tilde{E}_{y_k}^M\right)^2\Bigg)$$
$$+\frac{1}{4}\Bigg(\tilde{H}_{z_k}^{1/2}\left((n_x)_k\tilde{E}_{y_k}^0-(n_y)_k\tilde{E}_{x_k}^0-\frac{1}{2Y_k}\tilde{H}_{z_k}^{1/2}\right)$$
$$-\tilde{H}_{z_k}^{M+1/2}\left((n_x)_k\tilde{E}_{y_k}^M-(n_y)_k\tilde{E}_{x_k}^M-\frac{1}{2Y_k}\tilde{H}_{z_k}^{M+1/2}\right)\Bigg)ds,$$

which concludes the proof. □

Theorem 4.1. *Let us consider the leap-frog DG method* (3.2)-(3.4) *complemented with the discrete boundary conditions defined in Section* 3.1. *If the time step* Δt *is such that*

$$\Delta t < \frac{\min\{\underline{\epsilon},\underline{\mu}\}}{\max\{C_E,C_H\}}\min\{h_k\}, \tag{4.8}$$

where

$$C_E = \frac{1}{2}C_{inv}N^2+C_\tau^2(N+1)(N+2)\left(2+\beta_2+\frac{2\alpha+\beta_1}{2\min\{Z_k\}}\right),$$
$$C_H = \frac{1}{2}C_{inv}N^2+C_\tau^2(N+1)(N+2)\left(2+\beta_2+\frac{\alpha+\beta_2\beta_3}{\min\{Y_k\}}\right),$$

with C_τ *defined by* (A.4) *of Lemma* A.1 *and* C_{inv} *defined by* (A.6) *of Lemma* A.2, *and* $\beta_1=\alpha,\beta_2=0$ *for PEC,* $\beta_1=0,\beta_2=1$, $\beta_3=\alpha$ *for PMC, and* $\beta_1=\beta_2=\frac{1}{2}$, $\beta_3=1$ *for Silver-Müller boundary conditions, then the method is stable.*

Proof. From (4.3) and the previous lemmata, considering the Cauchy-Schwarz's inequal-

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ity and taking into account that $Z_i/(Z_i+Z_k)<1$, we obtain

$$\begin{aligned}
&\sum_{T_k\in\mathcal{T}_h}\left(\left(\epsilon\tilde{E}_k^M,\tilde{E}_k^M\right)_{T_k}+\left(\mu\tilde{H}_{z_k}^{M+1/2},\tilde{H}_{z_k}^{M+1/2}\right)_{T_k}\right)\\
\leq&\sum_{T_k\in\mathcal{T}_h}\left(\left(\epsilon\tilde{E}_k^0,\tilde{E}_k^0\right)_{T_k}+\left(\mu\tilde{H}_{z_k}^{1/2},\tilde{H}_{z_k}^{1/2}\right)_{T_k}\right)\\
&\quad+\Delta t\sum_{T_k\in\mathcal{T}_h}\left(\|\nabla\times\tilde{H}_{z_k}^{1/2}\|_{L^2(T_k)}\|\tilde{E}_k^0\|_{L^2(T_k)}+\|\nabla\times\tilde{H}_{z_k}^{M+1/2}\|_{L^2(T_k)}\|\tilde{E}_k^M\|_{L^2(T_k)}\right)\\
&\quad+2\Delta t\sum_{f_{ik}\in F^{int}}\left(\|\tilde{E}_k^M\|_{L^2(f_{ik})}\|[\tilde{H}_{z_k}^{M+1/2}]\|_{L^2(f_{ik})}+\|\tilde{E}_k^0\|_{L^2(f_{ik})}\|[\tilde{H}_{z_k}^{1/2}]\|_{L^2(f_{ik})}\right)\\
&\quad+\frac{\alpha\Delta t}{4\min\{Z_k\}}\sum_{f_{ik}\in F^{int}}\|[\tilde{E}_k^M]\|_{L^2(f_{ik})}^2+\frac{\alpha\Delta t}{4\min\{Y_k\}}\sum_{f_{ik}\in F^{int}}\|[\tilde{H}_{z_k}^{M+1/2}]\|_{L^2(f_{ik})}^2\\
&\quad+\frac{\beta_1\Delta t}{2\min\{Z_k\}}\sum_{f_k\in F^{ext}}\|\tilde{E}_k^M\|_{L^2(f_k)}^2+\frac{\beta_2\beta_3\Delta t}{\min\{Y_k\}}\sum_{f_k\in F^{ext}}\|\tilde{H}_{z_k}^{M+1/2}\|_{L^2(f_k)}^2\\
&\quad+2\beta_2\Delta t\sum_{f_k\in F^{ext}}\left(\|\tilde{H}_{z_k}^{1/2}\|_{L^2(f_k)}\|\tilde{E}_k^0\|_{L^2(f_k)}+\|\tilde{H}_{z_k}^{M+1/2}\|_{L^2(f_k)}\|\tilde{E}_k^M\|_{L^2(f_k)}\right).
\end{aligned}$$

Using the inequality (A.4) of Lemma A.1 and the inequality (A.6) of Lemma A.2 (both in Appendix), we get

$$\begin{aligned}
&\min\{\underline{\epsilon},\underline{\mu}\}\left(\|\tilde{E}^M\|_{L^2(\Omega)}^2+\|\tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}^2\right)\\
\leq&\max\{\bar{\epsilon},\bar{\mu}\}\left(\|\tilde{E}^0\|_{L^2(\Omega)}^2+\|\tilde{H}_z^{1/2}\|_{L^2(\Omega)}^2\right)\\
&\quad+\frac{\Delta t}{2}C_{inv}N^2\max\left\{h_k^{-1}\right\}\left(\|\tilde{H}_z^{1/2}\|_{L^2(\Omega)}^2+\|\tilde{E}^0\|_{L^2(\Omega)}^2+\|\tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}^2+\|\tilde{E}^M\|_{L^2(\Omega)}^2\right)\\
&\quad+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{2\alpha+\beta_1}{2\min\{Z_k\}}\right)\|\tilde{E}^M\|_{L^2(\Omega)}^2\\
&\quad+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{\alpha+\beta_2\beta_3}{\min\{Y_k\}}\right)\|\tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}^2\\
&\quad+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}(2+\beta_2)\left(\|\tilde{E}^0\|_{L^2(\Omega)}^2+\|\tilde{H}_z^{1/2}\|_{L^2(\Omega)}^2\right).
\end{aligned}$$

and so, taking $C_0=\frac{1}{2}C_{inv}N^2+C_\tau^2(N+1)(N+2)(2+\beta_2)$,

$$\begin{aligned}
&\left(\min\{\underline{\epsilon},\underline{\mu}\}-\Delta t\max\left\{h_k^{-1}\right\}\max\{C_E,C_H\}\right)\left(\|\tilde{E}^M\|_{L^2(\Omega)}^2+\|\tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}^2\right)\\
\leq&\left(\max\{\bar{\epsilon},\bar{\mu}\}+\Delta t\max\left\{h_k^{-1}\right\}C_0\right)\left(\|\tilde{E}^0\|_{L^2(\Omega)}^2+\|\tilde{H}_z^{1/2}\|_{L^2(\Omega)}^2\right),
\end{aligned}$$

which concludes the proof. □

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The stability condition (4.8) shows that the method is conditionally stable, which is natural since we are considering an explicit time discretization. Furthermore, it discloses the influence of the values of α, $h_{\min}$ and N on the bounds of the stable region. This is of utmost importance to balance accuracy *versus* stability.

For the completeness of the study of the method we present the error estimates in the following Theorem 4.2. To provide a proper functional setting, we need to define spaces involving time-dependent functions [8]. Let X denote a Banach space with norm $\|.\|_X$. The spaces $L^2(0,T;X)$ and $L^\infty(0,T;X)$ consist, respectively, of all measurable functions $v:[0,T]\to X$ with

$$\|v\|_{L^2(0,T;X)}=\Big(\int_0^T \|v(t)\|_X^2\,dt\Big)^{1/2}<\infty,\quad \|v\|_{L^\infty(0,T;X)}=\operatorname*{esssup}_{0\le t\le T}\|v(t)\|_X<\infty.$$

In what follows, X is shorthand for any of the usual Sobolev spaces $H^p(\Omega)$ or the Banach space $L^\infty(\Omega)$. By $\|.\|_{L^2(\Omega)}$ we denote the classical norm in $L^2(\Omega)$ or $(L^2(\Omega))^2$, depending on the context, and we use the same type of notation for the norms and semi-norms in other Sobolev spaces $H^p(\Omega)$.

Theorem 4.2. *Let us consider the leap-frog DG method* (3.2)-(3.4) *complemented with the discrete boundary conditions defined in Section* 3.1 *and suppose that the solution of the Maxwell's equations* (2.2)-(2.3) *complemented by* (2.5), (2.6) *or* (2.7) *has the following regularity:* $E_x,E_y,H_z\in L^\infty(0,T_f;H^{s+1}(\Omega))$, $\frac{\partial E_x}{\partial t},\frac{\partial E_y}{\partial t},\frac{\partial H_z}{\partial t}\in L^2(0,T_f;H^{s+1}(\Omega)\cap L^\infty(\partial\Omega))$ *and* $\frac{\partial^2 E_x}{\partial t^2},\frac{\partial^2 E_y}{\partial t^2},\frac{\partial^2 H_z}{\partial t^2}\in L^2(0,T_f;H^1(\Omega))$, $s\ge 0$. *If the time step* Δt *satisfies*

$$\Delta t\le\frac{\min\{\underline{\epsilon},\underline{\mu}\}}{\max\{C_E,C_H\}}\min\{h_k\}(1-\delta),\quad 0<\delta<1, \tag{4.9}$$

where C_E *and* C_H *are the constants defined in Theorem* 4.1, *then, for the case of PEC and PMC boundary conditions, holds*

$$\begin{aligned}&\max_{1\le m\le M}\Big(\|E^m-\tilde{E}^m\|_{L^2(\Omega)}+\|H_z^{m+1/2}-\tilde{H}_z^{m+1/2}\|_{L^2(\Omega)}\Big)\\&\le C(\Delta t^2+h^{\min\{s,N\}})+C\Big(\|E^0-\tilde{E}^0\|_{L^2(\Omega)}+\|H_z^{1/2}-\tilde{H}_z^{1/2}\|_{L^2(\Omega)}\Big)\end{aligned}$$

and, for the case of Silver-Müller absorbing boundary conditions, holds

$$\begin{aligned}&\max_{1\le m\le M}\Big(\|E^m-\tilde{E}^m\|_{L^2(\Omega)}+\|H_z^{m+1/2}-\tilde{H}_z^{m+1/2}\|_{L^2(\Omega)}\Big)\\&\le C(\Delta t+h^{\min\{s,N\}})+C\Big(\|E^0-\tilde{E}^0\|_{L^2(\Omega)}+\|H_z^{1/2}-\tilde{H}_z^{1/2}\|_{L^2(\Omega)}\Big),\end{aligned}$$

where C *is a generic constant independent of* Δt *and the mesh size* h.

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Proof. The key idea for the proof is to find a variational system for the difference between the numerical solution and a projection of (E_x,E_y,H_z) onto the space V_N.

Let $(\mathcal{P}_N E_x,\mathcal{P}_N E_y,\mathcal{P}_N H_z)\in V_N$ be an interpolant of (E_x,E_y,H_z) having the optimal approximation errors given by the *hp*-approximation properties (A.7) and (A.8). On the external boundary we define the jumps $[\mathcal{P}_N E_x]=2\mathcal{P}_N E_x$, $[\mathcal{P}_N E_y]=2\mathcal{P}_N E_y$, $[\mathcal{P}_N H_z]=0$ and $[\mathcal{P}_N E_x]=0$, $[\mathcal{P}_N E_y]=0$, $[\mathcal{P}_N H_z]=2\mathcal{P}_N H_z$, for PEC and PMC boundary conditions, respectively. For Silver-Müller absorbing boundary conditions we consider $[\mathcal{P}_N E_x]=\mathcal{P}_N E_x$, $[\mathcal{P}_N E_y]=\mathcal{P}_N E_y$, $[\mathcal{P}_N H_z]=\mathcal{P}_N H_z$. Let us consider the notations $\xi_x^m=\mathcal{P}_N E_x^m-\tilde{E}_x^m$, $\xi_y^m=\mathcal{P}_N E_y^m-\tilde{E}_y^m$, $\xi^m=(\xi_x^m,\xi_y^m)$, and $\eta_z^{m+1/2}=\mathcal{P}_N H_z^{m+1/2}-\tilde{H}_z^{m+1/2}$.

Taking into account lemmas A.1, A.2 and A.3, we obtain

$$
\begin{aligned}
&\min\{\epsilon,\underline{\mu}\}\left(\|\xi^M\|^2_{L^2(\Omega)}+\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}\right)\\
&\le\max\{\bar{\epsilon},\bar{\mu}\}\left(\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{1/2}\|^2_{L^2(\Omega)}\right)\\
&\quad+\frac{\Delta t}{2}C_{inv}N^2\max\left\{h_k^{-1}\right\}\left(\|\eta_z^{1/2}\|^2_{L^2(\Omega)}+\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}+\|\xi^M\|^2_{L^2(\Omega)}\right)\\
&\quad+\Delta t\delta(7+\beta_4)\left(\sum_{m=1}^{M-1}\|\xi^m\|^2_{L^2(\Omega)}+\sum_{m=1}^{M-1}\|\eta_z^{m+1/2}\|^2_{L^2(\Omega)}\right)\\
&\quad+\frac{\Delta t\delta}{2}(7+\beta_4)\left(\|\xi^0\|^2_{L^2(\Omega)}+\|\xi^M\|^2_{L^2(\Omega)}+\|\eta_{z_k}^{1/2}\|^2_{L^2(\Omega)}+\|\eta_{z_k}^{M+1/2}\|^2_{L^2(\Omega)}\right)\\
&\quad+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{2\alpha+\beta_1}{2\min\{Z_k\}}\right)\|\xi^M\|^2_{L^2(\Omega)}\\
&\quad+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}\left(2+\beta_2+\frac{2\alpha+2\beta_2\beta_3}{2\min\{Y_k\}}\right)\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}\\
&\quad+C_\tau^2(N+1)(N+2)\Delta t\max\left\{h_k^{-1}\right\}(2+\beta_2)\left(\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{1/2}\|^2_{L^2(\Omega)}\right)\\
&\quad+\frac{Ch^{2\sigma}\Delta t}{\delta N^{2p}}\left(\overline{\epsilon}\int_0^T\left\|\frac{\partial E}{\partial t}\right\|^2_{H^p(\Omega)}dt+\overline{\mu}\int_{\Delta t/2}^{T+\Delta t/2}\left\|\frac{\partial H_z}{\partial t}\right\|^2_{H^p(\Omega)}dt\right)\\
&\quad+\frac{2C\Delta t^4}{\delta}\left(\int_{\Delta t/2}^{T+\Delta t/2}\left|\frac{\partial^2 E}{\partial t^2}\right|^2_{H^1(\Omega)}dt+\int_0^T\left|\frac{\partial^2 H_z}{\partial t^2}\right|^2_{H^1(\Omega)}dt\right)\\
&\quad+\frac{2CTh^{2\sigma-2}}{\delta N^{2p+1}}\left(N^3+\left(1+\frac{\alpha}{\min\{Z_k^2\}}\right)C_\tau^2(N+1)(N+2)\right)\|E\|^2_{L^\infty(0,T;H^p(\Omega))}\\
&\quad+\frac{2CTh^{2\sigma-2}}{\delta N^{2p+1}}\left(N^3+\left(1+\frac{\alpha}{\min\{Y_k^2\}}\right)C_\tau^2(N+1)(N+2)\right)\|H_z\|^2_{L^\infty(0,T;H^p(\Omega))}\\
&\quad+\frac{\beta_4C_\tau^2(N+1)(N+2)\Delta t^2}{8\delta\min\{Z_k^2\}}\int_0^{T-\Delta t/2}\left\|\frac{\partial E}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt\\
&\quad+\frac{\beta_4C_\tau^2(N+1)(N+2)\Delta t^2}{16\delta\min\{Y_k^2\}}\int_{\Delta t/2}^{T}\left\|\frac{\partial H_z}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt,
\end{aligned}
$$

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where $p\geq 0$, $\sigma=\min(p,N+1)$, C and C_τ are constants independent of $(E_{x_k},E_{y_k},H_{z_k})$, h_k and N, δ is an arbitrary positive constant, $\beta_4=0$ for PEC and PMC boundary conditions and $\beta_4=1$ for Silver-Müller absorbing boundary conditions.

If (4.9) holds, using the discrete Gronwall's Lemma (see *e.g.* [7,22]), we obtain

$$\begin{aligned}
&\|\xi^M\|^2_{L^2(\Omega)}+\|\eta_z^{M+1/2}\|^2_{L^2(\Omega)}\\
\leq& C(\epsilon,\mu,N)\Bigg(\|\xi^0\|^2_{L^2(\Omega)}+\|\eta_z^{1/2}\|^2_{L^2(\Omega)}\\
&+\Delta t h^{2\sigma}\int_0^T\left\|\frac{\partial E}{\partial t}\right\|^2_{H^p(\Omega)}dt+\Delta t h^{2\sigma}\int_{\Delta t/2}^{T+\Delta t/2}\left\|\frac{\partial H_z}{\partial t}\right\|^2_{H^p(\Omega)}dt\\
&+\Delta t^4\int_0^T\left|\frac{\partial^2 E}{\partial t^2}\right|^2_{H^1(\Omega)}dt+\Delta t^4\int_{\Delta t/2}^{T+\Delta t/2}\left|\frac{\partial^2 H_z}{\partial t^2}\right|^2_{H^1(\Omega)}dt\\
&+h^{2\sigma-2}\|E^m\|^2_{L^\infty(0,T;H^p(\Omega))}+h^{2\sigma-2}\|H_z^{m+1/2}\|^2_{L^\infty(0,T;H^p(\Omega))}\\
&+\beta_4\Delta t^2\int_0^{T-\Delta t/2}\left\|\frac{\partial E}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt+\beta_4\Delta t^2\int_{\Delta t/2}^{T}\left\|\frac{\partial H_z}{\partial t}\right\|^2_{L^\infty(\partial\Omega)}dt\Bigg).
\end{aligned}$$

We complete the proof by using the triangle inequality and the hp-approximation properties of Lemma A.3 to estimate $\mathcal{P}_N E^M_{x_k}-E^M_{x_k}$, $\mathcal{P}_N E^M_{y_k}-E^M_{y_k}$ and $\mathcal{P}_N H^{M+1/2}_{z_k}-H^{M+1/2}_{z_k}$. □

Remark 4.1. We want to remark that in the case of Silver-Müller absorbing boundary conditions we only get first order convergence in time. A possible way to recover second order convergence is to consider a locally implicit time scheme (see *e.g.* [9]). In order to keep efficiency, we propose an alternative which is explicit and second order convergent in time: for each time step, solve (3.2)-(3.4) and save the solution in the variables $(\tilde{\tilde{E}}^{m+1}_{x_k},\tilde{\tilde{E}}^{m+1}_{y_k},\tilde{\tilde{H}}^{m+3/2}_{z_k})$. Then the numerical solution $(\tilde{E}^{m+1}_{x_k},\tilde{E}^{m+1}_{y_k},\tilde{H}^{m+3/2}_{z_k})$ is computed replacing in (3.2)-(3.4) the numerical flux by the following expression

$$\begin{pmatrix}
\frac{-n_y}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x\frac{[\tilde{E}_y^m]+[\tilde{\tilde{E}}_y^{m+1}]}{2}-n_y\frac{[\tilde{E}_x^m]+[\tilde{\tilde{E}}_x^{m+1}]}{2}\right)\right)\\
\frac{n_x}{Z^++Z^-}\left(Z^+[\tilde{H}_z^{m+1/2}]-\alpha\left(n_x\frac{[\tilde{E}_y^m]+[\tilde{\tilde{E}}_y^{m+1}]}{2}-n_y\frac{[\tilde{E}_x^m]+[\tilde{\tilde{E}}_x^{m+1}]}{2}\right)\right)\\
\frac{1}{Y^++Y^-}\left(Y^+\left(n_x[\tilde{E}_y^{m+1}]-n_y[\tilde{E}_x^{m+1}]\right)-\alpha\frac{[\tilde{H}_z^{m+1/2}]+[\tilde{\tilde{H}}_z^{m+3/2}]}{2}\right)
\end{pmatrix}.$$

5 Numerical results

In this section we present numerical results that support the theoretical results derived in the previous section.

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We will check numerically that (4.8) defines a sharp stability condition, in terms of the influence of N and $h_{\min}=\min\{h_k\}$. In our experiments, we computed C that satisfies

$$\Delta t_{\max}=\frac{C}{(N+1)(N+2)}h_{\min}, \tag{5.1}$$

where $\Delta t_{\max}$ is the maximum observed value of Δt such that the method is stable. For these tests, the domain is the square $\Omega=(-1,1)^2$, the simulation final time is fixed at $T=1$, we consider a symmetric and positive definite anisotropic constant permittivity tensor (2.4), with $\epsilon_{xx}=5$, $\epsilon_{xy}=\epsilon_{yx}=1$ and $\epsilon_{yy}=3$, and $\mu=1$. We consider Eqs. (2.2)-(2.3) with initial conditions $E_x(x,y,0)=0$, $E_y(x,y,0)=0$, $H_z(x,y,\Delta t/2)=\cos(\pi x)\cos(\pi y)\cos(\omega\Delta t/2)$, where $\omega=\pi\sqrt{1/\epsilon_{xx}+1/\epsilon_{yy}}$, for the case of PEC boundary conditions and $E_x(x,y,0)=0$, $E_y(x,y,0)=0$, $H_z(x,y,\Delta t/2)=\sin(\pi\Delta t/2)\sin(\pi xy)$ for the case of Silver-Müller absorbing boundary conditions.

In Table 1 and Table 2 the results are computed for different mesh sizes, considering respectively central and upwind fluxes in the DG method, for the case of PEC boundary conditions, while in Table 3 and Table 4, the results are computed for the case of Silver-Müller boundary conditions.

Table 1: $\Delta t_{\max}$ such that the method is stable and C computed by (5.1) for PEC and central flux.

$h_{\min}$	$N=1$		$N=2$		$N=3$		$N=4$		$N=5$	
	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C
0.5657	0.17	1.80	0.1	2.12	0.065	2.30	0.044	2.33	0.032	2.37
0.2828	0.088	1.87	0.05	2.12	0.031	2.20	0.021	2.23	0.016	2.37
0.1414	0.044	1.87	0.024	2.04	0.015	2.12	0.01	2.12	0.0078	2.32
0.0707	0.021	1.78	0.012	2.04	0.0078	2.20	0.0054	2.30	0.0038	2.26
0.0354	0.01	1.70	0.006	2.04	0.0039	2.20	0.0027	2.30	0.0019	2.26
0.0177	0.0054	1.83	0.003	2.04	0.0019	2.15	0.0013	2.21	0.00095	2.26

Table 2: $\Delta t_{\max}$ such that the method is stable and C computed by (5.1) for PEC and upwind flux.

$h_{\min}$	$N=1$		$N=2$		$N=3$		$N=4$		$N=5$	
	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C
0.5657	0.10	1.06	0.056	1.19	0.034	1.20	0.023	1.22	0.016	1.19
0.2828	0.047	1.00	0.026	1.10	0.016	1.13	0.011	1.17	0.0081	1.20
0.1414	0.023	0.98	0.012	1.02	0.008	1.13	0.0054	1.15	0.0039	1.16
0.0707	0.011	0.93	0.0062	1.05	0.0039	1.10	0.0026	1.10	0.0019	1.13
0.0354	0.0055	0.93	0.003	1.02	0.0019	1.07	0.0013	1.10	0.0009	1.07
0.0177	0.0027	0.92	0.0015	1.02	0.0009	1.02	0.0006	1.02	0.0004	0.95

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Table 3: $\Delta t_{\max}$ such that the method is stable and C computed by (5.1) for SM-ABC and central flux.

$h_{\min}$	$N=1$		$N=2$		$N=3$		$N=4$		$N=5$	
	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C
0.5657	0.18	1.91	0.1	2.12	0.064	2.26	0.044	2.33	0.031	2.30
0.2828	0.092	1.95	0.05	2.12	0.031	2.19	0.021	2.02	0.015	2.23
0.1414	0.044	1.87	0.024	2.04	0.015	2.12	0.01	2.12	0.0079	2.35
0.0707	0.021	1.78	0.012	2.04	0.0077	2.18	0.0053	2.25	0.0038	2.26
0.0354	0.01	1.70	0.006	2.04	0.0038	2.15	0.0026	2.21	0.0019	2.26
0.0177	0.0053	1.80	0.003	2.04	0.0018	2.04	0.0012	2.04	0.00095	2.26

Table 4: $\Delta t_{\max}$ such that the method is stable and C computed by (5.1) for SM-ABC and upwind flux.

$h_{\min}$	$N=1$		$N=2$		$N=3$		$N=4$		$N=5$	
	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C	$\Delta t_{\max}$	C
0.5657	0.11	1.17	0.057	1.21	0.035	1.24	0.023	1.22	0.016	1.19
0.2828	0.051	1.08	0.026	1.10	0.016	1.13	0.011	1.17	0.008	1.19
0.1414	0.023	0.98	0.012	1.02	0.008	1.13	0.0054	1.15	0.0039	1.16
0.0707	0.011	0.93	0.0061	1.04	0.0039	1.10	0.0026	1.10	0.0019	1.13
0.0354	0.0055	0.93	0.003	1.02	0.0018	1.07	0.0013	1.10	0.00097	1.15
0.0177	0.0027	0.92	0.0015	1.02	0.00097	1.10	0.00065	1.10	0.00045	1.07

As expected from the condition (4.8), the numerical examples show that the stability regions corresponding to central fluxes are slightly bigger when compared to the regions obtained using upwind fluxes. From all the examples presented, we may deduce that the right hand side of (4.8) is a sharp bound for $\Delta t_{\max}$. Moreover, we can also conclude that $\Delta t_{\max}$ is directly proportional to $h_{\min}$ and inversely proportional to $(N+1)(N+2)$.

To illustrate the error estimate, we consider the model problem (2.2)-(2.3) defined in the square $\Omega=(-1,1)^2$, complemented by initial conditions and Silver-Müller absorbing boundary conditions (2.7). In order to make it easier to find examples with known exact solution and consequently with the possibility to compute the error of the numerical solution, source terms were introduced. The simulation time is fixed at $T=1$ and in all tests we assume that $\mu=1$ and ϵ is given by (2.4), with $\epsilon_{xx}=4x^2+y^2+1$, $\epsilon_{yy}=x^2+1$ and $\epsilon_{xy}=\epsilon_{yx}=\sqrt{x^2+y^2}$.

For the computation of convergence rates in space, we herein use rate $=\log(\text{error}/\text{error}^*)/\log(h/h^*)$, where error and error* denote the respective L^2-errors, $\|E^M-\tilde{E}^M\|_{L^2(\Omega)}+\|H_z^{M+1/2}-\tilde{H}_z^{M+1/2}\|_{L^2(\Omega)}$, computed for two consecutive meshes of sizes h and h^*. For the computation of convergence rates in time we proceed in a correspondent way computing the errors of the numerical solutions for two different values of Δt.

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To illustrate the order of convergence in space, we fix $\Delta t = 10^{-5}$, except for $N=4$ where we consider $\Delta t = 10^{-6}$. In Table 5 we compute the discrete L^2-error considering different degrees for the polynomial approximation while the spatial mesh is refined for both central and upwind fluxes. For central flux we find that the numerical convergence rate is close to the value estimated in Theorem 4.2, $\mathcal{O}(h^N)$, and for upwind flux we observe higher order of convergence, up to $\mathcal{O}(h^{N+1})$ in some cases.

Table 5: The error and the order of convergence in space for SM-ABC.

N	K	h	error	rate	error	rate
			Central flux		Upwind flux	
1	32	7.07E-01	6.18E-01		2.83E-01	
	50	5.66E-01	4.51E-01	1.42	1.70E-01	2.28
	200	2.83E-01	2.28E-1	0.99	3.75E-02	2.18
	800	1.41E-01	1.16E-1	0.97	6.90E-03	2.45
2	32	7.07E-01	1.61E-01		4.97E-02	
	50	5.66E-01	9.74E-02	2.27	2.31E-02	3.43
	200	2.83E-01	2.23E-02	2.13	2.90E-03	2.98
	800	1.41E-01	5.00E-03	2.14	3.12E-04	3.23
3	32	7.07E-01	2.93E-02		9.90E-03	
	50	5.66E-01	1.42E-02	3.24	3.80E-03	4.26
	200	2.83E-01	2.40E-03	2.59	2.48E0-4	3.94
	800	1.41E-01	3.16E-04	2.90	1.58E-05	3.97
4	32	7.07E-01	4.06E-03		1.40E-03	
	50	5.66E-01	1.90E-03	3.91	4.38E-04	5.04
	200	2.83E-01	1.61E-04	3.57	1.36E-05	5.01
	800	1.41E-01	9.39E-06	4.11	8.45E-07	4.01

We can visualise the convergence order in time in Table 6, where the polynomials degree and the number of elements have been set to $N=8$ and $K=800$, respectively, for upwind flux. The results exhibit first order convergency in time for the leap-frog method

Table 6: The error and the order of convergence in time for SM-ABC.

Δt	error	rate	error	rate
	Leap-frog		Modified leap-frog	
1.00E-03	7.04E-04		1.27E-04	
5.00E-04	3.52E-04	1.00	2.84E-05	2.16
2.50E-04	1.76E-04	1.00	6.79E-06	2.07
1.25E-04	8.79E-05	1.00	1.66E-06	2.03
6.25E-05	4.39E-05	1.00	4.10E-07	2.02

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and show that the second order is recovered when the modified leap-frog method is considered.

6 Stability of the 3 D model

In this section we extend the analysis in Section 4 for the TE mode of Maxwell's equations in two-dimensions to the full three-dimensional time-dependent Maxwell equations (2.1), defined on a bounded polyhedral domain $\Omega\subset\mathbb{R}^3$.

We assume that electric permittivity and the magnetic permeability tensors ϵ and μ are symmetric and uniformly positive definite for almost every $(x,y,z)\in\Omega$, and are uniformly bounded with a strictly positive lower bound, *i.e.*, there are constants $\underline{\epsilon}>0$, $\overline{\epsilon}>0$ and $\underline{\mu}>0$, $\overline{\mu}>0$ such that, for almost every $(x,y,z)\in\Omega$,

$$\underline{\epsilon}|\xi|^2\leq\xi^T\epsilon(x,y,z)\xi\leq\overline{\epsilon}|\xi|^2,\qquad \underline{\mu}|\xi|^2\leq\xi^T\mu(x,y,z)\xi\leq\overline{\mu}|\xi|^2,\qquad \forall\xi\in\mathbb{R}^3.$$

Let us define an effective permeability (in the same way as the effective permittivity) by $\mu_{eff}=\det(\mu)/(n^T\mu n)$. Now the speed with which a wave travels along the direction of the unit normal is given by $c=\sqrt{1/(\mu_{eff}\epsilon_{eff})}$.

We assume that Ω is partitioned into K disjoint tetrahedral elements T_k. The leap-frog discontinuous Galerkin method is the natural extension of the formulation (3.2)-(3.4) to the three-dimensional domain, with the numerical flux defined by

$$\begin{pmatrix}\frac{-1}{Z^++Z^-}n\times\left(Z^+[\tilde{H}]-\alpha n\times[\tilde{E}]\right)\\ \frac{1}{Y^++Y^-}n\times\left(Y^+[\tilde{E}]+\alpha n\times[\tilde{H}]\right)\end{pmatrix}.$$

We start by noticing that the following equalities hold

$$\begin{aligned}&\left(\epsilon\tilde{E}_k^{m+1},\tilde{E}_k^{m+1}\right)_{T_k}-\left(\epsilon\tilde{E}_k^{m},\tilde{E}_k^{m}\right)_{T_k}\\ =&2\Delta t\left(\nabla\times\tilde{H}_k^{m+1/2},\tilde{E}_k^{[m+1/2]}\right)_{T_k}\\ &-2\Delta t\left(\frac{1}{Z^++Z^-}n\times\left(Z^+[\tilde{H}^{m+1/2}]-\alpha n\times[\tilde{E}^m]\right),\tilde{E}_k^{[m+1/2]}\right)_{\partial T_k}\end{aligned}\tag{6.1}$$

and

$$\begin{aligned}&\left(\mu\tilde{H}_k^{m+3/2},\tilde{H}_k^{m+3/2}\right)_{T_k}-\left(\mu\tilde{H}_k^{m+1/2},\tilde{H}_k^{m+1/2}\right)_{T_k}\\ =&-2\Delta t\left(\nabla\times\tilde{E}_k^{m+1},\tilde{H}_k^{[m+1]}\right)_{T_k}\\ &+2\Delta t\left(\frac{1}{Y^++Y^-}n\times\left(Y^+[\tilde{E}^{m+1}]+\alpha n\times[\tilde{H}^{m+1/2}]\right),\tilde{H}_k^{[m+1]}\right)_{\partial T_k},\end{aligned}\tag{6.2}$$

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where $(\cdot,\cdot)_{T_k}$ and $(\cdot,\cdot)_{\partial T_k}$ denote the inner-products in $(L^2(T_k))^3$ and $(L^2(\partial T_k))^3$, respectively. Using the identity,

$$\left(\nabla\times\tilde{E}_k^{m+1},\tilde{H}_k^{[m+1]}\right)_{T_k}=\left(\nabla\times\tilde{H}_k^{[m+1]},\tilde{E}_k^{m+1}\right)_{T_k}+\left(n\times\tilde{E}_k^{m+1},\tilde{H}_k^{[m+1]}\right)_{\partial T_k},$$

summing (6.1) and (6.2) from $m=0$ to $m=M-1$, and integrating by parts, we get

$$\begin{aligned}&\left(\epsilon\tilde{E}_k^M,\tilde{E}_k^M\right)_{T_k}+\left(\mu\tilde{H}_k^{M+1/2},\tilde{H}_k^{M+1/2}\right)_{T_k}\\&=\left(\epsilon\tilde{E}_k^0,\tilde{E}_k^0\right)_{T_k}+\left(\mu\tilde{H}_k^{1/2},\tilde{H}_{z_k}^{1/2}\right)_{T_k}+\Delta t\left(\nabla\times\tilde{H}_k^{1/2},\tilde{E}_k^0\right)_{T_k}\\&\quad-\Delta t\left(\nabla\times\tilde{H}_k^{M+1/2},\tilde{E}_k^M\right)_{T_k}+2\Delta t\sum_{m=0}^{M-1}A_k^m,\end{aligned}\tag{6.3}$$

where

$$\begin{aligned}A_k^m=&-\left(\frac{1}{Z^++Z^-}n\times\left(Z^+[\tilde{H}^{m+1/2}]-\alpha n\times[\tilde{E}^m]\right),\tilde{E}_k^{[m+1/2]}\right)_{\partial T_k}\\&\left(\frac{1}{Y^++Y^-}n\times\left(Y^+[\tilde{E}^{m+1}]+\alpha n\times[\tilde{H}^{m+1/2}]\right),\tilde{H}_k^{[m+1]}\right)_{\partial T_k}-\left(n\times\tilde{E}_k^{m+1},\tilde{H}_k^{[m+1]}\right)_{\partial T_k}.\end{aligned}$$

Let us consider the following decomposition $\sum_{T_k\in\mathcal{T}_h}A_k^m=B_1^m+B_2^m$, where

$$\begin{aligned}B_1^m=\sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\Bigg(&\frac{-1}{Z_i+Z_k}n_{ki}\times\left(Z_i[\tilde{H}_k^{m+1/2}]-\alpha n_{ki}\times[\tilde{E}_k^m]\right)\cdot\tilde{E}_k^{[m+1/2]}\\&+\frac{1}{Z_i+Z_k}n_{ki}\times\left(Z_k[\tilde{H}_i^{m+1/2}]+\alpha n_{ki}\times[\tilde{E}_i^m]\right)\cdot\tilde{E}_i^{[m+1/2]}\\&+\frac{1}{Y_i+Y_k}n_{ki}\times\left(Y_i[\tilde{E}_k^{m+1}]+\alpha n_{ki}\times[\tilde{H}_k^{m+1/2}]\right)\cdot\tilde{H}_k^{[m+1]}\\&-\frac{1}{Y_i+Y_k}n_{ki}\times\left(Y_k[\tilde{E}_i^{m+1}]-\alpha n_{ki}\times[\tilde{H}_i^{m+1/2}]\right)\cdot\tilde{H}_i^{[m+1]}\\&-n_{ki}\times\tilde{E}_k^{m+1}\cdot\tilde{H}_k^{[m+1]}+n_{ki}\times\tilde{E}_i^{m+1}\cdot\tilde{H}_i^{[m+1]}\Bigg)ds,\end{aligned}\tag{6.4}$$

and

$$\begin{aligned}B_2^m=\sum_{f_k\in F^{ext}}\int_{f_k}\Bigg(&-\frac{1}{2Z_k}n_k\times\left(Z_k[\tilde{H}_k^{m+1/2}]-\alpha n_k\times[\tilde{E}_k^m]\right)\cdot\tilde{E}_k^{[m+1/2]}\\&+\frac{1}{2Y_k}n_k\times\left(Y_k[\tilde{E}_k^{m+1}]+\alpha n_k\times[\tilde{H}_k^{m+1/2}]\right)\cdot\tilde{H}_k^{[m+1]}-n_k\times\tilde{E}_k^{m+1}\cdot\tilde{H}_k^{[m+1]}\Bigg)ds.\end{aligned}\tag{6.5}$$

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We will now estimate B_1^m and B_2^m. Summing from $m=0$ to $m=M-1$, using the equalities $u\times v\cdot w=-u\times w\cdot v$, and $u\times(v\times w)=v(u\cdot w)-w(u\cdot v)$ and since $I-n_{ki}n_{ik}^T$ is a positive semidefinite matrix, where I is the identity matrix, we obtain

$$\begin{aligned}\sum_{m=0}^{M-1}B_1^m \le \sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\Bigg(&\frac{\alpha}{4(Z_i+Z_k)}[\tilde{E}_k^M]^T\left(I-n_{ki}n_{ik}^T\right)[\tilde{E}_k^M]\\&+\frac{\alpha}{4(Y_i+Y_k)}[\tilde{H}_k^{M+1/2}]^T\left(I-n_{ki}n_{ik}^T\right)[\tilde{H}_k^{M+1/2}]\\&+\frac{Z_k}{2(Z_i+Z_k)}\left(n_{ki}\times[\tilde{H}_k^{1/2}]\cdot[E_k^0]-n_{ki}\times[\tilde{H}_k^{M+1/2}]\cdot[\tilde{E}_k^M]\right)\\&+\frac{1}{2}\left(n_{ki}\times[\tilde{H}_k^{M+1/2}]\cdot\tilde{E}_k^M-n_{ki}\times[\tilde{H}_k^{1/2}]\cdot\tilde{E}_k^0\right)\Bigg)ds.\end{aligned}$$

The proof follows from the fact that, since the matrix $I-n_{ki}n_{ik}^T$ is an orthogonal projector, $x^T(I-n_{ki}n_{ik}^T)x\le x\cdot x$, for all vector x, and then

$$\begin{aligned}\sum_{m=0}^{M-1}B_1^m \le \sum_{f_{ik}\in F^{int}}\int_{f_{ik}}\Bigg(&\frac{\alpha}{4(Z_i+Z_k)}[\tilde{E}_k^M]\cdot[\tilde{E}_k^M]+\frac{\alpha}{4(Y_i+Y_k)}[\tilde{H}_k^{M+1/2}]\cdot[\tilde{H}_k^{M+1/2}]\\&+\frac{Z_k}{2(Z_i+Z_k)}\left(n_{ki}\times[\tilde{H}_k^{1/2}]\cdot[E_k^0]-n_{ki}\times[\tilde{H}_k^{M+1/2}]\cdot[\tilde{E}_k^M]\right)\\&+\frac{1}{2}\left(n_{ki}\times[\tilde{H}_k^{M+1/2}]\cdot\tilde{E}_k^M-n_{ki}\times[\tilde{H}_k^{1/2}]\cdot\tilde{E}_k^0\right)\Bigg)ds.\end{aligned}\tag{6.6}$$

For B_2^m be defined by (6.5) we obtain

$$\begin{aligned}\sum_{m=0}^{M-1}B_2^m \le \sum_{f_k\in F^{ext}}\int_{f_k}\Bigg(&\frac{\beta_1}{4Z_k}\left(-\left(n_k\times\tilde{E}_k^0\right)\cdot\left(n_k\times\tilde{E}_k^0\right)+\left(n_k\times\tilde{E}_k^M\right)\cdot\left(n_k\times\tilde{E}_k^M\right)\right)\\&+\frac{\beta_2}{2}\left(n_k\times\tilde{E}_k^0\cdot\tilde{H}_k^{1/2}-n_k\times\tilde{E}_k^M\cdot\tilde{H}_k^{M+1/2}\right)+\frac{\beta_3}{4Y_k}\Bigg(-\left(n_k\times\tilde{H}_k^{1/2}\right)\cdot\left(n_k\times\tilde{H}_k^{1/2}\right)\\&+\left(n_k\times\tilde{H}_k^{M+1/2}\right)\cdot\left(n_k\times\tilde{H}_k^{M+1/2}\right)\Bigg)\Bigg)ds,\end{aligned}\tag{6.7}$$

where $\beta_1=\alpha$, $\beta_2=0$ for PEC, $\beta_1=0$, $\beta_2=1$, $\beta_3=\alpha$ for PMC, and $\beta_1=\beta_2=\beta_3=\frac{1}{2}$ for Silver-Müller boundary conditions.

As for the 2D case, from (6.3), (6.6), (6.7), considering the Cauchy-Schwarz's and triangular inequality and using the inequality (A.5) of Lemma A.1 and the inequality (A.6) of Lemma A.2, we get the following stability result.

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Theorem 6.1. *Let us consider the leap-frog DG method* (3.2)-(3.4) *complemented with the discrete boundary conditions defined in Section* 3.1. *If the time step* Δt *is such that*

$$\Delta t < \frac{\min\{\underline{\epsilon},\underline{\mu}\}}{\max\{C_E,C_H\}}\min\{h_k\}, \tag{6.8}$$

where

$$C_E = \frac{1}{2}C_{inv}N^2 + C_\tau^2(N+1)(N+3)\left(3+\frac{\beta_2}{2}+\frac{\alpha+\beta_1}{2\min\{Z_k\}}\right),$$
$$C_H = \frac{1}{2}C_{inv}N^2 + C_\tau^2(N+1)(N+3)\left(3+\frac{\beta_2}{2}+\frac{\alpha+\beta_3}{2\min\{Y_k\}}\right),$$

with C_τ *defined by* (A.5) *of Lemma* A.1 *and* C_{inv} *defined by* (A.6) *of Lemma* A.2, *and* $\beta_1=\alpha$, $\beta_2=0$ *for PEC,* $\beta_1=0$, $\beta_2=1$, $\beta_3=\alpha$ *for PMC, and* $\beta_1=\beta_2=\frac{1}{2}$, $\beta_3=1$ *for Silver-Müller boundary conditions, then the method is stable.*

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A Technical lemmata

We consider the following trace inequalities (see *e.g.* [18]).

Lemma A.1. *Let* T_k *be an element of* $\mathcal{T}_h$ *with diameter* h_k *and let* f_k *be an edge or a face of* T_k. *There exists a positive constant* C *independent of* h_k *such that, for any* $u\in H^1(T_k)$,

$$\|u\|_{L^2(f_k)} \le C\sqrt{\frac{|f_k|}{|T_k|}}\left(\|u\|_{L^2(T_k)}+h_k\|\nabla u\|_{L^2(T_k)}\right). \tag{A.1}$$

Moreover, the following inequality holds for any $u\in P_N(T_k)$

$$\textit{in } 2D: \quad \|u\|_{L^2(f_k)} \le \sqrt{\frac{(N+1)(N+2)}{2}\frac{|f_k|}{|T_k|}}\|u\|_{L^2(T_k)}, \tag{A.2}$$

$$\textit{in } 3D: \quad \|u\|_{L^2(f_k)} \le \sqrt{\frac{(N+1)(N+3)}{3}\frac{|f_k|}{|T_k|}}\|u\|_{L^2(T_k)}. \tag{A.3}$$

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Consequently, there exists a positive constant C_τ independent of h_k and N but dependent on the shape-regularity h_k/τ_k, where τ_k is the diameter of the largest inscribed ball contained in T_k (see (3.1)), such that, for any $u\in P_N(T_k)$,

$$\text{in } 2D: \quad \|u\|_{L^2(\partial T_k)} \le C_\tau \sqrt{(N+1)(N+2)}\, h_k^{-1/2} \|u\|_{L^2(T_k)}, \tag{A.4}$$

$$\text{in } 3D: \quad \|u\|_{L^2(\partial T_k)} \le C_\tau \sqrt{(N+1)(N+3)}\, h_k^{-1/2} \|u\|_{L^2(T_k)}. \tag{A.5}$$

The next result is an inverse-type estimate [5, 10], where we present explicitly the dependence of the constant on the polynomials degree.

Lemma A.2. *Let us consider $T_k\in\mathcal{T}_h$ with diameter h_k. There exists a positive constant C_{inv} independent of h_k and N such that, for any $u\in P_N(T_k)$,*

$$\|u\|_{H^q(T_k)} \le C_{inv} N^{2q} h_k^{-q} \|u\|_{L^2(T_k)}, \tag{A.6}$$

where $q\ge 0$.

The reader can refer to [3] or [19] for the following approximation properties.

Lemma A.3. *Let $T_k\in\mathcal{T}_h$ and $u\in H^p(T_k)$. Then there exits a constant C depending on p and on the shape-regularity of T_k but independent of u, h_k and N and a sequence $\mathcal{P}_N u\in P_N(T_k)$, $N=1,2,\cdots$, such that, for any $0\le q\le p$*

$$\|u-\mathcal{P}_N u\|_{H^q(T_k)} \le C \frac{h_k^{\sigma-q}}{N^{p-q}} \|u\|_{H^p(T_k)}, \quad p\ge 0, \tag{A.7}$$

$$\|u-\mathcal{P}_N u\|_{L^2(f_k)} \le C \frac{h_k^{\sigma-1/2}}{N^{p-1/2}} \|u\|_{H^p(T_k)}, \quad p>\frac{1}{2}, \tag{A.8}$$

where $\sigma=\min(p,N+1)$ and f_k is an edge of T_k.

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