INTERSECTION GROUPS AND RINGS

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In a previous paper(2) the two-dimensional intersection group of an algebraic surface has been introduced. This group can be defined as a factor group of the two-dimensional homology group of the surface, and it is of particular interest because, unlike the homology group, it is invariant under all birational transformations, and not merely under topological ones*. It may therefore be of interest to give a generalization of the notion of intersection group to an arbitrary finite absolute orientable n-dimensional manifold M^n (the terminology is that of (1), p. 197). In making this extension we are led to the concept of an *intersection ring* which is a quotient ring of the homology ring ((1), p. 207).

We shall denote the p-dimensional integral homology group of M^n by $\mathfrak{H}^p$, and an arbitrary element of the group by Γ^p. It will be convenient to refer to the classes Γ^p simply as 'cycles'. We are concerned only with integral homology groups, and the homology ring over the ring of integers; the extension of the ideas of this note to groups over any ring is perfectly straightforward.

The two-dimensional intersection group $\mathfrak{J}^2$ of an algebraic surface M^4 can be defined as follows. First introduce the group $\mathfrak{R}^2$ of cycles Γ^2 defined by the property $\Gamma^3 . \Gamma^2 = 0$ for all 3-cycles Γ^3. $\mathfrak{J}^2$ is the factor group $\mathfrak{H}^2/\mathfrak{R}^2$†. The isomorphism between this group and the intersection group as previously defined is obvious (cf. (2), p. 419). We shall extend this idea in a simple way to more general cases.

Let $\mathfrak{R}$ be the homology ring of a manifold M^n. $\mathfrak{R}$ is the direct sum of the homology groups with cycle-intersection as the ring multiplication. Let D be an arbitrary set of cycles (possibly of quite different dimensions) and $\mathfrak{R}^p$ the group of cycles Γ^p such that $\Gamma^p . \Delta = 0$ for every cycle Δ of D. On analogy with the intersection group of an algebraic surface, in which case D may be taken as a base for the 3-cycles, we define the p-dimensional intersection group $\mathfrak{J}^p$ as the quotient group $\mathfrak{H}^p/\mathfrak{R}^p$.

If instead of the set D we had used to define $\mathfrak{R}^p$ the two-sided ideal $\mathfrak{d}$ of $\mathfrak{R}$ generated by the set D, we should obtain the same group $\mathfrak{R}^p$. (This is a simple consequence of the associativity of $\mathfrak{R}$ and of the commutation rule for intersections ((1), p. 207).) We shall therefore in future consider an ideal $\mathfrak{d}$ of $\mathfrak{R}$ instead of the arbitrary set D, and will also write $\mathfrak{J}^p_{\mathfrak{d}}$ and $\mathfrak{R}^p_{\mathfrak{d}}$ to emphasize their dependence on $\mathfrak{d}$. It is easy to see that the pairing of $\mathfrak{H}^p$ and $\mathfrak{H}^q$ to $\mathfrak{H}^{p+q-n}$ defined by the intersections, gives rise to a pairing of $\mathfrak{J}^p_{\mathfrak{d}}$ and $\mathfrak{J}^q_{\mathfrak{d}}$ to $\mathfrak{J}^{p+q-n}_{\mathfrak{d}}$. The direct sum of the $\mathfrak{J}^p_{\mathfrak{d}}$ $(p = 0, 1, \ldots, n)$ can therefore be regarded as a ring $\mathfrak{R}_{\mathfrak{d}}$ with this pairing as the ring multiplication. This ring we call the

* In (2) we proved not merely the birational invariance of the intersection group $\mathfrak{J}^2$, but also the invariance of a particular realization of $\mathfrak{J}^2$ by matrices defining intersections.

† We shall, following the procedure in (1), write the factor group in this form, even though the group operation is addition.

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intersection ring of M^n relative to the ideal $\mathfrak{d}$. We could equally easily define $\mathfrak{R}_\mathfrak{d}$ as a quotient ring. If $\mathfrak{A}_\mathfrak{d}$ is the ideal of $\mathfrak{R}$ which annihilates $\mathfrak{d}$, i.e. the direct sum of the $\mathfrak{R}^q_\mathfrak{d}$, then $\mathfrak{R}_\mathfrak{d}$ is the quotient ring $\mathfrak{R}/\mathfrak{A}_\mathfrak{d}$.

If the ideal $\mathfrak{d}$ contains no cycles of dimension $\geqslant n-p$, then for $0 \leqslant q \leqslant p$ we have $\mathfrak{R}^q = \mathfrak{H}^q$ and $\mathfrak{J}^q = 0$. In particular, $\mathfrak{J}^0 = 0$ unless the ideal $\mathfrak{d}$ contains a multiple of the unit I of $\mathfrak{R}$, where I is the generator of the group $\mathfrak{H}^n$. This point will be brought out in some of the examples which follow.

Example 1. $\mathfrak{d}$ is the unit ideal (I). Then $\mathfrak{A}_\mathfrak{d} = 0$ and $\mathfrak{R}_\mathfrak{d} \simeq \mathfrak{R}$.

Example 2. $\mathfrak{d}$ is the principal ideal (tI), where t is divisible by all the torsion numbers of M^n. $\mathfrak{A}_\mathfrak{d}$ is then the *torsion ring*, the direct sum of all the torsion groups of M^n, and $\mathfrak{R}_\mathfrak{d}$ is the *Betti ring* of M^n, i.e. the ring which replaces the homology ring if strict homology is replaced, as the fundamental equivalence relation, by weak homology.

The above example is of interest as it shows that the correspondence between ideals $\mathfrak{d}$ and intersection rings $\mathfrak{R}_\mathfrak{d}$ is not one-one, since, for any integer k, the ideal (ktI) defines the same intersection ring as (tI). We can make the correspondence one-one, however, if instead of the ideal $\mathfrak{d}$ we take the ideal $\bar{\mathfrak{d}}$ defined as the largest ideal whose annihilator is $\mathfrak{A}_\mathfrak{d}$. It is easy to show that this ideal is in fact the annihilator of the annihilator of $\mathfrak{d}$. We are then naturally led to formulate the problem as to whether every homomorph of the ring $\mathfrak{R}$ is isomorphic to an intersection ring, and we find that this is not the case. For any homomorph $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ is isomorphic with a quotient ring $\mathfrak{R}/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathfrak{R}$. The ring $\mathfrak{R}/\mathfrak{a}$ is an intersection ring only if there is an ideal $\mathfrak{d}$ such that $\mathfrak{a} = \mathfrak{A}_\mathfrak{d}$. It is simple enough to show that $\mathfrak{a}$ is of this type only if $\mathfrak{a}$ is the annihilator of its own annihilator. An example of an ideal not of this type is the ideal (ktI) already referred to, the annihilator of whose annihilator contains its proper factor (tI). So that the quotient ring $\mathfrak{R}/(ktI)$ is *not* an intersection ring of the homology ring over the integers. This ring is, of course, the homology ring over the ring of integer residue classes mod kt.

Example 3. M^4 is the Riemannian of an algebraic surface and $\mathfrak{d}$ is generated by a base for the 3-cycles of M^4. Then $\mathfrak{J}^1_\mathfrak{d} \simeq \mathfrak{H}^1$ and $\mathfrak{J}^3_\mathfrak{d} \simeq \mathfrak{H}^3$. But $\mathfrak{H}^1$ and $\mathfrak{H}^3$ are orthogonally paired to $\mathfrak{H}^0$ by the multiplication in $\mathfrak{R}$, whereas $\mathfrak{J}^1_\mathfrak{d}$ and $\mathfrak{J}^3_\mathfrak{d}$ annihilate one another in $\mathfrak{R}_\mathfrak{d}$. The pairing of $\mathfrak{J}^3_\mathfrak{d}$ and $\mathfrak{J}^2_\mathfrak{d}$ to $\mathfrak{J}^1_\mathfrak{d}$ by the multiplication of $\mathfrak{R}_\mathfrak{d}$ is orthogonal, as is the self-pairing of $\mathfrak{J}^3_\mathfrak{d}$ to $\mathfrak{J}^2_\mathfrak{d}$. We can, of course, replace $\mathfrak{J}^1_\mathfrak{d}$ and $\mathfrak{J}^3_\mathfrak{d}$ in these pairings by their isomorphs $\mathfrak{H}^1$ and $\mathfrak{H}^3$. The former of these pairings is, in effect, the chief subject matter of (2).

Example 4. $\mathfrak{d}$ is generated by a base for the algebraic two-cycles of M^4. Then $\mathfrak{J}^0_\mathfrak{d} = 0$, $\mathfrak{J}^1_\mathfrak{d} = 0$, $\mathfrak{J}^2_\mathfrak{d}$ is isomorphic to the group of homology classes of the algebraic two-cycles of M^4, and $\mathfrak{J}^3_\mathfrak{d} \simeq \mathfrak{H}^3$. The self-pairing of $\mathfrak{J}^3_\mathfrak{d}$ to $\mathfrak{J}^2_\mathfrak{d}$ set up by the ring multiplication in $\mathfrak{R}_\mathfrak{d}$ (or more strictly the pairing of the isomorphic groups), is also discussed in (2).

Example 5. $\mathfrak{d}$ is generated by a base for the transcendental two-cycles of M^4. Then $\mathfrak{J}^0_\mathfrak{d} = 0$, $\mathfrak{J}^1_\mathfrak{d} = 0$, $\mathfrak{J}^2_\mathfrak{d}$ is isomorphic to the group of homology classes of the transcendental cycles, but we do not know, in the present state of the theory, whether or not $\mathfrak{J}^3_\mathfrak{d} \simeq \mathfrak{H}^3$.

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