Proceedings of the Edinburgh Mathematical Society (2016) **59**, 759–776
DOI:10.1017/S0013091515000401

A STATE POLYTOPE DECOMPOSITION FORMULA

DONGHOON HYEON[1,2] AND JAEKWANG KIM[3]

[1] *Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea* (dhyeon@snu.ac.kr)

[2] *Center for Geometry and Its Applications, POSTECH, Pohang, Gyungbuk 790-784, Republic of Korea*

[3] *Department of Mathematics, POSTECH, Pohang, Gyungbuk 790-784, Republic of Korea* (kjk429@postech.ac.kr)

(Received 7 January 2014)

Abstract We give a decomposition formula for computing the state polytope of a reducible variety in terms of the state polytopes of its components: if a polarized projective variety X is a chain of subvarieties X_i satisfying some further conditions, then the state polytope of X is the Minkowski sum of the state polytopes of X_i translated by a vector τ, which can be readily computed from the ideal of X_i. The decomposition is in the strongest sense in that the vertices of the state polytope of X are precisely the sum of vertices of the state polytopes of X_i translated by τ. We also give a similar decomposition formula for the Hilbert–Mumford index of the Hilbert points of X. We give a few examples of the state polytope and the Hilbert–Mumford index computation of reducible curves, which are interesting in the context of the log minimal model program for the moduli space of stable curves.

Keywords: geometric invariant theory; state polytope; initial ideal; Gröbner basis

2010 *Mathematics subject classification:* Primary 14L24

1. Introduction

The state polytope of an ideal encodes much information about the scheme it defines. Let V be a vector space over an algebraically closed field k of characteristic zero. Given a rational representation W of $\mathrm{SL}(V)$ and a maximal torus $T \subset \mathrm{SL}(V)$, Kempf [**7**, § 3] defined the *state* of $w \in W$ (with respect to T) to be the set of the characters $\chi \in X(T)$ such that $w_\chi \neq 0$, where w_χ is the projection of w in the weight space W_χ. Given a projective variety $X \subset \mathbb{P}(V)$ and a choice of homogeneous coordinates, Bayer and Morrison [**2**] defined the *mth state polytope* of $X \subset \mathbb{P}(V)$ to be the convex hull of the states of (any affine point over) the mth Hilbert point $[X]_m \in \mathbb{P}(\bigwedge^{Q(m)} S^m V^*)$ defined by

$$[(I_X)_m \to S^m V^*] \in \mathrm{Gr}_{Q(m)} S^m V^* \hookrightarrow \mathbb{P}\left(\bigwedge^{Q(m)} S^m V^*\right),$$

where I_X is the saturated homogeneous ideal of X, $Q(t)$ is the dimension of the tth graded piece $(I_X)_t$, and m is a positive integer bigger than or equal to the Castelnuovo–Mumford

© 2015 The Edinburgh Mathematical Society

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

regularity of X. The state polytope of a homogeneous ideal is the state polytope of the projective variety it defines. The relation between the state polytopes and the Gröbner theory is described by the following theorem.

Theorem 1.1 (Bayer and Morrison [2, Theorem 3.1]). *For m sufficiently large there is a natural one-to-one correspondence between saturations of initial ideals and the vertices of the state polytope.*

Let T be the maximal torus of $\mathrm{SL}(V^*)$ diagonalized by $x_0, \ldots, x_n$. Then, by considering the T-weight space decomposition of $\bigwedge^{Q(m)} S^m V^*$, one can naturally associate the characters in the state of a Hilbert point $[(I_X)_m \to S^m V^*]$ and the *monomials* $x^{\alpha(1)} \wedge \cdots \wedge x^{\alpha(Q(m))}$ whose associated Plücker coordinate does not vanish at $[X]_m$. See [**8**, § 3.1] for this correspondence as well as for a very nice exposition on Kempf's theory of the worst one-parameter subgroup and basics on state polytopes. In particular, the trivial character corresponds to the *barycentre* of which coordinates are all $mQ(m)/\dim V$.

In view of this correspondence, we take the following definition of the state polytope [**11**, (2.7)].

Definition 1.2. Given a homogeneous ideal $I \subset S = k[x_0, \ldots, x_n]$ and $m \geqslant \mathrm{reg}(I)$, the *$m$th state polytope* is defined and denoted by

$$\mathcal{P}_m(I) := \mathrm{Conv}\left\{ \sum_{x^\alpha \in \mathrm{in}_\prec(I)_m} \alpha \;\middle|\; \prec \text{ is a monomial order} \right\}. \tag{1.1}$$

Here, 'Conv' means taking the convex hull, $\mathrm{reg}(I)$ is the Castelnuovo–Mumford regularity of I, and the condition $m \geqslant \mathrm{reg}(I)$ ensures that the mth Hilbert point of the scheme X cut out by I is well defined.

Apparent from the definition is that the state polytope can be computed from the universal Gröbner basis. More importantly, $\mathcal{P}_m(I_X)$ determines the semi-stability of the Hilbert point $[X]_m$ with respect to the chosen basis. This is a direct consequence of Mumford's numerical criterion [**9**]: $[X]_m$ is T-semi-stable (respectively, T-stable) if and only if the state polytope in $X(T)$ (respectively, the interior of the polytope) contains the trivial character. This condition is equivalent to the state polytope (1.1) (respectively, its interior) containing the barycentre, where the coordinates $x_0, \ldots, x_n$ are chosen so that T acts on them via characters, i.e. they diagonalize the T-action. The upshot is that by computing the universal Gröbner basis (with a computer algebra system if and when convenient), one can determine the semi-stability with respect to the given coordinates, i.e. with respect to the associated maximal torus T. If $[X]_m$ is T-unstable for some T, then $[X]_m$ is geometric invariant theory (GIT) unstable.

Of course, to prove the semi-stability of $[X]_m$, one has to prove its T-semi-stability for *all* maximal tori T, so in that regard the state polytope formulation of GIT semi-stability may not seem too much of a help. But in the special case when $X \subset \mathbb{P}(V)$ and V is a multiplicity-free representation of a linearly reductive subgroup Γ of $\mathrm{Aut}(X)$, the (semi-)stability of $[X]_m$ is equivalent to the T-(semi-)stability with respect to any

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

maximal torus T that preserves the Γ-irreducibles of V [**8**, Proposition 4.7]. This was the key idea of Morrison and Swinarski that allowed them to prove the m-Hilbert semi-stability of various curves for small m [**8**, § 7]. It was also the starting point for Alper *et al.* to obtain their results on the m-Hilbert semi-stability of canonical and bicanonical images of generic smooth curves for $m \geqslant 2$ [**1**], which should prove essential in carrying out the log minimal model program for $\overline{M}_g$ (the *Hassett-Keel program*). In fact, Alper *et al.* do not rely on the state polytope technique: instead, they work out *by hand* a collection of basis members for the (bi)canonical system and deduce from them the semi-stability directly, which is in every manner very impressive.

Inspired by these exciting developments, we shall consider in this paper how one can more efficiently compute the state polytope of certain reducible varieties. More precisely, we give a formula for the state polytope of a variety in terms of the state polytopes of its subvarieties. We say that X is a *chain of subvarieties* X_i if $X = \bigcup_{i=1}^{\ell} X_i$ and X_i meets X_j when and only when $|i-j| = 1$.

Theorem 1.3. *Let X be a chain of subvarieties $X_1, \ldots, X_\ell$ defined by a homogeneous ideal $I_X = \bigcap_i I_{X_i}$. Suppose that there is a homogeneous coordinate system $x_0, \ldots, x_n$ and a sequence $n_0 = 0 < n_1 < \cdots < n_\ell = n$ such that*

$$X_i \subset \{x_0 = \cdots = x_{n_{i-1}-1} = 0,\ x_{n_i+1} = x_{n_i+2} = \cdots = x_n = 0\}.$$

Then the state polytope of X is given by the following decomposition formula:

$$\mathcal{P}_m(I_X) = \sum_{i=1}^{\ell} \mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}]) + \sum_{i=1}^{\ell-1} \mathcal{P}_m(T_i \cap k[x_{n_{i-1}}, \ldots, x_n]), \qquad (\dagger)$$

where

$$T_i = \langle x_{n_{i-2}}, \ldots, x_{n_{i-1}-1} \rangle \langle x_{n_i+1}, \ldots, x_n \rangle \quad \text{for } 2 \leqslant i \leqslant \ell - 1,$$

and

$$T_1 = \langle x_{n_1+1}, x_{n_1+2}, \ldots, x_n \rangle \quad \text{and} \quad T_\ell = \langle x_{n_{\ell-2}}, x_{n_{\ell-2}+1}, \ldots, x_{n_{\ell-1}-1} \rangle.$$

Remark 1.4.

(1) Here, $\mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}])$ is regarded as a convex polytope in the subspace

$$\{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \mid a_0 = \cdots = a_{n_{i-1}-1} = 0,\ a_{n_i+1} = a_{n_i+2} = \cdots = a_n = 0\}.$$

Similarly, $\mathcal{P}_m(T_i \cap k[x_{n_{i-1}}, \ldots, x_n])$ is also regarded as a convex polytope in the relevant vector subspace.

(2) Note that the second term of (†) is zero dimensional since the T_i are monomial ideals. We shall reserve the letter τ to denote it.

In fact, the polytope decomposition in Theorem 1.3 is sharp in the following sense.

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

Corollary 1.5. *Retain the notation from Theorem 1.3. Let $\mathcal{V}_i$ denote the set of vertices of $\mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \dots, x_{n_i}])$, $i = 1, \dots, \ell$. Then the vertices of $\mathcal{P}_m(I_X)$ are precisely*

$$\left\{ \tau + \sum_{i=1}^{\ell} v_i \;\middle|\; v_i \in \mathcal{V}_i \right\}.$$

The Hilbert–Mumford index can also be computed by a similar decomposition formula. To be consistent with our main reference [**5**], we state the formula in terms of the *dual Hilbert point*

$$[X]^{\star}_m = [S^m V^* \to S^m V^*/I_m] \in \mathrm{Gr}^{P(m)} S^m V^* \hookrightarrow \mathbb{P}\Big(\bigwedge^{P(m)} S^m V^* \Big),$$

where $P(t) \in \mathbb{Q}[t]$ is the Hilbert polynomial of X.

Proposition 1.6. *Let X be as in Theorem 1.3, let $\rho\colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup of GL_{n+1} diagonalized by $\{x_0, \dots, x_n\}$ with weights $(r_0, \dots, r_n)$, and let ρ_i be the restriction of ρ to $\mathrm{GL}(kx_{n_{i-1}} + \cdots + kx_{n_i})$. Then the Hilbert–Mumford index $\mu([X]^{\star}_m, \rho)$ of the mth Hilbert point of X with respect to ρ is given by*

$$\mu([X]^{\star}_m, \rho) = \sum_{i=1}^{\ell} \mu([X_i]^{\star}_m, \rho_i) - \sum_{i=1}^{\ell} \left(\frac{mP_i(m)}{n_i - n_{i-1} + 1} \sum_{k=n_{i-1}}^{n_i} r_k \right) + \frac{mP(m)}{n+1} \sum_{i=0}^{n} r_i + m \sum_{i=1}^{\ell-1} r_{n_i},$$

where $P(m)$ is the Hilbert polynomial of $I_X \subset k[x_0, \dots, x_n]$, and $P_i(m)$, the Hilbert polynomial of $I_{X_i} \cap k[x_{n_{i-1}}, \dots, x_{n_i}]$ regarded as an ideal in $k[x_{n_{i-1}}, \dots, x_{n_i}]$.

We shall give proofs of Theorem 1.3 and Proposition 1.6 in § 4. Corollary 1.5 will be established in § 3.

2. Basic examples

Before proving the main results we shall give a few basic examples at the far ends of the spectrum, namely, monomial ideals and hypersurfaces (plane curves, to be more specific).

Example 2.1 (monomial ideals). Let $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^3$ be a chain of $\mathbb{P}^1$s:

$$X_1 = \{x_2 = x_3 = 0\}, \qquad X_2 = \{x_0 = x_3 = 0\}, \qquad X_3 = \{x_0 = x_1 = 0\}.$$

Then $I_{X_1} \cap k[x_0, x_1] = 0$ and it does not contribute to the state polytope. The other components do not contribute for the same reason, but the mixed terms

$$\{x_0x_2, x_0x_3\} \cup \{x_0x_3, x_1x_3\}$$

are precisely the monomial generators of the ideal of X, and the formula (†) holds.

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

More generally, if $X = \bigcup X_i$ as in Theorem 1.3 and each of the X_i are defined by monomials $M_{i\alpha}$,

$$\mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}]) = \left\{ \sum_{\alpha} \log_x M_{i\alpha} \right\},$$

where $\log_x x^\alpha = \alpha$. The formula (†) implies that

$$\mathcal{P}_m(I_X) = \left\{ \sum_i \sum_\alpha \log_x M_{i\alpha} + \sum_{x^\beta \in \sum_i T_i} \beta \right\}.$$

This is the same as

$$\mathcal{P}_m(I_X) = \left\{ \sum_{x^\alpha \in (I_X)_m} \alpha \right\}$$

since

$$I_X = \bigcap_i I_{X_i} = \bigcap_i \langle \{M_{i\alpha}\}_\alpha, x_0, \ldots, x_{n_{i-1}-1}, x_{n_i+1}, \ldots, x_n \rangle = \langle \{M_{i\alpha}\}_{i,\alpha} \rangle + \sum_i T_i.$$

Note that the inclusion $M_{i\alpha} \in \bigcap_i I_{X_i}$ follows from the assumption that $M_{i\alpha}$ is not a power of x_{n_i} or of $x_{n_{i-1}}$.

Example 2.2 (plane curves). We consider a simple example of two plane curves

$$E_1 = \{b^2c = a(a-c)(a-2c),\ d = e = 0\} \quad \text{and} \quad E_2 = \{d^2c = e^2(e+c),\ a = b = 0\}$$

meeting in one node. Let C denote the union of E_1 and E_2. The third state polytope of E_1 has three vertices

$$\{(3,0,0,0,0), (1,0,2,0,0), (0,2,1,0,0)\} \tag{2.1}$$

and that of E_2 has two vertices

$$\{(0,0,1,2,0), (0,0,0,0,3)\}. \tag{2.2}$$

Indeed, since the E_i are hypersurfaces (in suitable linear subspaces) of degree 3, their third state polytopes are precisely the Newton polytopes.

To compute the state polytope of C we first take the sums of a point from (2.1) and a point from (2.2):

$$\{(3,0,1,2,0), (3,0,0,0,3), (1,0,3,2,0), (1,0,2,0,3), (0,2,2,2,0), (0,2,1,0,3)\}.$$

By the decomposition formula, the vertices themselves are these translated by τ, which is the sum of the exponent vectors of

$$T = \{a^2d, abd, acd, ad^2, ade, a^2e, abe, ace, ae^2, b^2d, bcd, bd^2, bde, b^2e, bce, be^2\}.$$

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

We compute $\tau = (11, 11, 4, 11, 11)$, and hence the vertices of $\mathcal{P}_3(I_C)$ are

$$(14, 11, 5, 13, 11), \quad (14, 11, 4, 11, 14), \quad (12, 11, 7, 13, 11),$$
$$(12, 11, 6, 11, 14), \quad (11, 13, 6, 13, 11), \quad (11, 13, 5, 11, 14).$$

This agrees with the direct computation of the third state polytope of the ideal of C:

$$\langle be, ae, bd, ad, -cd^2 + e^3 + e^2, a^3 - 3a^2c - b^2c + 2ac^2 \rangle.$$

Here we demonstrate the output of the MACAULAY 2 [**3**] package `StatePolytope`, written by Swinarski.

```
i2: R = QQ[a,b,c,d,e];
i3: I1 = ideal(b^2*c - a*(a-c)*(a-2*c))
3 2 2 2
o3 = ideal(- a + 3a c + b c - 2a*c)
o3: Ideal of R
i4: I2 = ideal(d^2*c-e^2*(e+1))
2 3 2
o4 = ideal(c*d - e - e)
o4: Ideal of R
i5: I = intersect(I1+ideal(d,e),I2+ideal(a,b))
2 3 2 3 2 2 2
o5 = ideal (b*e, a*e, b*d, a*d, - c*d + e + e, a - 3a c - b c + 2a*c)
o5: Ideal of R
i6: statePolytope(3,I)
LP algorithm being used: "cddgmp".
polymake: used package cddlib
Implementation of the double description method of Motzkin \et
Copyright by Komei Fukuda.
http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html

VERTICES
1 14 11 5 13 11
1 14 11 4 11 14
1 12 11 6 11 14
1 11 13 5 11 14
1 12 11 7 13 11
1 11 13 6 13 11

o6 = {{14, 11, 5, 13, 11}, {14, 11, 4, 11, 14}, {12, 11, 6, 11, 14},
------------------------------------------------------------
{11, 13, 5, 11, 14}, {12, 11, 7, 13, 11}, {11, 13, 6, 13, 11}}
```

Example 2.3. This example is non-trivial in comparison to the previous two. We shall consider a particular genus 4 curve with a genus 2 tail. Let R be a rational curve with a rhamphoid cusp. It is of arithmetic genus 2 and admits a $\mathbb{G}_m$ action with two fixed points, one of which is the cusp. The action comes from the automorphism of its normalization. Let C be a genus 2 curve obtained by attaching two copies R_1, R_2 of R at the smooth fixed points, say $p_i \in R_i$, $i = 1, 2$. (See Figure 1.)

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

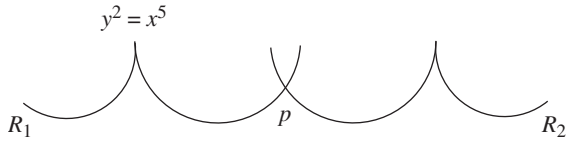


Figure 1. A rational rhamphoid cuspidal tail.

We bicanonically embed C in $\mathbb{P}^8$ and consider its state polytope. Note that R_1 can be parametrized by

$$\begin{aligned}\mathbb{P}^1 &\to \mathbb{P}^8\\ [s,t] &\mapsto [s^6, s^4t^2, s^2t^4, st^5, t^6, \mathbf{0}_4],\end{aligned}$$

which has a rhamphoid cusp at $[1,0,\ldots,0]$ and $p := [\mathbf{0}_4, 1, \mathbf{0}_4]$ is fixed under the automorphism, where

$$\mathbf{0}_i = (\underbrace{0,\ldots,0}_{i}).$$

R_2 is parametrized similarly, from which we can compute its defining saturated ideal. The sixth state polytope of $I_{R_1} \cap k[x_0,\ldots,x_4]$ has 51 vertices

$$\{(216,191,206,206,231,\mathbf{0}_4),(216,191,191,236,216,\mathbf{0}_4),\ldots,(181,248,210,180,231,\mathbf{0}_4)\},$$

which are obtained by Swinarski's `StatePolytope`.

The state polytope of $I_{R_2} \cap k[x_4,\ldots,x_8]$ is the mirror image of the above with respect to x_4. By the decomposition formula, the sixth state polytope P of C is the τ-translate of the convex hull of $51 \times 51 = 2601$ vertices obtained by choosing one from each state polytope and adding them up. Being the state polytope of a product of ideals of linear subspaces, τ can be readily computed either by hand or a computer algebra system:

$$\tau = (1750,1750,1750,1750,1504,1750,1750,1750,1750).$$

The barycentre of this state polytope is $v = (1956,1956,\ldots,1956)$. How can we use the decomposition formula to check whether v is in the state polytope? A natural thing to do is to try to decompose $v-\tau$ into the two symmetric points $v_1 = (206,206,206,206,226,\mathbf{0}_4)$ and $v_2 = (\mathbf{0}_4,226,206,206,206,206)$ and see if v_i is contained in the sixth state polytope of R_i. But this can be easily checked by using the function `contains` of the `Polyhedra` package in Macaulay 2 as follows:

```
i1: loadPackage("StatePolytope");
i2: R=QQ[a,b,c,d,e];
i3: Q=QQ[s,t];
i4: f=map(Q,R,{s^6,s^4*t^2,s^2*t^4,s*t^5,t^6});
o4: RingMap Q <--- R
i5: I=ker f;
o5: Ideal of R
i6: L=statePolytope(6,I);
i7: loadPackage("Polyhedra");
```

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

```
i8: P1=convexHull transpose matrix L;
i9: v1=transpose matrix {{206,206,206,206,226}};
5 1
o9: Matrix ZZ <--- ZZ
i10: contains(P1,v1)
o10 = true
```

Hence, we conclude that v is contained in P. In fact, the ideal of C is simple enough that its state polytope can be directly computed by using the state polytope package of MACAULAY 2, which agrees with the result obtained above.

The assumption on the existence of the coordinate system as in Theorem 1.3 may seem quite restrictive, but we shall see that it is satisfied by an important class of varieties, namely, the pluricanonical images of the generic members of the boundary of $\overline{M}_g$. We shall give a few interesting examples in this vein in § 6.

3. Decomposition formula for initial ideals

First we shall prove a key lemma on initial ideals from which the main theorem follows with some simple observations regarding the monomial orders. Let Y and Z be closed subvarieties in $\mathbb{P}^n$ defined by homogeneous ideals I_Y and I_Z of $k[x_0, \ldots, x_n]$, respectively, and let X be the projective variety defined by $I_X := I_Y \cap I_Z$. Suppose that with respect to the homogeneous coordinate system $x_0, x_1, \ldots, x_n$ the subvarieties Y and Z are contained in linear subspaces as follows:

$$Y \subset L_1 := \{x_0 = \cdots = x_{l-1} = 0\} \cong \mathbb{P}^{n-l},$$
$$Z \subset L_2 := \{x_{l+1} = \cdots = x_n = 0\} \cong \mathbb{P}^l.$$

In particular, $Y \cap Z = \{p\}$, where p is the unique point in $L_1 \cap L_2$ whose coordinates are all zero except x_l.

Lemma 3.1. *Let $\prec$ be a monomial order. The initial ideal of I with respect to $\prec$ is given by*

$$\mathrm{in}_\prec(I_X) = \langle \mathrm{in}_\prec(I_Y \cap k[x_l, \ldots, x_n]) \rangle + \langle \mathrm{in}_\prec(I_Z \cap k[x_0, \ldots, x_l]) \rangle + T, \qquad (\dagger\dagger)$$

where $T = \langle x_0, \ldots, x_{l-1} \rangle \langle x_{l+1}, \ldots, x_n \rangle$. Note that $\mathrm{in}_\prec(I_Y \cap k[x_l, \ldots, x_n])$ is computed as an ideal of $k[x_l, \ldots, x_n]$ and $\langle \mathrm{in}_\prec(I_Y \cap k[x_l, \ldots, x_n]) \rangle$ is the ideal in $k[x_0, \ldots, x_n]$ it generates. A similar statement for Z is noted.

Proof. We first note that T is contained in $\mathrm{in}_\prec(I_X)$ since it is a monomial ideal contained in I_X.

Let $x^\alpha = \prod x_i^{\alpha_i}$ be a monomial in $\mathrm{in}_\prec(I_X)_m$, i.e. $x^\alpha = \mathrm{in}_\prec(f)$ for some $f \in I_X$. If $x^\alpha \notin T$, then x^α is contained in $k[x_0, \ldots, x_l]$ or $k[x_l, \ldots, x_n]$. If $x^\alpha \in k[x_0, \ldots, x_l]$, then $\mathrm{in}_\prec(g) = x^\alpha$, where $g(x_0, \ldots, x_l) = f(x_0, \ldots, x_l, 0, \ldots, 0)$. But $g \in I_Z \cap k[x_0, \ldots, x_l]$, so x^α is contained in $\mathrm{in}_\prec(I_Z \cap k[x_0, \ldots, x_l])_m$. Similarly, if $x^\alpha \in k[x_l, \ldots, x_n]$, then $x^\alpha \in \mathrm{in}_\prec(I_Y \cap k[x_l, \ldots, x_n])_m$. This proves that the left-hand side is contained in the right.

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

To see the other inclusion, suppose that $x^\alpha \in \mathrm{in}_\prec(I_Y \cap k[x_l, \ldots, x_n])_m$, i.e. $x^\alpha = \mathrm{in}_\prec(f)$ for some $f \in I_Y \cap k[x_l, \ldots, x_n]$. Monomials of $k[x_l, \ldots, x_n]$ that do not vanish at p are of the form x_l^i. But since f is homogeneous and vanishes at $p \in Y$, it cannot have the term $x_l^{\deg(f)}$, so each term of f is divisible by x_i for some $i > l$. This implies that f vanishes on Z so that $f \in I_X$ and $x^\alpha \in \mathrm{in}_\prec(I_X)$. Hence, $\mathrm{in}_\prec(I_Y \cap k[x_l, \ldots, x_n]) \subset \mathrm{in}_\prec(I_X)$, and by a similar argument $\mathrm{in}_\prec(I_Z \cap k[x_0, \ldots, x_l]) \subset \mathrm{in}_\prec(I_X)$. □

We obtain the following corollary by induction.

Corollary 3.2. *Retain the notation from Theorem 1.3. For any monomial order $\prec$ on $k[x_0, \ldots, x_n]$ we have*

$$\begin{aligned}\mathrm{in}_\prec(I_X) &= \sum_{i=1}^{\ell} \langle \mathrm{in}(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}]) \rangle + \sum_{i=1}^{\ell-1} T_i \\ &= \sum_{i=1}^{\ell} \langle \mathrm{in}(I_X \cap k[x_{n_{i-1}}, \ldots, x_{n_i}]) \rangle + \sum_{i=1}^{\ell-1} T_i.\end{aligned}$$

Proof. Suppose that the formula holds for $\ell - 1$. Regarding X as the union of $Z := \bigcup_{i=1}^{\ell-1} X_i$ and $Y := X_\ell$ and applying Lemma 3.1 and the induction hypothesis, we obtain

$$\mathrm{in}_\prec(I_X) = \sum_{i=1}^{\ell-1} \langle \mathrm{in}(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}]) \rangle + \sum_{i=1}^{\ell-2} T_i + \langle \mathrm{in}_\prec(I_Z \cap k[x_{n_{\ell-1}}, \ldots, x_n]) \rangle + T,$$

where $T = \langle x_0, \ldots, x_{n_{\ell-1}-1} \rangle \langle x_{n_{\ell-1}+1}, \ldots, x_n \rangle$ may be replaced by

$$T_{\ell-1} = \langle x_{n_{\ell-2}}, \ldots, x_{n_{\ell-1}-1} \rangle \langle x_{n_{\ell-1}+1}, \ldots, x_n \rangle$$

since $\langle x_0, \ldots, x_{n_{\ell-2}-1} \rangle \langle x_{n_{\ell-1}+1}, \ldots, x_n \rangle$ is contained in $\sum_{i=1}^{\ell-2} T_i$. □

Now we can give the proof of Corollary 1.5.

Proof of Corollary 1.5. This is an immediate consequence of Theorem 1.1 and Corollary 3.2. □

To prove the decomposition formula of the Hilbert–Mumford index, we need to find the standard monomial of each component. The following lemma gives the relation between standard monomials.

Lemma 3.3 (standard monomials). *Let $\Sigma_{X,m}$, $\Sigma_{Y,m}$ and $\Sigma_{Z,m}$ be defined as*

$$\begin{aligned}\Sigma_{X,m} &= \{\textit{monomials in } k[x_0, \ldots, x_n]_m\} \setminus \mathrm{in}_\prec(I_X)_m, \\ \Sigma_{Y,m} &= \{\textit{monomials in } k[x_l, \ldots, x_n]_m\} \setminus \mathrm{in}_\prec(I_Y \cap k[x_l, \ldots, x_n])_m, \\ \Sigma_{Z,m} &= \{\textit{monomials in } k[x_0, \ldots, x_l]_m\} \setminus \mathrm{in}_\prec(I_Z \cap k[x_0, \ldots, x_l])_m.\end{aligned}$$

Then $\Sigma_{X,m} = \Sigma_{Y,m} \cup \Sigma_{Z,m}$ and $\Sigma_{Y,m} \cap \Sigma_{Z,m} = \{x_l^m\}$.

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

Proof. Let x^α be a degree m monomial and let T be as in Lemma 3.1. Since $T \subset I_X$, if x^α is not in $\mathrm{in}_\prec(I_X)$, it is necessarily in $k[x_0,\dots,x_l]$ or in $k[x_l,\dots,x_n]$. Such a monomial x^α is in the initial ideal if and only if it is in $\mathrm{in}_\prec(I_Y \cap k[x_l,\dots,x_n])_m$ or in $\mathrm{in}_\prec(I_Z \cap k[x_0,\dots,x_l])_m$, and the equality $\Sigma_{X,m} = \Sigma_{Y,m} \cup \Sigma_{Z,m}$ follows. That $\Sigma_{Y,m} \cap \Sigma_{Z,m} = \{x_l^m\}$ is clear from the fact that x_l^m is the only degree m monomial not vanishing at p, and thus not contained in any of the initial ideals involved in the discussion. □

4. State polytope decomposition formula

We prove our main results, Theorem 1.3 and Proposition 1.6, in this section.

Proof of Theorem 1.3. Surely, it suffices to prove Theorem 1.3 for the $\ell = 2$ case, as the general case would follow from it by a simple induction as in the proof of Corollary 3.2. So, let $X = Y \cup Z$ and let T be as in Lemma 3.1. We shall prove that

$$\mathcal{P}_m(I_X) = \mathcal{P}_m(I_Y \cap k[x_l,\dots,x_n]) + \mathcal{P}_m(I_Z \cap k[x_0,\dots,x_l]) + \tau, \tag{4.1}$$

where $\tau = \sum_{x^\alpha \in T} \alpha$.

Let $\sum_{x^\alpha \in \mathrm{in}_\prec(I_X)_m} \alpha$ be a vertex of $\mathcal{P}_m(I_X)$ induced by some monomial order $\prec$ on $k[x_0,\dots,x_n]$. Define

$$\Sigma^c_{Y,m} = \mathrm{in}_\prec(I_Y \cap k[x_l,\dots,x_n])_m, \qquad \Sigma^c_{Z,m} = \mathrm{in}_\prec(I_Z \cap k[x_0,\dots,x_l])_m.$$

By Lemma 3.1, we have

$$\mathrm{in}_\prec(I_X) = \langle \mathrm{in}_\prec(I_Y \cap k[x_l,\dots,x_n]) \rangle + \langle \mathrm{in}_\prec(I_Z \cap k[x_0,\dots,x_l]) \rangle + T,$$

which implies that

$$\sum_{x^\alpha \in \mathrm{in}_\prec(I_X)_m} \alpha = \sum_{x^\alpha \in \Sigma^c_{Y,m}} \alpha + \sum_{x^\alpha \in \Sigma^c_{Z,m}} \alpha + \sum_{x^\alpha \in T} \alpha.$$

Since $\sum_{x^\alpha \in \Sigma^c_{Y,m}} \alpha$ and $\sum_{x^\alpha \in \Sigma^c_{Z,m}} \alpha$ are vertices of $\mathcal{P}_m(I_Y \cap k[x_l,\dots,x_n])$ and $\mathcal{P}_m(I_Z \cap k[x_0,\dots,x_l])$, respectively, $\sum_{x^\alpha \in \mathrm{in}_\prec(I_X)_m} \alpha$ is contained in the right-hand side of (4.1).

Conversely, let α_1 be a vertex of $\mathcal{P}_m(I_Y \cap k[x_l,\dots,x_n])$ induced by the monomial order $\prec_1$ on $k[x_l,\dots,x_n]$, and let α_2 be a vertex of $\mathcal{P}_m(I_Z \cap k[x_0,\dots,x_l])$ induced by the monomial order $\prec_2$ on $k[x_0,\dots,x_l]$. We claim that there is a monomial order $\prec$ on $k[x_0,\dots,x_n]$ that induces the initial ideals with respect to the given orders $\prec_1$ on $k[x_l,\dots,x_n]$ and $\prec_2$ on $k[x_0,\dots,x_l]$. There are vectors $v \in \mathbb{N}^{n-l+1}$ and $v' \in \mathbb{N}^{l+1}$ such that (see [**11**, Proposition 1.11])

$$\mathrm{in}_{\prec_v}(I_Y \cap k[x_l,\dots,x_n]) = \mathrm{in}_{\prec_1}(I_Y \cap k[x_l,\dots,x_n]),$$
$$\mathrm{in}_{\prec_{v'}}(I_Z \cap k[x_0,\dots,x_l]) = \mathrm{in}_{\prec_2}(I_Z \cap k[x_0,\dots,x_l]),$$

where $\prec_v$ and $\prec_{v'}$ are the weight orders given by v and v', respectively. In general, an integral vector only gives rise to a partial order, but the choice of v and v' is made such

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

that they give total orders. This is possible, again by [**11**, Proposition 1.11]. By modifying the first entry of v (without affecting the order it defines), we may assume that it equals the last entry of v'. For instance, we may simply add $v'_l - v_1$ to all entries of v, which does not change the monomial order. Let $w = (v'_1, \dots, v'_l = v_1, v_2, \dots, v_{n-l+1})$. In general, this only defines a partial order, in which case we may employ any tie-breaking device, for instance the Lex order, to define a total order: declare $M = \prod_{i=0}^n x_i^{\alpha_i} \prec M' = \prod_{i=0}^n x_i^{\beta_i}$ if

(i) $w \cdot \alpha < w \cdot \beta$ or

(ii) $w \cdot \alpha = w \cdot \beta$ and $M \prec' M'$,

where $\prec'$ is the chosen tie breaking. Note that $\mathrm{in}_\prec(I_Y \cap k[x_l, \dots, x_n]) = \mathrm{in}_{\prec_1}(I_Y \cap k[x_l, \dots, x_n])$ since $\prec$ induces the weight order given by v on $k[x_l, \dots, x_n]$. A similar statement holds for I_Z and $\prec_2$. Then, by Lemma 3.1, we have

$$\mathrm{in}_\prec(I_X) = \langle \mathrm{in}_{\prec_1}(I_Y \cap k[x_l, \dots, x_n]) \rangle + \langle \mathrm{in}_{\prec_2}(I_Z \cap k[x_0, \dots, x_l]) \rangle + T$$

and this shows that $\alpha_1 + \alpha_2 + \tau = \sum_{x^\alpha \in \mathrm{in}_\prec(I_X)_m} \alpha \in \mathcal{P}_m(I_X)$. This completes the proof of Theorem 1.3. □

We now prove the decomposition formula for the Hilbert–Mumford index.

Lemma 4.1. *Let X, Y, Z be as in Lemma 3.1, let $\rho\colon \mathbb{G}_m \to \mathrm{GL}_{n+1}$ be a one-parameter subgroup diagonalized by $\{x_0, \dots, x_n\}$ with weights $(r_0, r_1, \dots, r_n)$, and let ρ', ρ'' be the restrictions of ρ to $\mathrm{GL}(kx_l + \cdots + kx_n)$ and $\mathrm{GL}(kx_0 + \cdots + kx_l)$, respectively. Then the Hilbert–Mumford index $\mu([X]^\star_m, \rho)$ of the mth Hilbert point of X with respect to ρ is given by*

$$\mu([X]^\star_m, \rho) = \mu([Y]^\star_m, \rho') + \mu([Z]^\star_m, \rho'') - \frac{mP_Z(m)}{n+1-l} \sum_{i=l}^n r_i - \frac{mP_Y(m)}{l+1} \sum_{i=0}^l r_i + \frac{mP_X(m)}{n+1} \sum_{i=0}^n r_i + mr_l,$$

where

$$P_Y(m) = \dim_k(k[x_l, \dots, x_n]/\mathrm{in}_{\prec_{\rho'}}(I_Y \cap k[x_l, \dots, x_n]))_m$$

and

$$P_Z(m) = \dim_k(k[x_0, \dots, x_l]/\mathrm{in}_{\prec_{\rho''}}(I_Z \cap k[x_0, \dots, x_l]))_m$$

are the Hilbert polynomials of Y and Z regarded as closed subvarieties of

$$\{x_0 = \cdots = x_{l-1} = 0\} \simeq \mathbb{P}^{n-l} \quad \text{and} \quad \{x_{l+1} = \cdots = x_n = 0\} \simeq \mathbb{P}^l,$$

respectively.

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

Proof. Let $\Sigma_{X,m}$, $\Sigma_{Y,m}$ and $\Sigma_{Z,m}$ be as in Lemma 3.3. By [**5**], the Hilbert–Mumford index can be computed by the formula

$$\mu([X]^\star_m, \rho) = -\sum_{x^\alpha \notin \mathrm{in}_{\prec_\rho}(I_X)_m} wt(x^\alpha) + \frac{mP_X(m)}{n+1}\sum_{i=0}^{n} r_i. \tag{4.2}$$

Since $\Sigma_{X,m} = \Sigma_{Y,m} \cup \Sigma_{Z,m}$ and $\Sigma_{Y,m} \cap \Sigma_{Z,m} = \{x_l^m\}$ by Lemma 3.3, we have

$$-\sum_{x^\alpha \notin \mathrm{in}(I_X)_m} wt(x^\alpha) = -\sum_{x^\alpha \in \Sigma_{Y,m}} wt(x^\alpha) - \sum_{x^\alpha \in \Sigma_{Z,m}} wt(x^\alpha) + wt(x_l^m).$$

By using [**5**] again, we have

$$\mu([Y]^\star_m, \rho') = \frac{mP_Y(m)}{n-l+1}\sum_{i=l}^{n} r_i - \sum_{x^\alpha \in \Sigma_{Y,m}} wt(x^\alpha),$$

$$\mu([Z]^\star_m, \rho'') = \frac{mP_Z(m)}{l+1}\sum_{i=0}^{l} r_i - \sum_{x^\alpha \in \Sigma_{Z,m}} wt(x^\alpha),$$

and plugging these into (4.2) produces the desired formula. □

In terms of monomial weights, Proposition 1.6 takes the following form:

$$\mu([X]^\star_m, \rho) = -\sum_{x^\alpha \in \Sigma_{Y,m}} wt_{\rho'}(x^\alpha) - \sum_{x^\alpha \in \Sigma_{Z,m}} wt_{\rho''}(x^\alpha) + \frac{mP(m)}{n+1} + mr_l. \tag{4.3}$$

Using the lemma inductively as in the proof of Corollary 3.2, we obtain the general case, Proposition 1.6, immediately.

5. Decomposition of the barycentre

Let H be a hyperplane in $\mathbb{R}^{n+1}$ defined by the equation $\sum_{i=0}^{n} x_i = m$ for some $m \in \mathbb{R}$. For any given sequence of integers $n_0 = 0 < n_1 < \cdots < n_\ell = n$ and a subset $\{m_1, m_2, \ldots, m_\ell\}$ of $\mathbb{R}$ such that $m = m_1 + m_2 + \cdots + m_\ell$, we aim to show that H can be decomposed into a sum of affine subspaces $H_i \subset \mathbb{R}^{n+1}$, $i = 1, \ldots, \ell$, defined by

$$H_1 = \left\{ \boldsymbol{a} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=0}^{n_1} a_j = m_1;\ a_j = 0,\ n_1 + 1 \leqslant j \leqslant n \right\},$$

$$H_i = \left\{ \boldsymbol{a} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=n_{i-1}}^{n_i} a_j = m_i;\ a_k = 0,\ 0 \leqslant j \leqslant n_{i-1} - 1,\ n_i + 1 \leqslant j \leqslant n \right\}$$

$$\text{for } i = 2, 3, \ldots, \ell - 1,$$

$$H_\ell = \left\{ \boldsymbol{a} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=n_{\ell-1}}^{n} a_j = m_\ell;\ a_j = 0,\ 0 \leqslant j \leqslant n_{\ell-1} - 1 \right\}.$$

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

Proposition 5.1. *Let H and H_i be given as above, then $H = \sum_{i=1}^{\ell} H_i$. Moreover, any point of H has a unique decomposition as a sum of elements in H_i.*

Proof. Since the sum of the coordinates of any point in H_i is m_i, the Minkowski sum $\sum_{i=1}^{\ell} H_i$ is contained in H. Conversely, for any given point $p = (p_0, p_1, \ldots, p_n)$ we can decompose p as follows. First, let

$$q_{1j} = \begin{cases} p_j & \text{if } 0 \leqslant j \leqslant n_1 - 1, \\ m_1 - \sum_{i=0}^{n_1-1} p_i & \text{if } j = n_1, \\ 0 & \text{else,} \end{cases}$$

and note that $q_1 = (q_{10}, q_{11}, \ldots, q_{1n})$ is an element of H_1. Now, consider $p - q_1$ as an element of the hyperplane in $\{x \in \mathbb{R}^n \mid x_0 = x_1 = \cdots = x_{n_1-1} = 0\} \simeq \mathbb{R}^{n-n_1+1}$ defined by $\sum_{i=n_1}^{n} x_i = m - m_1$. In the same manner, we find $q_2 \in H_2$ such that $p - q_1 = q_2 + (p - q_1 - q_2)$ such that $p - q_1 - q_2$ is in the hyperplane of $\{x_0 = x_1 = \cdots = x_{n_2-1} = 0\}$ defined by $\sum_{i=n_2}^{n} x_i = m - m_1 - m_2$. It is plain that repeating this procedure inductively produces a decomposition $p = q_1 + q_2 + \cdots + q_\ell$, where $q_i \in H_i$.

It remains to show the uniqueness of the decomposition. Let p be a point in H and suppose that $p = q_1 + q_2 + \cdots + q_\ell = r_1 + r_2 + \cdots + r_\ell$, where $q_i, r_i \in H_i$ for all $i = 1, 2, \ldots, \ell$. Rearranging the terms, we have

$$q_1 - r_1 = (r_2 + r_3 + \cdots + r_n) - (q_2 + q_3 + \cdots + q_n),$$

which implies that $(q_1)_i = (r_1)_i$ for $0 \leqslant i \leqslant n_1 - 1$ because the ith coordinate of the right-hand side is zero when $i < n_1$. But since $\sum_{i=0}^{n}(q_1)_i = \sum_{i=0}^{n}(r_1)_i = m_1$ and $(q_1)_i = (r_1)_i = 0$ for $i > n_1$, it follows that $(q_1)_{n_1} = (r_1)_{n_1}$, and hence $q_1 = r_1$. In the same manner, one can easily show that $q_i = r_i$ for all $i \geqslant 2$ and this completes the proof. □

Corollary 5.2. *Retain the notation H_i from Proposition 5.1. Let P_i be a polytope contained in H_i for $i = 1, 2, \ldots, \ell$. We say that H_i is the* supporting hyperplane *for P_i. Let $p \in H$ and let $p = q_1 + q_2 + \cdots + q_\ell$ be the decomposition of p as a sum of elements in H_i given by Proposition 5.1. Then any p is contained in $\sum_{i=1}^{\ell} P_i$ if and only if each q_i is contained in P_i.*

Proof. Suppose that p is contained in $\sum_{i=1}^{\ell} P_i$. Then, by definition of the Minkowski sum, there are elements $r_i \in P_i \subset H_i$ such that $p = r_1 + r_2 + \cdots + r_\ell$. But by the uniqueness of the decomposition, $r_i = q_i$ for all $i = 1, 2, \ldots, \ell$. The other implication holds trivially. □

In the case of the state polytopes, as in Theorem 1.3, the state polytopes

$$P_i := \mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}])$$

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

are supported by the hyperplanes

$$H_i := \left\{ x_0 = \cdots = x_{n_{i-1}-1} = 0,\ x_{n_i+1} = \cdots = x_n = 0,\ \sum_{i=0}^{n} x_i = mQ_i(m) \right\},$$

where $Q_i(m) = \dim_k(I_{X_i})_m$. Likewise, the state polytope of X is supported by $H = \{\sum_{i=0}^{n} x_i = mQ(m)\}$, $Q(m) = \dim_k(I_X)_m$. Let γ be the barycentre of the state polytope of X, and let τ be as in Remark 1.4. Applying Corollary 5.2, we obtain the following corollary.

Corollary 5.3. *Let $\gamma - \tau = q_1 + q_2 + \cdots + q_\ell$ be the decomposition into a sum of elements in the supporting hyperplanes H_i. Then γ is contained in $\mathcal{P}_m(I_X)$ if and only if each q_i is contained in $\mathcal{P}_m(I_{X_i} \cap k[x_{n_{i-1}}, \ldots, x_{n_i}])$.*

6. GIT of Hilbert points of pluricanonical curves

Our main application is to the study of GIT of pluricanonical curves.

6.1. Bi-canonical elliptic bridge

We revisit the state polytope analysis in [**8**, Example 8.4]. Morrison and Swinarski consider the state polytope of a genus 5 curve of the form $C = \mathcal{W}_2 \cup E \cup \mathcal{W}_2$, where $\mathcal{W}_g$ denotes the Wiman curve of genus g [**8**, § 6.2] and E is the elliptic curve $y^2 = x^3 - x$. According to the direct computation using the ideal of C, it has 500,094 initial ideals. By using the decomposition formula and Macaulay 2 we can compute its state polytope rather easily, since the state polytopes of $\mathcal{W}_2$ and E are fairly small. Here, we give a complete description of the second state polytope:

$$\mathcal{P}_2(I_C) = \tau + \overbrace{\begin{matrix}(2,2,1,1,2,\mathbf{0}_7)\\(2,2,0,3,1,\mathbf{0}_7)\\(2,1,1,4,0,\mathbf{0}_7)\\(2,0,3,3,0,\mathbf{0}_7)\\(0,2,3,3,0,\mathbf{0}_7)\\(0,0,3,4,1,\mathbf{0}_7)\\(0,3,1,4,0,\mathbf{0}_7)\\(0,1,1,5,1,\mathbf{0}_7)\\(0,4,0,3,1,\mathbf{0}_7)\\(0,2,0,4,2,\mathbf{0}_7)\\(2,1,3,0,2,\mathbf{0}_7)\\(2,0,4,1,1,\mathbf{0}_7)\\(0,2,4,1,1,\mathbf{0}_7)\\(0,0,4,2,2,\mathbf{0}_7)\\(0,3,3,0,2,\mathbf{0}_7)\\(0,1,3,1,3,\mathbf{0}_7)\\(0,4,1,1,2,\mathbf{0}_7)\\(0,2,1,2,3,\mathbf{0}_7)\end{matrix}}^{\text{Conv}} + \overbrace{\begin{matrix}(\mathbf{0}_4,2,0,1,1,\mathbf{0}_4)\\(\mathbf{0}_4,1,0,3,0,\mathbf{0}_4)\\(\mathbf{0}_4,0,0,3,1,\mathbf{0}_4)\\(\mathbf{0}_4,1,0,1,2,\mathbf{0}_4)\\(\mathbf{0}_4,1,2,0,1,\mathbf{0}_4)\\(\mathbf{0}_4,0,2,2,0,\mathbf{0}_4)\end{matrix}}^{\text{Conv}} + \overbrace{\begin{matrix}(\mathbf{0}_7,2,1,1,2,2)\\(\mathbf{0}_7,3,2,1,2,0)\\(\mathbf{0}_7,2,1,1,4,0)\\(\mathbf{0}_7,1,3,0,4,0)\\(\mathbf{0}_7,2,4,0,2,0)\\(\mathbf{0}_7,1,5,1,1,0)\\(\mathbf{0}_7,0,4,1,3,0)\\(\mathbf{0}_7,1,3,0,2,2)\\(\mathbf{0}_7,0,4,1,1,2)\\(\mathbf{0}_7,2,0,3,3,0)\\(\mathbf{0}_7,1,1,4,2,0)\\(\mathbf{0}_7,3,1,3,1,0)\\(\mathbf{0}_7,2,0,3,1,2)\\(\mathbf{0}_7,1,1,4,0,2)\\(\mathbf{0}_7,2,2,4,0,0)\\(\mathbf{0}_7,1,4,3,0,0)\\(\mathbf{0}_7,0,3,3,2,0)\\(\mathbf{0}_7,0,3,3,0,2)\end{matrix}}^{\text{Conv}}$$

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

The second state polytope of C is the $\tau = (7,7,7,7,4,8,8,4,7,7,7,7)$-translate of the Minkowski sum as given above. The three columns are the (second) state polytopes of the three components $\mathcal{W}_2$, E and $\mathcal{W}_2$. Moreover, by Corollary 1.5, the vertices of the Minkowski sum are precisely the sums of three vertices obtained by choosing one from each column. Hence, the second state polytope of C has $18 \times 18 \times 6 = 1944$ vertices.

Using Proposition 5.1, we consider the decomposition of the barycentre. The barycentre of $\mathcal{P}_2(I_C)$ is $\gamma = (\frac{25}{3}, \frac{25}{3}, \ldots, \frac{25}{3})$ and $\gamma - \tau$ can be uniquely decomposed into a sum of elements in the supporting hyperplanes as follows:

$$\gamma - \tau = (\tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}, \tfrac{8}{3}, \mathbf{0}_7) + (\mathbf{0}_4, \tfrac{5}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{5}{3}, \mathbf{0}_4) + (\mathbf{0}_7, \tfrac{8}{3}, \tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}).$$

Using MACAULAY 2, we verify that the summands are not contained in the second state polytope of W_2 and E. Therefore, the bicanonical image of C is 2-Hilbert unstable.

6.2. 4-canonical curve with a cuspidal tail

Let $Y = D \cup_p R$ be a pseudo-stable curve [**10**] consisting of a non-singular curve of genus $g-1$, $g \geqslant 2$, and a rational curve R with a cusp meeting in a node. We revisit the computation in [**6**, Lemma 3], where the m-Hilbert instability for any $m \geqslant 2$ of the 4-canonical image of Y,

$$Y \hookrightarrow \mathbb{P}(\Gamma(\omega_Y^{\otimes 4})) \simeq \mathbb{P}^n,$$

is proved. Here, $n = 7(g-1)$ and the isomorphism is given by choosing sections (homogeneous coordinates) such that

$$R \subset \{x_0 = \cdots = x_{7g-11} = 0\},$$
$$D \subset \{x_{7g-9} = \cdots = x_n = 0\}.$$

Let $l = 7g - 10$ and apply Proposition 1.6. Let ρ be the one-parameter subgroup with weight $(4,4,\ldots,4,3,2,0)$. The total ρ-weight of the degree 2 monomials in $x_l, \ldots, x_n$ that are not in $\mathrm{in}_{\prec_\rho}(R)$ is 35. The degree 2 monomials in $x_0, \ldots, x_l$ not in $\mathrm{in}_{\prec_\rho}(D)$ contribute $15g - 22$ to the total weight. Lastly, $mP(m)/(n+1) = m(8m-1)(4g-5)$. Putting all these together through Proposition 1.6 or the monomial weight version (4.3), we obtain

$$\mu([Y]_2, \rho) = -(35 + 2 \times 4 \times (15g - 22)) + 30 \times (4g - 5) + 2 \times 4 = -1.$$

Likewise,

$$\mu([Y]_3, \rho) = -(77 + 3 \times 4 \times (23g - 34)) + 3 \times 23 \times (4g - 5) + 3 \times 4 = -2,$$

from which it is deduced that $\mu([Y]_m, \rho) = -m + 1$.

6.3. Open rosaries

We revisit the Hilbert–Mumford index computation of the special curves called '*rosaries*'. Recall from [**4**, Definition 6.1] that an open rosary of genus r is $R = L_1 \cup_{a_1} L_2 \cup_{a_2} \cdots \cup_{a_r} L_{r+1}$, where L_i are smooth genus zero curves and the a_is are

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

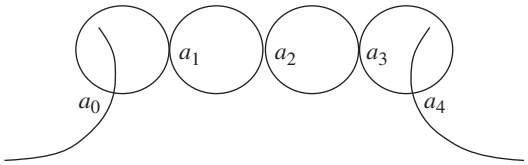


Figure 2. An open rosary of genus 3.

tacnodes (Figure 2). Each L_i intersects with L_j if and only if $|i-j|=1$. In [**4**, § 8.1] the Hilbert–Mumford index of an open rosary (with respect to a one-parameter subgroup coming from its automorphism group) embedded by $\omega_R^{\otimes 2}(2a_0+2a_{r+1})$ is computed, where a_0 and a_{r+1} are smooth points of L_1 and L_{r+1}, respectively.

Note that since $a_1,\dots,a_r$ are tacnodes, we are not able to apply the decomposition formula directly. However, in the proposition below, we shall demonstrate that a similar argument can be used to obtain a systematic analysis of the initial ideal.

Proposition 6.1. *The initial ideal of R with respect to the ρ-weighted Lex order satisfies the following decomposition: let T_l^d denote the set of degree d monomials in $x_0,\dots,x_{3l+2}$ (respectively, $x_0,\dots,x_{3r}$) that involve x_i and x_j for $i<3l-2$ and $j>3l-1$ for $l=1,\dots,r-1$ (respectively, $l=r$). We define $x_l=x_0$ if $l<0$ and $x_l=x_{3r}$ if $l>3r$.*

(i) *Degree* 2 *piece:*

$$(\operatorname{in} I_R)_2 \cup \{x_{3l-2}^2\}_{l=1}^r = \bigcup_{l=1}^{r+1} \operatorname{in}(I_{L_l} \cap k[x_{3l-5},\dots,x_{3l-1}])_2 \cup \Bigg(\bigcup_{l=1}^{r} T_l^2\Bigg).$$

(ii) *Degree* 3 *piece:*

$$(\operatorname{in} I_R)_3 \cup \{x_{3l-2}^3, x_{3l-2}^2 x_{3l-1}\}_{l=1}^r = \bigcup_{l=1}^{r+1} \operatorname{in}(I_{L_l} \cap k[x_{3l-5},\dots,x_{3l-1}])_3 \cup \Bigg(\bigcup_{l=1}^{r} T_l^3\Bigg).$$

Proof. The proof follows the idea of Lemma 3.1 closely, but more care needs to be exercised because of the additional overlapping coordinates.

(i) Suppose that $x^\alpha \in (\operatorname{in} I_R)_2$, then $x^\alpha = \operatorname{in} f$ for some homogeneous quadratic $f \in I_R$. If x^α is not contained in $\bigcup_{l=1}^r T_l$, then x^α is contained in $k[x_{3l-5},\dots,x_{3l-1}]$ for some l. Let $g = f(0,\dots,0,x_{3l-5},\dots,x_{3l-1},0,\dots,0)$. Then $g \in I_{L_l} \cap k[x_{3l-5},\dots,x_{3l-1}]$, and therefore x^α is contained in $(\operatorname{in} I_{L_l} \cap k[x_{3l-5},\dots,x_{3l-1}])_2$. Since $x_{3l-2}^2 - x_{3l-1}x_{3l} \in I_{L_{l+1}}$, we have $x_{3l-2}^2 \in \operatorname{in} I_{L_l+1}$.

For the other inclusion, suppose that $x^\alpha \in (\operatorname{in} I_{L_l} \cap k[x_{3l-5},\dots,x_{3l-1}])_2$ for some $l=1,2,\dots,r+1$. If $l=1$ or $r+1$, then $x^\alpha = x_0x_2$ or x_{3r-2}^2, respectively, but we know that $x_0x_2 = \operatorname{in}(x_0x_2 - x_1^2 - x_2x_3) \in \operatorname{in} I_R$. Hence, x^α is contained in the left-hand side. If $l=2,3,\dots,r$, then

$$I_{L_l} = \langle x_{3l-2}^2 - x_{3l-3}x_{3l-1}, x_{3l-3}x_{3l-2} - x_{3l-5}x_{3l-1}, x_{3l-5}x_{3l-2} - x_{3l-4}x_{3l-1},\\ x_{3l-3}^2 - x_{3l-4}x_{3l-1}, x_{3l-5}x_{3l-3} - x_{3l-4}x_{3l-2}, x_{3l-5}^2 - x_{3l-4}x_{3l-3}\rangle.$$

Let us denote the generators of I_{L_l} by f_i for $i=1,2,\dots,6$ in the order shown above.

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

The initial ideal of L_l is

$$\operatorname{in} I_{L_l} = \langle x_{3l-3}x_{3l-1}, x_{3l-4}x_{3l-1}, x_{3l-4}x_{3l-2}^2, x_{3l-5}x_{3l-1}, x_{3l-5}x_{3l-2}, x_{3l-5}x_{3l-3}, x_{3l-5}^2\rangle.$$

This shows that x^α comes from the initial term of some generator of I_{L_l}. That is, $x^\alpha = \operatorname{in}(f_i)$ for some i. But $f_i \in I_R$ for $i = 2, 3, 4, 5$, so it suffices to consider the $i = 1, 6$ cases. For the $i = 1$ case, $x^\alpha = x_{3l-3}x_{3l-1}$ and so $x^\alpha \in \operatorname{in}(x_{3l-2}^2 - x_{3l-3}x_{3l-1} - x_{3l-1}x_{3l}) \in \operatorname{in} I_R$. For $i = 6$, $x^\alpha = x_{3l-5}^2$. Obviously, $\bigcup_{l=1}^r T_l$ is contained in $\operatorname{in} I_R$.

(ii) Since $x_{3l-2}^2 \in \operatorname{in} I_{L_{l+1}}$, x_{3l-2}^3 and $x_{3l-2}^2 x_{3l-1}$ are also elements in $I_{L_{l+1}}$. If $x^\alpha \in \operatorname{in} I_R$ and not in $\bigcup_{l=1}^r T_l$, we can easily show that x^α is contained in the right-hand side by the same proof of the degree 2 case.

For the other inclusion, suppose that x^α is a degree 3 monomial in $\operatorname{in}(I_{L_l} \cap k[x_{3l-5}, \ldots, x_{3l-1}])$ for some l. If $l = 1$, then $x^\alpha = x_i(x_0x_2)$ for some $i = 0, 1, 2$. Hence, $x^\alpha = \operatorname{in}(x_i(x_0x_2 - x_1^2 - x_2x_3)) \in \operatorname{in} I_R$. If $l = r + 1$, then $x^\alpha = x_i(x_{3r-2})^2$ for some $i = 3r - 2, 3r - 1, 3r$. For $i = 3r$, we have $x^\alpha = \operatorname{in}(x_{3r}(x_{3r-2}^2 - x_{3r-1}x_{3r})) \in \operatorname{in} I_R$.

Suppose that $l = 2, 3, \ldots, r$. Then, by using the same notation as in the degree 2 case, we know that $x^\alpha = \operatorname{in}(x_i f_j)$ for $j = 1, 2, \ldots, 6$ and $i = 3l - 5, \ldots, 3l - 1$ except for the case in which $x^\alpha = x_{3l-4}x_{3l-2}^2$. If $j = 2, 3, 4, 5$, then $x_i f_j \in I_R$ and so $x^\alpha \in \operatorname{in} I_R$. If $j = 1$, then $x^\alpha = \operatorname{in}(x_i(x_{3l-2}^2 - x_{3l-3}x_{3l-1} - x_{3l-1}x_{3l})) \in \operatorname{in} I_R$. Hence, it remains to consider the case in which $j = 6$ and $x^\alpha = x_{3l-4}x_{3l-2}^2$.

For the $j = 6$ case, suppose that $x^\alpha = \operatorname{in}(x_i f_6) = x_i x_{3l-5}^2$. For $i = 3l - 5, 3l - 4$, there is nothing to prove. If $i > 3l - 4$, then $x^\alpha = \operatorname{in}(x_{3l-5}f_j)$ for $j = 2, 3, 5$. Hence, $x^\alpha \in \operatorname{in} I_R$ since $f_j \in I_R$, where $j = 2, 3, 5$. For the case in which $x^\alpha = x_{3l-4}x_{3l-2}^2$, $x^\alpha = x_{3l-4}x_{3l-2}^2 = \operatorname{in}\{x_{3l-4}(f_1 - x_{3l-1}x_{3l}) + x_{3l-3}(f_4)\} \in \operatorname{in} I_R$ since $f_1 - x_{3l-1}x_{3l}$ and f_4 are elements in I_R. Finally, $\bigcup_{l=1}^r T_l$ is contained in $\operatorname{in} I_R$. This completes the proof. □

By using the above proposition, we can compute inductively the sum of ρ-weights of degree 2 and 3 monomials in the initial ideal of I_R. Let $w_i(r)$ be the sum $\sum_{x^\alpha \in \operatorname{in}(I_R)_i} wt_\rho(x^\alpha)$ for $i = 2, 3$, where R is the open rosary of length $r + 1$. Then

$$w_2(r) = \begin{cases} w_2(r-2) + (72r - 110) & \text{for odd } r > 1, \\ w_2(r-2) + (72r - 92) & \text{for even } r > 2. \end{cases}$$

For the degree 3 case,

$$w_3(r) = \begin{cases} w_3(r-2) + (162r^2 - 162r - 57) & \text{for odd } r > 1, \\ w_3(r-2) + (162r^2 - 108r - 66) & \text{for even } r > 2. \end{cases}$$

Since $w_2(1) = 6$, $w_2(2) = 52$, $w_3(1) = 34$, $w_3(2) = 366$, we can easily compute the $w_i(r)$ as follows:

$$w_2(r) = \begin{cases} 18r^2 - 19r + 7 & \text{for odd } r, \\ 18r^2 - 10r & \text{for even } r, \end{cases}$$

and

$$w_3(r) = \begin{cases} 27r^3 + \frac{81}{2}r^2 - \frac{111}{2}r + 22 & \text{for odd } r, \\ 27r^3 + 54r^2 - 33r & \text{for even } r, \end{cases}$$

which is consistent with the results in [**4**, § 8.1].

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press

Acknowledgements. D.H. was partly supported by the following grants funded by the government of Korea: NRF Grant 2011-0030044 (SRC-GAIA), NRF-2013R1A1A 2010649 and the Korea Institute for Advanced Study (KIAS) grant. J.K. was partly supported by the following grants funded by the government of Korea: NRF Grants 2011-0030044 (SRC-GAIA) and 2011-0005072.

References

1. J. Alper, M. Fedorchuk and D. I. Smyth, Singularities with G_m-action and the log minimal model program for $\overline{M}_g$, *J. Reine Angew. Math.* DOI:10.1515/crelle-2014-0063.
2. D. Bayer and I. Morrison, Standard bases and geometric invariant theory, I, Initial ideals and state polytopes, *J. Symb. Computat.* **6**(2–3) (1988), 209–217.
3. D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry (available at www.math.uiuc.edu/Macaulay2/).
4. B. Hassett and D. Hyeon, Log minimal model program for the moduli space of curves: the first flip, *Annals Math. (2)* **177**(3) (2013), 911–968.
5. B. Hassett, D. Hyeon and Y. Lee, Stability computation via Gröbner basis, *J. Korean Math. Soc.* **47**(1) (2010), 41–62.
6. D. Hyeon and I. Morrison, Stability of tails and 4-canonical models, *Math. Res. Lett.* **17**(4) (2010), 721–729.
7. G. R. Kempf, Instability in invariant theory, *Annals Math. (2)* **108**(2) (1978), 299–316.
8. I. Morrison and D. Swinarski, Gröbner techniques for low-degree Hilbert stability, *Exp. Math.* **20**(1) (2011), 34–56.
9. D. Mumford, *Geometric invariant theory*, 1st edn, Ergebnisse der Mathematik und ihrer Grenzgebiete (Springer, 1965).
10. D. Schubert, A new compactification of the moduli space of curves, *Compositio Math.* **78**(3) (1991), 297–313.
11. B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, Volume 8 (American Mathematical Society, Providence, RI, 1996).

https://doi.org/10.1017/S0013091515000401 Published online by Cambridge University Press