J. Plasma Physics (2015), *vol.* 81, 905810101 © Cambridge University Press 2014
doi:10.1017/S0022377814000269

Transition from a coherent three wave system to turbulence with application to the fluid closure

Jan Weiland[1]†, Chuan S. Liu[2] and Anatoly Zagorodny[3]

[1]Chalmers University of Technology and EURATOM-VR Association, S-41296 Gothenburg, Sweden
[2]East-West Space Science Center, University of Maryland, College Park, MD, USA
[3]Bogoliubov Institute for Theoretical Physics, Kiev, Ukraine

(Received 19 March 2014; revised 15 April 2014; accepted 22 April 2014;
first published online 9 June 2014)

We start from a Mattor–Parker system and its generalization to include diffusion and derive the Random Phase equations. It is shown that the same type of fluid closure holds in the coherent and turbulent regimes. This is due to the fact that the Random Phase levels ($1/I_1 = 1/I_2 + 1/I_3$), where I_j is the intensity of wave packet 'j', are attractors for the wave dynamics both in the coherent and incoherent cases. Focus here is on the wave dynamics with phase velocities varying due to nonlinear frequency shifts. Thus a Maxwellian distribution function is kept in all cases.

1. Introduction

It is well known that the confinement time in fusion machines like tokamaks, is much shorter than expected from classical or neoclassical transport. Very strong evidence indicates that the source of transport is turbulence. A very fundamental reason for turbulence is instabilities driven by inhomogeneities in density and temperature since such inhomogeneities are directly linked to the confinement itself. A fundamental problem in the study of turbulence and transport in Tokamaks is that the typical phase velocities of waves are comparable to the thermal velocities of particles. This, in principle, means that a kinetic description is needed. However, fully nonlinear kinetic codes can still not be run as transport codes on the transport timescale. Of course, this problem was an even larger problem in the 1990s when first principles models started to be used to predict the performance of fusion reactors like the International Thermonuclear Experimental Reactor (ITER). This led to the development of fluid models with advanced closures. Since one of these models was very pessimistic about the performance of the previous ITER design (ITER Final Design Report (FDR) with a large radius of 8 m), Department of Energy (DOE) initiated a test of advanced fluid models against fully nonlinear kinetic codes on the turbulence timescale in the Cyclone project (Dimits et al. 2000). This test showed that the fluid model retaining linear Landau damping in the nonlinear regime gave about a factor 3 to high transport. In this system sources of density and temperature were understood to maintain fixed gradients in configuration space but no other sources were included. In order to obtain an understanding of these results, Mattor and Parker studied a three-wave system of slab ion temperature gradient (ITG) modes introducing a nonlinear closure at the third moment. An interesting aspect of the Mattor–Parker

† Email address for correspondence: elfjw@chalmers.se

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

system was that they replaced integration over particle orbits by integration over wave phases, thus avoiding the problem of particle stochasticity. The result was pretty much in agreement with the Cyclone results. The system with the nonlinear closure, on the other hand, was in good agreement with drift kinetic calculations but gave transport about a factor 3 below the fluid model with linear Landau damping. This was because the nonlinear frequency shift turned Landau growth into Landau damping at high amplitude levels. Although the specific calculation of the dynamics for the Mattor–Parker system and its generalization by Holod et al. (2002) is quite complicated, general procedures for introducing nonlinear frequency shifts in three-wave interaction give qualitatively the same result. The dynamics, in close agreement with a drift kinetic calculation, was explained as due to particle trapping which, in this coherent limit meant that the *wave–particle energy exchange* was completely *averaged out*. Later, a generalized system by Holod et al. (2002) with nonlinear closure in the fifth moment and with the option of adding diffusion to simulate energy transfer to background waves confirmed the good performance of the nonlinear closure. As it turned out, this system was also in very good agreement with the corresponding reactive system, i.e. with closure at the fourth moment without the dissipative kinetic resonance. This was to be expected on comparison with the Mattor–Parker system where this feature was interpreted as trapping. However, particle trapping acts back on the waves and we can see the dynamics as due to a nonlinear frequency shift, changing the phase velocity of the wave and turning Landau growth into Landau damping at high amplitude levels. This follows from the formula for the growth rate of the Universal instability. Thus, we need only to consider wave dynamics in a Maxwellian particle distribution. The reason why the explanation by Mattor and Parker was not generally accepted as the reason for the discrepancy between the fluid results with linear Landau damping and the kinetic simulations in the Cyclone simulations was that the Mattor–Parker system is coherent while the Cyclone simulations and tokamak turbulence are incoherent. However, we have also studied the turbulent limit by deriving a Fokker–Planck equation (Zagorodny and Weiland 2009) which has an exact solution (Chandrasekhar 1943) where there is no energy exchange between waves and resonant particles on the average.

The aim of the present paper is to bridge the coherent and turbulent systems to get a general description of the fluid closure. In doing so we find it useful to focus interest on quasi attractors of the system. With ‘quasi attractors’ we mean states which the system tends to approach if we leave it to itself without external sources.

Again we make an exemption for sources in configuration space, maintaining fixed gradients of density and temperature in the configuration space, i.e. fixed temperature and density gradients are a part of our isolated system. This means that we exclude instabilities caused by fast particles like Fishbone instabilities or Toroidal Alfve'n waves. This was exactly the situation in the Cyclone studies published in Dimits et al. (2000) and also in Mattor and Parker (1997) and Holod et al. (2002). However, it should be pointed out that other transport models using the same or similar types of gyro-Landau closure as the model with linear Landau damping in the Cyclone work, still need a fitting factor to nonlinear kinetics, to compensate for the large transport. This shows that the difference in transport levels found in the Cyclone work, i.e. in the case we study here is a realistic property of present day tokamak experiments.

Our manuscript will start with a general overview of the mode coupling equations. Then we will focus in turn on two ‘quasi attractors’ describing the states isolated systems tend to approach. Again, here it is important to exclude resonant sources in velocity space. Finally we will draw conclusions from these results.

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

2. Mode coupling equations

We will start by considering general forms of wave–wave interaction equations as given by Kadomtsev (1965), Dupree (1966), Hasegawa (1975), and Weiland and Wilhelmsson (1977). The simplest case of wave–wave interaction is three-wave interaction. This just means that a wave can be driven at the beat wave frequency and wave number of two other waves thus corresponding to quadratic nonlinearity. A natural extension is here to also include third-order nonlinearities (Weiland and Wilhelmsson 1977) since these always contribute resonant terms with nonlinear frequency shifts proportional to intensities. However, we will start with only quadratic nonlinearities.

2.1. *Three-wave mode coupling equations*

We start from coherent three-wave interaction and use the main formulas of Weiland and Wilhelmsson (1977).

$$\frac{\partial u_1}{\partial t} = s u_2 u_3 \cos \Phi, \tag{1a}$$

$$\frac{\partial u_2}{\partial t} = u_1 u_3 \cos \Phi, \tag{1b}$$

$$\frac{\partial u_3}{\partial t} = u_1 u_2 \cos \Phi, \tag{1c}$$

$$\frac{\partial \Phi}{\partial t} = \Delta\omega + u_1 u_2 u_3 \left[s \frac{1}{u_1^2} - \frac{1}{u_2^2} - \frac{1}{u_3^2} \right] \sin \Phi. \tag{1d}$$

Here s depends on the sign of the wave energy. Thus $s = -1$ corresponds to positive wave energy of all three waves. The book by Weiland and Wilhelmsson (1977) considers also the case $s = 1$ which can occur if there is free energy in the system. It corresponds to the interaction of one negative energy mode with two positive energy modes. For this case cubic nonlinearities are needed in order to avoid explosive instability. However, once these cubic terms are included the dynamics becomes quite similar to the case with only positive energy waves.

A general complex field is now written as $\boldsymbol{a}_\mathrm{j} = u_\mathrm{j} e_\mathrm{j}^{i\varphi}$, and we introduced

$$\Phi = \phi_1 - \phi_2 - \phi_3, \tag{1e}$$

$$\Delta\omega = \omega_1 - \omega_2 - \omega_3, \tag{1f}$$

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3. \tag{1g}$$

This system has the following Hamiltonian:

$$u_1 u_2 u_3 \sin \Phi - \frac{1}{2} \Delta\omega u_1^2 = \Gamma. \tag{2}$$

It also has the Manly–Rowe constants:

$$s u_1^2 - u_2^2 = M_1, \quad u_2^2 - u_3^2 = M_2. \tag{3}$$

In the following we will assume that both frequency mismatch and dissipation are small so that the basic dynamics is given by the perfectly matched conservative interaction.

Combining (1)–(3) we can obtain an exact analytical solution in terms of elliptic functions (Weiland and Wilhelmsson 1977). This can be expressed by (7).

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

2.2. *Dissipation*

In the presence of dissipation we can only obtain an analytical solution in the case when all waves have equal damping. In the general case, system (1) turns into

$$\frac{\partial u_1}{\partial t} + \nu_1 u_1 = -u_2 u_3 \cos\Phi, \tag{4a}$$

$$\frac{\partial u_2}{\partial t} + \nu_2 u_2 = u_1 u_3 \cos\Phi, \tag{4b}$$

$$\frac{\partial u_3}{\partial t} + \nu_2 u_2 = u_1 u_2 \cos\Phi, \tag{4c}$$

$$\frac{\partial \Phi}{\partial t} = \Delta\omega + u_1 u_2 u_3 \left[\frac{1}{u_1^2} - \frac{1}{u_2^2} - \frac{1}{u_3^2}\right] \sin\Phi. \tag{4d}$$

We will in the following consider only positive wave energy. Then $s = -1$ and the variation in time of the Hamiltonian (2) due to dissipation is

$$\begin{aligned}\frac{\partial}{\partial t}\left[u_1 u_2 u_3 \sin\Phi\right] &= \sin\Phi \left\{-(\nu_1+\nu_2+\nu_3)u_1u_2u_3 + (u_1u_2u_3)^2 \cos\Phi \left[\frac{1}{u_2^2} + \frac{1}{u_3^2} - \frac{1}{u_1^2}\right]\right\} \\ &\quad + u_1u_2u_3 \cos\Phi \left\{-\Delta\omega + u_1u_2u_3 \sin\Phi \left[\frac{1}{u_1^2} - \frac{1}{u_2^2} - \frac{1}{u_3^2}\right]\right\} \\ &= -(\nu_1+\nu_2+\nu_3)u_1u_2u_3 \sin\Phi - \Delta\omega u_1u_2u_3 \cos\Phi. \end{aligned} \tag{5}$$

Now since

$$u_1u_2u_3 \cos\Phi = u_1 \left(\frac{\partial u_1}{\partial t} + \nu_1 u_1\right),$$

we obtain

$$\frac{\partial}{\partial t}\left[u_1u_2u_3 \sin\Phi - \frac{1}{2}\Delta\omega u_1^2\right] = -(\nu_1+\nu_2+\nu_3)u_1u_2u_3 \sin\Phi + \Delta\omega\nu_1 u_1^2. \tag{6}$$

Of course we recover (2) from (6) in the absence of dissipation.

3. Quasi attractors

As explained above we will now consider quantities which are attractors of the system in the sense that the system tends to approach these states when they are isolated. Since we study inhomogeneous systems, we include in these attractors the 'external' action of sources in configuration space which have no other influence than to maintain fixed gradients in density and temperature. These attractors will be called 'Quasi attractors'.

The first case of an attractor is the minimum of the potential of three-wave interactions without dissipation:

$$\frac{1}{2}\left(\frac{\partial u_j}{\partial t}\right)^2 + \Pi(u_j) = 0, \tag{7}$$

where (7) is derived from (2) and (3) and Π; is an effective potential. From (7) it is straightforward to derive analytical solutions of (1) in terms of elliptic functions (Weiland and Wilhelmsson 1977). The typical dynamics is displayed in Fig. 1.

Of course, also the motion of a trapped particle in a wave potential will have a similar dynamics.

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

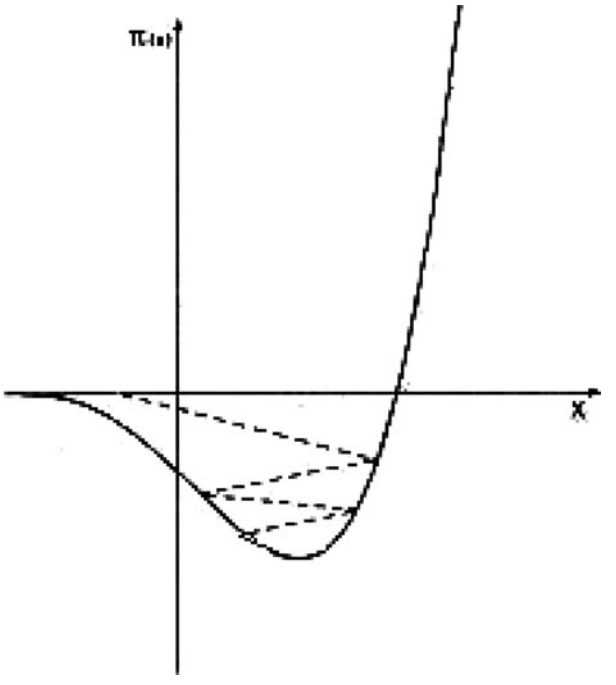


FIGURE 1. Potential in (7). The dotted line indicates motion in the presence of dissipation.

4. The first quasi attractor

In a nontrivial asymptotic equilibrium all time derivatives vanish. The phase (1a) is then

$$\sin \Phi = -\frac{\Delta\omega}{u_1 u_2 u_3 \left[\frac{1}{u_1^2} - \frac{1}{u_2^2} - \frac{1}{u_3^2}\right]}. \tag{8a}$$

Thus, the phase Φ depends strongly on the sign of

$$\Theta = \frac{1}{u_1^2} - \frac{1}{u_2^2} - \frac{1}{u_3^2}. \tag{8b}$$

Here $\Theta = 0$ is the quasi attractor we are looking for. This state actually coincides with the minimum point of potential Π in the absence of mismatch which is our lowest-order case. When we add dissipation the system will follow the dotted line in Fig. 1. This is most easily seen if we consider the solution of (6) without mismatch as seen in (10).

We may rewrite (6) as

$$\frac{\partial}{\partial t}\left[u_1 u_2 u_3 \sin\Phi - \frac{1}{2}\Delta\omega u_1^2\right] = -(\nu_1+\nu_2+\nu_3)u_1 u_2 u_3 \sin\Phi + \frac{1}{2}\Delta\omega(\nu_1+\nu_2+\nu_3)u_1^2 + \frac{1}{2}\Delta\omega(\nu_1-\nu_2-\nu_3)u_1^2$$

or

$$e^{-(\nu_1+\nu_2+\nu_3)t}\frac{\partial}{\partial t}\left\{\left[u_1 u_2 u_3 \sin\Phi - \frac{1}{2}\Delta\omega u_1^2\right] e^{(\nu_1+\nu_2+\nu_3)t}\right\} = \frac{1}{2}\Delta\omega(\nu_1-\nu_2-\nu_3)u_1^2. \tag{9}$$

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

If $\Delta\omega = 0$ or the linear combinations of dampings in the right hand side of (9) are small, we can integrate (9) analytically. In case $\Delta\omega = 0$, we obtain

$$u_1 u_2 u_3 \sin \Phi = \Gamma e^{-(\nu_1+\nu_2+\nu_3)t}. \tag{10}$$

The damping of the integral of motion corresponds to the dotted line in Fig. 1, describing an approach to the equilibrium point. This point is close to the balance $\Theta = 0$ or

$$\frac{1}{u_1^2} = \frac{1}{u_2^2} + \frac{1}{u_3^2}. \tag{11}$$

Equation (11) also corresponds to the random phase levels, i.e. the relation approached by the intensities in the random phase equations for long times.

5. The second quasi attractor

Our second quasi attractor also can be seen as the minimum of a potential according to (7). However, this is the usual particle trapping in a wave which we here replace by trapping of a wave in a particle distribution due to an oscillating nonlinear frequency shift. This will become obvious from our numerical results. Since we will consider wave dynamics with a spread in wave numbers we will take averages over the wave packets. This does, however, not change the general picture.

In the three-wave system in Holod et al. (2002), the damping of the waves was due to viscosity and diffusion. Clearly the initial growth was there due to the ITG instability (fluid); so for turbulence to be created in the first place we must assume that the linear growth of the most unstable mode exceeds its damping. On the other hand, an approach to a final stationary state, shown by Fig. 1, requires that the sum of the linear growths and dampings for the three waves is positive.

6. Cubic nonlinearities and wave-particle resonances

We are here making a comparatively general study of three-wave systems. However, a major application will be to questions of fluid closure as discussed by Mattor and Parker and Holod et al. (2002). The simplest case of wave–particle resonances is for the Landau growth as occurring in the Universal instability. This application is also present in Mattor and Parker (1997) and Holod et al. (2002). As an example the simple Landau growth due to electron Landau damping (isothermal regime) is (*c.f.* Weiland 2012)

$$\gamma = \left(\frac{\pi}{2}\right)^{1/2} \omega_{*e} \frac{\omega - \omega_{*e}}{k_{||}\, v_{te}} e^{-\omega^2/(k_{||}\, v_{te})}. \tag{12}$$

The same type of solution is obtained also for ion Landau damping since we have a type of symmetry between ion and electron dynamics here (compare with Liu (1969)).

The interpretation of this result is that the inhomogeneity introduces a free energy which opens the possibility of Landau growth. Whether this is a growth or damping clearly depends on if the frequency is larger or smaller than the drift frequency. For the simple slab ITG mode, as studied by Mattor and Parker and Holod et al. it is obvious that the linear ITG growth rate is increased by the linear wave particle resonance. This is what leads to the too high saturation level for the Hammett–Perkins closure, where the linear growth rate was kept by Mattor and Parker. When a nonlinear frequency shift is included it will cause the kinetic resonance part of the growth rate to change sign so that the kinetic resonance stabilizes at high amplitude (Fig. 2). This is obvious

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

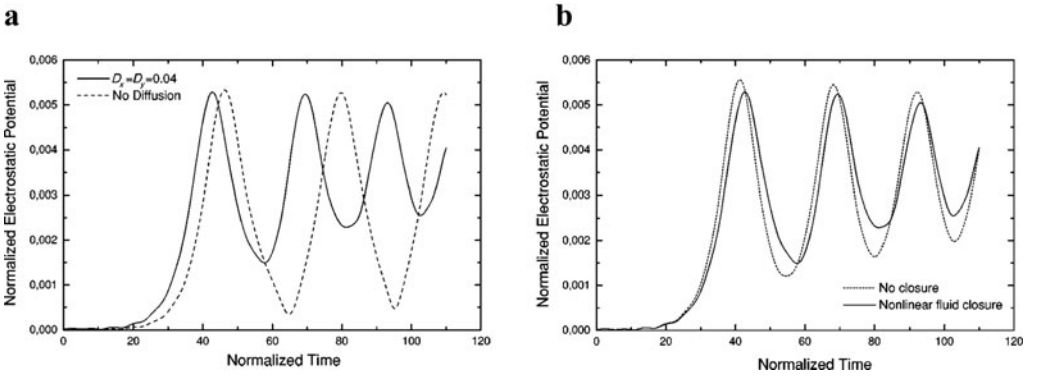


FIGURE 2. Time variation of the generalized Mattor–Parker system. Figure 2(a) compares the case without (dotted) and with diffusive damping. Figure 2(b) compares the cases with and without (dotted) closure (dissipative kinetic resonance) both with diffusive damping causing an approach to the random phase levels (Fig. 1). We note that the dissipative kinetic damping is destabilizing near minima and stabilizing near maxima. (From Holod et al. Phys. Plasmas **9**,1217 (2002), with the permission of AIP).

from Holod et al. where the oscillatory behavior of the wave particle resonance is just like particle trapping as emphasized by Mattor and Parker. However, the dynamics is exactly the same in the two papers (Fig. 2(a) without diffusive damping is very similar to the Mattor–Parker result). Thus, we can conclude that the oscillation of the wave phase in the Maxwellian distribution, due to the inhomogeneity in configuration space, averages out the energy exchange between waves and particles in time just as particle trapping would. The instability is ultimately stabilized by the three-wave interaction. However, for transport in the general case we should also include the effect of zonal flows which are stronger for a reactive closure. The Weiland model in Dimits et al. (2000) did not have zonal flows explicitly included. However, the normalization was made for absorbing boundaries in k-space, thus requiring zonal flows to enter for long wavelengths.

The growth rate equation (12) is a special case of the more general expression (Hasegawa 1975)

$$\gamma = -\frac{\mathrm{Im}D(\omega,\mathbf{k})}{\partial \mathrm{Re}D(\omega,\mathbf{k})/\partial\omega}, \tag{13}$$

$$D(\omega,\mathbf{k}) = \omega\varepsilon(\omega,\mathbf{k})$$
$$= \omega\left\{1+\frac{4\pi e^2}{k^2}\left[1+\int dv \frac{\omega-\omega_{*e}\left[1+\eta_{i_i}(m_i v^2/2T_i-3/2)\right]}{\omega-k_{||}\,v_{||}-\omega_D(v_{||}^2+v_\perp^2/2)/v_{\mathrm{thi}}^2} J_0^2(\xi_k)F_M\right]\right\}, \tag{14a}$$

where $\xi_k = kv/\Omega_c$.

We note that the difference between frequency and drift frequency again appears but here inside the integral. Thus, equation (12) is still qualitatively correct. According to the Landau rule it will appear evaluated at the resonance.

$$\int \frac{g(\omega,\mathbf{v})}{\omega-r(\mathbf{v})}dv = P\int \frac{g(\omega,\mathbf{v})}{\omega-r(\mathbf{v})}dv + i\pi\,(\mathrm{g}(\omega,\mathbf{v}))_{|r(v)=\omega}\,. \tag{14b}$$

We note that a nonlinear frequency shift of the real part of the frequency, through the wave–particle interaction, now also will create effective nonlinear dissipation as discussed in Weiland and Wilhelmsson (1977). The situation here is the same as

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

in the Mattor–Parker case and thus particles will perform oscillations in the wave potential or, as discussed above, the phase velocity of the wave will oscillate in the particle distribution leading to averaging out of energy exchange between waves and particles.

We thus conclude that according to (13) and (14) the sign of the wave–particle energy transfer will always be determined by the difference between frequency and drift frequency. As we could see from Fig. 2 this sign will oscillate in time due to the nonlinear frequency shift in three-wave interaction. Moreover, the oscillations will disappear asymptotically in time when the system has come close to the random phase levels. This is also consistent with Fig. 2. Now the frequency was defined in an eikonal fashion both by Mattor and Parker (1997) and Holod et al. (2002). Since the derivations involve several nonlinearities we have to consider the effects of both quadratic and cubic nonlinearities. Although cubic nonlinearities should be smaller they have a more clear character of nonlinear frequency shifts since their resonant contributions enter as magnitude squares. They are also phase independent. Thus, we should add cubic nonlinearities to our systems (1) and (4). A real frequency shift enters as a frequency mismatch (Weiland and Wilhelmsson 1977), i.e.

$$\delta\omega_{\mathrm{nl}} = \sum_j \alpha_j u_j{}^2. \tag{15}$$

As it turns out, the system (1) and the conserved quantity (2) remain valid if we make the replacement

$$\Delta\omega \to \Delta\omega + \delta\omega_{\mathrm{nl}}. \tag{16}$$

We now recall that a nonlinear three-wave system is the fundamental building block of turbulence. The systems in Mattor and Parker (1997) and Holod et al. (2002) moreover include a zonal flow so the system really mimics the situation in a tokamak. In the absence of viscosity and diffusion, the system is conservative and can be seen as describing what happens when we have reflecting walls in k-space. With viscosity and diffusion the system approaches an asymptotic stationary state resembling stationary turbulence.

Thus, we have reason to consider these results as fundamental. The fact that the wave–particle resonance is sometimes stabilizing and sometimes destabilizing is a natural consequence of the dynamics where wave amplitudes oscillate. However, here it is primarily the phase velocity of the waves that oscillates due to nonlinear frequency shifts.

$$v_f \to \frac{\omega + \delta\omega}{k} \tag{17}$$

Figure 2 here shows that the energy exchange really tends to cancel out in time just as for particle trapping. However, the nature of oscillating phase velocity here means that we do not have to worry about flattening of the distribution function even on the average! What we may then worry about is the transition to a many-wave system. We have already noted that the levels (equation (11)) which are an important limit of the coherent system actually are the *random phase levels.*

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

7. Multiple three-wave systems

We will now look at a multiple three-wave system. It can be described by (Weiland and Wilhelmsson 1977):

$$\begin{aligned}
\frac{\partial u_{1j}}{\partial t} &= -\sum_{k,l} u_{2k}u_{3l}\cos(\Phi_{jkl} + \Delta\omega_{jkl}t),\\
\frac{\partial u_{2k}}{\partial t} &= \sum_{jl} u_{1j}u_{3l}\cos(\Phi_{jkl} + \Delta\omega_{jkl}t),\\
\frac{\partial u_{3l}}{\partial t} &= \sum u_{1j}u_{2k}\cos(\Phi_{jkl} + \Delta\omega_{jkl}t),\\
\frac{\partial \phi_{1,j}}{\partial t} &= \sum_{k,l} \frac{u_{2k}u_{3l}}{u_{1j}}\sin(\Phi_{jkl} + \Delta\omega_{jkl}t),\\
\frac{\partial \phi_{2,k}}{\partial t} &= \sum_{j,l} \frac{u_{1j}u_{3l}}{u_{2k}}\sin(\Phi_{jkl} + \Delta\omega_{jkl}t),\\
\frac{\partial \phi_{3,l}}{\partial t} &= \sum_{j,k} \frac{u_{1j}u_{2k}}{u_{3l}}\sin(\Phi_{jkl} + \Delta\omega_{jkl}t).
\end{aligned} \tag{18}$$

The individual phase equations are the same as obtained for the coherent system (1) and the phase Φ is a linear combination of those. The most significant quantities are now the intensities of the wave packets. They are written as

$$I_j = \sum_k u_{jk}{}^2. \tag{19}$$

These will fulfill

$$\frac{\partial I_j}{\partial t} = \mp 2\sum_{kl} u_{1j}u_{2k}u_{3l}\cos\Phi_{jkl}, \tag{20}$$

where the minus sign is valid for mode 1 and + for modes 2 and 3.

We then obtain the Manley–Rowe relations:

$$\begin{aligned}
I_1 + I_2 &= M_1\\
I_2 - I_3 &= M_2.
\end{aligned} \tag{21}$$

By summing over all individual couplings we also obtain a generalization of the Hamiltonian (2) (Weiland and Wilhelmsson 1977), i.e.

$$\sum_{jkl} u_{1j}u_{2k}u_{3l}\cos\Phi_{jkl} + \frac{1}{2}\sum_{j,k}\Delta\omega_{jk}u_{jk}{}^2 = \Gamma. \tag{22}$$

It is also natural to define

$$\langle\frac{\partial\Phi_{jkl}}{\partial t}\rangle = \langle\frac{\partial\phi_1}{\partial t}\rangle - \langle\frac{\partial\phi_2}{\partial t}\rangle - \langle\frac{\partial\phi_3}{\partial t}\rangle. \tag{23}$$

Then

$$\frac{\partial \mathrm{I_j}}{\partial \mathrm{t}} = \mp 2n\langle u_{1j}u_{2k}u_{3l}\cos\Phi_{jkl}\rangle, \tag{24}$$

where $\langle\ \rangle$ denotes ensemble average and n is the number of coupling terms.

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

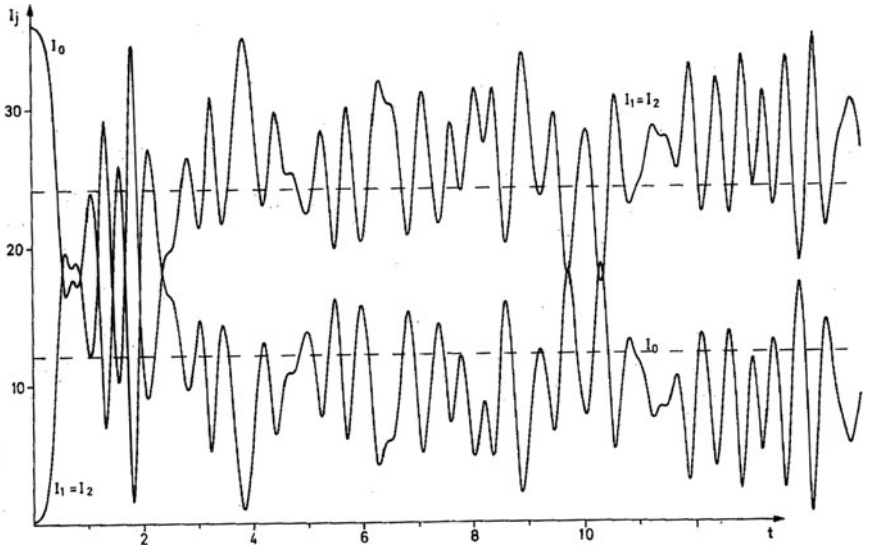


FIGURE 3. The approach to the random phase levels (dashed) of the intensities of three interacting wave packets with four waves in each packet. Note that the levels of intensities I_0 and $I_1 = I_2$ are swapped from the initial conditions. Here the constant of motion $\Gamma = 0.62$. From Weiland and Wilhelmsson (1977). (Copyright J. Weiland).

Now

$$\langle \frac{\partial \Phi_{jkl}}{\partial t} \rangle = \langle \Delta\omega_{jkl} \rangle + \left(\Gamma - \frac{1}{2} \sum_{j,k} \Delta\omega_{jk} |u_{jk}|^2 \right) \left(\frac{1}{I_1} - \frac{1}{I_2} - \frac{1}{I_3} \right), \tag{25}$$

where

$$\Gamma = n \langle u_{1j} u_{2k} u_{3l} \sin \Phi_{jkl} \rangle + \frac{1}{2} \sum \Delta\omega_{jk} u_{jk}^2. \tag{26}$$

Figure 3 shows the oscillations around the random phase levels for a partially coherent system.

Equation (26) is a direct generalization of the corresponding Hamiltonian for a single three-wave system (Weiland and Wilhelmsson 1977) and thus gives a conservation relation for our multi-wave system. Equation (25) shows that the random phase levels are *attractors* for the multiwave system as expected. These levels correspond exactly to levels (equation (11)) for the coherent system. As we have seen, a linear damping makes the coherent system approach levels (equation (11)). Actually the linear damping was mainly introduced to correspond to energy being absorbed by other waves. This is exactly what happens with system (18). This gives an approach to the random phase levels due to (25) as shown in Fig. 2. However, with a finite number of waves the system has a Poincare cycle and eventually returns close to the initial state. With infinite number of waves the system approaches the random phase levels irreversibly (Hasegawa 1975). Thus, on the average we have the same behavior for the incoherent system as for the coherent system with dissipation. It is also straight forward to add nonlinear frequency shifts, leading to

$$\Delta\omega_{jk} \to \Delta\omega_{jk} + \delta\omega_{nljk}. \tag{27}$$

This, again, gives us back the coherent system where the numerous couplings to other waves give us the approach to the random phase levels on the average. The

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

continuum limit of partial waves is taken in the book by Hasegawa (1975) and this continuum limit applies also to our first quasi attractor.

7.1. *Compatibility of attractors*

We are now in a weakly nonlinear state where only the nonlinear frequency shift corresponds to a strongly nonlinear effect. The random phase levels (equation (11)) will remain the same if we multiply all amplitudes by the same factor while the second attractor, i.e. the tendency for the nonlinear frequency shift to change the sign of the dissipation (13), (14) is sensitive to such a factor. Thus, we have no problems to fulfill both these conditions at the same time (compare with Fig. 2(b)). Note that we no longer need the viscosity and diffusion originally introduced in the coherent system. However, we need to add the linear growth rate which here will also include the kinetic resonances as described by (13) and (14). Due to these resonances, the nonlinear frequency shifts, (15) will give both real and imaginary contributions.

8. Damping or growth

When we include damping or growth we have to consider averages over the wave packets, i.e.

$$\bar{\nu}_j = \sum \nu_{jk} u_{jk}{}^2 / I_j, \tag{28a}$$

and thus

$$\frac{\partial I_j}{\partial t} + \bar{\nu}_j I_j = \mp 2 \sum_{jkl} u_{1j} u_{2k} u_{3l} \cos \Phi_{jkl}. \tag{28b}$$

Here the damping or growth $\bar{\upsilon}$ as given by (28a) again will be nonlinear and include the effects of nonlinear frequency shifts as included in (27). This will, however, work in the same way as for the coherent case. Thus, the limiting case where $\bar{\upsilon}$ is only the reactive fluid, growth rate will be an attractor.

By including sideband couplings in the Vlasov equation and substituting these into our main mode equation, keeping only resonant terms, we obtain nonlinear frequency shifts (as discussed above) and a Fokker–Planck equation (Dupree 1966; Hasegawa 1975). Here the diffusion coefficient and friction are due to cubic nonlinearities.

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right) f(\mathbf{r}, \mathbf{v}, \mathrm{t}) = \frac{\partial}{\partial \mathbf{v}} \cdot \left[\beta \mathbf{v} + \mathrm{D}^v \frac{\partial}{\partial \mathbf{v}} \right] f(\mathbf{r}, \mathbf{v}, \mathrm{t}), \tag{29a}$$

$$\beta = \sum_j \beta_j u_j{}^2, \quad D^v = \sum_j d_j u_j{}^2, \tag{29b}$$

where now friction β and diffusion in velocity space D^v are due to turbulence and we introduced a non-specified turbulent amplitude u_j. We can also derive (29) by the method of transition probabilities introduced at the Bogoliubov Institute of Theoretical Physics (Zagorodny and Weiland 2009). Here the derivations in Dupree (1966) and Hasegawa (1975) are more focused on mode coupling while the method by Zagorodny and Weiland (2009) is more focused on velocity space. However, the result is the same! Of course, the mode coupling derivation by Dupree and Hasegawa connects directly to our multi-wave system, so it is clear that the Fokker–Planck equation comes out of our multi-wave system in the phase mixed limit.

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

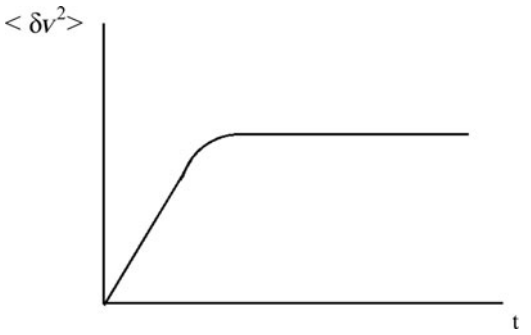


FIGURE 4. Mean square velocity deviation from analytical solution of (29) for constant coefficients. The initial development is quasilinear while the final is due to strongly nonlinear effects. The transition is at $t = 1/\beta$.

Equation (29) leads asymptotically to a state where there is no energy exchange between waves and particles. For constant coefficients (29) has an exact analytical solution (Chandrasekhar 1943).

The saturation in time of the mean square velocity deviation is sometimes called Dupree damping after the original derivation (Dupree 1966). Here Fig. 3 is only qualitative. Detailed calculations are shown in Holod et al. (2005). In general the friction coefficient β is complex and this leads to an oscillative approach to the asymptote as shown in simulations in Figs. 3 and 4 of Holod et al. (2005). We also recall that Kadomtsev's weak coupling approximation (Kadomtsev 1965) introduces a nonlinear frequency shift in the random phase equations. This is actually a strongly nonlinear effect just as our nonlinear frequency shifts introduced here

9. Particle trapping

It is natural that particle trapping is considered in a theory of fluid closure. However, in the present theory, where emphasis is put on wave dynamics, we can include also magnetic particle trapping in a fairly simple way. The main difference is in (14). The differences occur in the resonant denominator where the parallel motion has been averaged out and the magnetic drift becomes averaged over the trapped orbit (precession frequency).

$$D(\omega, \mathbf{k}) = \omega\varepsilon(\omega, \mathbf{k})$$
$$= \omega\left\{1 + \frac{4\pi e^2}{k^2}\left[1 + \int dv \frac{\omega - \omega_{*e}\left[1 + \eta_{i_i}(m_i\mathbf{v}^2/2T_i - 3/2)\right]}{\omega - < \omega_D(v_{||}^2 + v_{\perp}^2/2)/v_{\mathrm{thi}}^2 >} J_0^2(\xi_k)F_M\right]\right\} \tag{30}$$

Here only the resonance point is modified so (14b) is still valid in its general form and nonlinear frequency shifts will still be able to change the sign of the resonance. The frequency shift will remain such as to maintain momentum in particle trapping. As outlined initially the dissipation is assumed to be small but the tendency to change the growth rate in such a way as to reduce dissipation will remain. Thus, kinetic resonances will still tend to be averaged out. The Fokker–Planck equation will also remain in the same form. Actually, the numerical results from Mattor and Parker and Holod et al. were for only the parallel resonance. Now, the Fokker–Planck equation can be derived also with the magnetic drift resonance (Holod et al. 2005) which is the first generalization needed. Then, to instead use the bounce averaged magnetic drift is a smaller modification. The friction, and accordingly the timescale may change but

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

the final asymptotic state will still correspond to a situation where there is no energy transfer between waves and particles on the average.

10. The L-H transition

As we have seen in Weiland (2012), the Waltz condition (Waltz et al. 1994) works quite well for the L-H transition and also for the formation of internal transport barriers in our fluid model.

$$\gamma \rightarrow \gamma - \omega_{E\times B}. \tag{31}$$

Here $\omega_{\mathrm{E}\times\mathrm{B}}$ is the flow shear rate. It was found by Waltz et al. (1994) by turbulence simulations. Later a coefficient around 0.7 was included in front of the flow shear rate. However, later the result equation (31) was derived analytically by a nonlinear kinetic method (Zagorodny and Weiland 2009). The growth rate used in (31) should be the linear growth rate and that is higher for a fluid model than for a kinetic model in the toroidal case. Nevertheless, only half the Dimits nonlinear upshift (Dimits et al. 2000) was obtained with the Landau fluid model in the Cyclone tests while we have several indications that our fluid model gives the right upshift (Moestam et al. 2004). The reason is that the nonlinearly generated rotation (Zonal flow) contains the fluid resonance in the denominator. In a fluid model which includes the linear dissipative kinetic resonance, this dissipation gets added in the resonant denominator, thus reducing the rotation. Thus, we expect the flow shear rate to be stronger in a fluid model which keeps the fluid resonances but uses a reactive closure. Thus, the success of simulation of the L-H transition with the reactive fluid model in Weiland (2012) is not surprising. Another aspect of stronger zonal flows in the fluid model is that the zonal flow damping is stronger for long wavelengths. Thus, the requirements on radial box size for absorbing boundary for long wavelengths are less restrictive for a reactive fluid model. Thus, the fluid model in Weiland (2012) was derived assuming absorbing boundaries for both long and short wavelengths. Thus, we can apply a Large Eddy Simulation (LES) for short wavelengths and a Small Eddy Simulation for long wavelengths. The validity of the small eddy simulation requires large enough radial box size, actually sometimes of the order of the system size. Transport barriers are obtained when zonal flows are important at the correlation length.

11. Discussion

We have, in this work, tied together the studies of the fluid closure for simple three-wave systems with a partially coherent multi-wave system which in the continuum limit gives the random phase equations or the Fokker–Planck equation. We have thereby kept the focus on the wave dynamics, which is much simpler than the particle dynamics. Another great advantage with this is that we do not have to rely on flattening of the distribution function. Instead we use only a background Maxwellian distribution. This is possible since the nonlinear frequency shift makes the phase velocity of the waves oscillate in the particle distribution. Although we have only been able to show that the energy exchange between waves and particles vanishes approximately for the coherent system (it would disappear exactly for coherent trapping) our final Fokker–Planck equation indeed gives exact cancellation of wave–particle energy transfer. An important aspect is that the *random phase levels are the same in the coherent and incoherent systems and act as an attractor which is compatible with the attractor which is obtained for the case where the nonlinear frequency shift*

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

exactly cancels the dissipative kinetic resonance. We have also shown that the multi-wave system, on the average, behaves exactly as the coherent system with damping, where the phase mixing in the multi-wave system has the same effect as the damping in the coherent system. Now we have considered only turbulence driven by gradients in configuration space. In cases where we have resonant sources in velocity space we have to add these in the Fokker–Planck equation and this case is not considered here. This happens, e.g. in Fishbone modes and toroidicity-induced shear Alfvén eigenmode (TAE) modes but also for strong thermal excitation as shown in Rosenbluth and Liu (1976). The advantages with a fluid model with a good closure is that it saves enormous amounts of computing time for the simulation of tokamak transport as compared to a fully nonlinear kinetic code. Then it also improves our understanding of several important phenomena. A particularly striking case is that of particle and heat pinches. Wagner and Stroth (1993) have given a particularly good review of tokamak transport where the need for a particle pinch has been emphasized. Already the first version of our fluid transport model recovered both the electron heat and particle pinches (Weiland and Nordman 1993) in a DIII-D discharge with off–axis electron–cyclotron resonance (ECR) heating. In particular the particle pinch is sensitive to the fluid closure since it is completely absent at the typical correlation length of ITG modes in a quasilinear kinetic model (linear real frequency; Weiland et al. 2011). In the Fokker–Planck equation due to turbulent collisions, friction, giving the Dupree damping, also enters as a nonlinear frequency shift. Particle pinches may occur also in space plasmas and this was discussed also for our fluid model (Weiland 2012) in Weiland (2010). The model described in Weiland (2012) has developed considerably over the years and has for a long time been a part of the Multi-Mode Model. The latest achievements were published by Rafiq et al. (2013).

Acknowledgements

The authors are grateful to Ihor Holod for discussions.

REFERENCES

Chandrasekhar, S. 1943 Stochastic problems in Physics and Astronomy. *Rev. Mod. Phys.* **15**(1), 1–89.

Dimits, A. M. et al. 2000 Comparison and physics basis of tokamak transport models and turbulence simulations. *Phys. Plasmas.* **7**(3), 969–983.

Dupree, T. H. 1966 A perturbation method for strong plasma turbulence. *Phys. Fluids* **9**, 1773.

Hasegawa, A. 1975 *Plasma Instabilities and Nonlinear Effects*. New York: Springer.

Holod, I., Weiland, J. and Zagorodny, A. 2002 Nonlinear fluid closure: three-mode slab ion temperature gradient problem with diffusion. *Phys. Plasmas* **9**, 1217.

Holod, I., Zagorodny, A. and Weiland, J. 2005 Anisotropic diffusion across an external magnetic field and large-scale fluctuations in magnetized plasmas. *Phys. Rev.* E **71**, 046401.

Kadomtsev, B. B. 1965 *Plasma Turbulence*. New York: Academic Press.

Liu, C. S. 1969 Temperature gradient instabilities in axisymmetric systems. *Phys. Fluids* **12**, 1489.

Mattor, N. and Parker, S. E. 1997 Nonlinear kinetic fluid equations. *Phys. Rev. Lett.* **79**, 3419.

Moestam, R., Shaikh, D. and Weiland, J. 2004 Self consistent theory of zonal flows in ion temperature gradient turbulence. *Phys. Plasmas* **11**, 4801.

Rafiq, T., Kritz, A. H., Weiland, J., Pankin, A. Y. and Luo, L. 2013 Physics basis of multi-mode anomalous transport module. *Phys. Plasmas* **20**, 032506.

Rosenbluth, M. N. and Liu, C. S. 1976 Cross-field energy transport by plasma waves. *Phys. Fluids* **19**, 815.

Wagner, F. and Stroth, U. 1993 Transport in toroidal devices-the experimentalists view. *PPCF* **35**, 1321.

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press

Waltz, R. E., Kerbel, G. O. and Milovich, J. 1994 Toroidal gyro-Landau fluid model turbulence simulations in a nonlinear ballooning mode representation with radial modes. *Phys. Plasmas* **1**, 2229.

Weiland, J. 2010 Turbulence at a pinch. *Nature Phys.* **6**, 167.

Weiland, J. 2012 *Stability and Transport in Magnetic Confinement Systems*. New York: Springer.

Weiland, J. and Nordman, H. 1993 Drift wave model for inward energy transport in tokamak plasmas. *Phys. Fluids* **B5**, 1669.

Weiland, J. and Wilhelmsson, H. 1977 *Coherent Nonlinear Interaction of Waves in Plasmas*. Oxford: Pergamon Press.

Weiland, J. and Zagorodny, A. 2011 Fluid closure, relation to particle pinches, fluid resonances. In *Proceedings of the IFP-CNR – Chalmers Workshop on Nonlinear Phenomena in Fusion Plasmas*, Villa Monastero, Varenna, June 8–10 2011, AIP Conference Proceedings Vol 1392, pp. 33.

Zagorodny, A. and Weiland, J. 2009 Non-Markovian renormalization of kinetic coefficients for drift type turbulence in magnetized plasmas. *Phys. Plasmas* **16**, 052308.

https://doi.org/10.1017/S0022377814000269 Published online by Cambridge University Press