Ergod. Th. & Dynam. Sys. (2020), **40**, 2467–2481 © *Cambridge University Press, 2019*
doi:10.1017/etds.2019.23

An answer to Furstenberg's problem on topological disjointness

WEN HUANG, SONG SHAO and XIANGDONG YE

Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences and Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, PR China

(e-mail: wenh@mail.ustc.edu.cn, songshao@ustc.edu.cn, yexd@ustc.edu.cn)

(*Received* 6 *December* 2018 *and accepted in revised form* 13 *February* 2019)

Abstract. In this paper we give an answer to Furstenberg's problem on topological disjointness. Namely, we show that a transitive system (X, T) is disjoint from all minimal systems if and only if (X, T) is weakly mixing and there is some countable dense subset D of X such that for any minimal system (Y, S), any point $y \in Y$ and any open neighbourhood V of y, and for any non-empty open subset $U \subset X$, there is $x \in D \cap U$ such that $\{n \in \mathbb{Z}_+ : T^n x \in U, S^n y \in V\}$ is syndetic. Some characterization for the general case is also given. By way of application we show that if a transitive system (X, T) is disjoint from all minimal systems, then so are $(X^n, T^{(n)})$ and (X, T^n) for any $n \in \mathbb{N}$. It turns out that a transitive system (X, T) is disjoint from all minimal systems if and only if the hyperspace system $(K(X), T_K)$ is disjoint from all minimal systems.

Key words: topological dynamics, joining, disjoint, weak mixing, central set
2010 Mathematics Subject Classification: 37B05 (Primary); 54H20 (Secondary)

1. *Introduction*

The notion of *disjointness* of two dynamical systems, both in ergodic theory and in topological dynamics, was introduced by Furstenberg in his seminal paper [**5**]. This notion plays an important role for ergodic systems; see, for instance, [**7**]. Compared with ergodic theory, there still remain in topological dynamics some basic problems to settle. We refer to [**2**, **3**, **8**, **11**–**16**] for recent developments.

By a *topological dynamical system* (t.d.s.) we mean a pair (X, T), where X is a compact metric space (with metric d) and $T : X \to X$ is continuous and surjective. Let (X, T) and (Y, S) be two topological dynamical systems. We say that $J \subset X \times Y$ is a *joining* of X and Y if J is a non-empty, closed, invariant set, which is mapped onto X and Y by the respective coordinate projections. The product $X \times Y$ is always a joining and when it is

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

the only joining we say that (X, T) and (Y, S) are *disjoint*, denoted by $(X, T) \perp (Y, S)$ or $X \perp Y$. Note that if $(X, T) \perp (Y, S)$ then one of them is minimal [**5**], and if in addition (Y, S) is minimal then the set of recurrent points of (X, T) is dense in X [**11**].

Furstenberg [**5**] showed that each totally transitive system with dense set of periodic points is disjoint from any minimal system; and each weakly mixing system is disjoint from any minimal distal system. He left the following question.

Problem [**5**, Problem G]. Describe the system which is disjoint from all minimal systems.

Let $\mathscr{T}$ be a class of topological dynamical systems and (X, T) be a t.d.s. If $(X, T) \perp (Y, S)$ for every $(Y, S) \in \mathscr{T}$, then we denote it by $(X, T) \perp \mathscr{T}$, and let $\mathscr{T}^{\perp} = \{(X, T) : (X, T) \perp \mathscr{T}\}$. Let $\mathscr{M}$ be the class of all minimal systems. Hence Furstenberg's problem can be restated as follows: Describe the class $\mathscr{M}^{\perp}$.

We say that (X, T) has dense small periodic sets if for any non-empty open subset U of X there exist a closed subset Y of U and $k \in \mathbb{N}$ such that $T^k Y \subset Y$. Clearly, every transitive system with dense set of periodic points (so-called P-system) has dense small periodic sets. In [**11**], it was shown that a transitive t.d.s. disjoint from all minimal systems has to be a weakly mixing M-system; and each weakly mixing system with dense small periodic sets is disjoint from any minimal system. Then in [**3**, **14**], this result was generalized as follows: every weakly mixing system with dense distal points is disjoint from any minimal system. A further effort was made in [**12**]: if (X, T) is weakly mixing and $(K(X), T_K)$ has dense distal points, then (X, T) is disjoint from all minimal systems, where $(K(X), T_K)$ is the hyperspace system of (X, T). In [**13**] an example (X, T) was constructed such that $(K(X), T_K)$ has dense distal points, and at the same time (X, T) does not have dense distal points.

Let (X, T) be a t.d.s. Recently, Oprocha [**15**] gave the following result. Suppose that (X, T) is weakly mixing and that for every minimal system (Y, S) there exists a countable set $D \subseteq X$ such that for every non-empty open set U of X the following condition holds:

(1) for any $y \in Y$ and any open neighbourhood V of y there is $x \in D \cap U$ such that the set of transfer times $N_{T\times S}((x, y), U \times V) = \{n \in \mathbb{Z}_+ : T^n x \in U, S^n y \in V\}$ is syndetic.

Then (X, T) is disjoint from every minimal system.

Oprocha asked whether the assumptions in his theorem are also a necessary condition for M-systems to be disjoint from all minimal systems.

In this paper, we show the following theorem.

THEOREM 1.1. *Let (X, T) be a transitive t.d.s. Then (X, T) is disjoint from all minimal systems if and only if (X, T) is weakly mixing and there is some countable dense subset D of X (consisting of minimal points) such that for any minimal system (Y, S), any point $y \in Y$ and any open neighbourhood V of y, and for any non-empty open subset $U \subset X$, there is $x \in D \cap U$ such that $N_{T\times S}((x, y), U \times V)$ is syndetic.*

Hence we give an answer to Furstenberg's problem, and also answer the question by Oprocha affirmatively in some sense. Note that the countable set in our theorem is universal for any minimal system.

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

Central sets were introduced by Furstenberg, and they have very rich combinatorial properties [**6**]. A subset A of $\mathbb{Z}_+$ is called a *dynamical syndetic set* if there exist a minimal system (Y, S), $y \in Y$ and an open neighbourhood V_y of y such that $A \supset N_S(y, V_y) = \{n \in \mathbb{Z}_+ : S^n y \in V_y\}$. We will show (Theorem 3.7) that a set S is a central set if and only if $S = A \cap B$, where A is thick and B is dynamical syndetic. Using the notion of central sets we can give the following theorem.

THEOREM 1.2. *Let (X, T) be a transitive t.d.s. Then (X, T) is disjoint from all minimal systems if and only if (X, T) is weakly mixing and there is a countable dense subset D of X such that for each non-empty open subset U of X and each central set $S = A \cap B$, we can find $x = x(B) \in D \cap U$ (independent of A) such that $N_T(x, U) \cap S \neq \emptyset$, where A is thick and B is dynamical syndetic.*

It is clear that the above two theorems can be stated for a transitive system disjoint from a given minimal system. The following theorem was proved in [**11**, Corollary 4.6], which can be considered as an answer to Furstenberg's problem in the general case.

THEOREM 1.3. *Let (X, T) be a t.d.s. Then $(X, T) \perp \mathscr{M}$ if and only if for any minimal system (Y, S) there exist countably many transitive subsystems of (X, T) such that their union is dense in X and each of them is disjoint from Y.*

As a direct application we have the following proposition.

PROPOSITION 1.4. *Let (X, T) be a weakly mixing system. If for each non-empty open subset U of X there is $x \in U$ such that $N_T(x, U)$ is a C^*-set, then (X, T) is disjoint from all minimal systems, that is, $(X, T) \in \mathscr{M}^{\perp}$.*

In other applications we show that if a transitive system (X, T) is disjoint from all minimal systems, then so are $(X^n, T^{(n)})$ and (X, T^n) for any $n \in \mathbb{N}$, where $T^{(n)} = T \times \cdots \times T$ (n times). Combining this result with other results, it turns out that a transitive system (X, T) is disjoint from all minimal systems if and only if the hyperspace system $(K(X), T_K)$ is disjoint from all minimal systems.

Remark 1.5. We remark that in [**11**] Huang and Ye gave a necessary and sufficient condition for a transitive system being disjoint from all minimal systems through the notion of m-sets (see Proposition 2.2). The condition given in this paper is easier to handle than that one.

This paper is organized as follows. In §2 we give some preliminaries and in §3 we prove our main results. Then in §4 we give some applications of our results.

2. *Preliminary*

In this paper, integers, non-negative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$, respectively.

2.1. *Topological dynamical system.*

By a *topological dynamical system* we mean a pair (X, T), where X is a compact metric space (with metric d) and $T : X \to X$ is continuous and surjective. A non-empty closed invariant subset $Y \subset X$ naturally defines a *subsystem* (Y, T) of (X, T).

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

The *orbit* of x, $\mathrm{orb}(x, T)$ (or simply $\mathrm{orb}(x)$), is the set $\{T^n x : n \in \mathbb{Z}_+\}$. A t.d.s. (X, T) is *transitive* if for each pair of non-empty open subsets U and V,

$$N_T(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n} V \neq \emptyset\}$$

is infinite. Equivalently, (X, T) is transitive if and only if there exists $x \in X$ such that $\overline{\mathrm{orb}(x, T)} = X$; such x is called a *transitive point*, and the set of transitive points is denoted by $Tran_T$. It is well known that if a system (X, T) is transitive then $Tran_T$ is a dense G_δ set. A system (X, T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive.

A t.d.s. (X, T) is *minimal* if $Tran_T = X$. Equivalently, (X, T) is minimal if and only if it contains no proper subsystems. A point $x \in X$ is *minimal* or *almost periodic* if the subsystem $(\overline{\mathrm{orb}(x, T)}, T)$ is minimal.

A t.d.s. (X, T) is an *M-system* if it is transitive and the set of minimal points is dense.

Let (X, T) be a t.d.s. and $(x, y) \in X^2$. It is a *proximal* pair if there is a sequence $\{n_i\}$ in $\mathbb{Z}_+$ such that $\lim_{i\to+\infty} T^{n_i} x = \lim_{i\to+\infty} T^{n_i} y$; and it is a *distal* pair if it is not proximal. Denote by $P(X, T)$ or P_X the set of all proximal pairs of (X, T). For a point $x \in X$, $P[x] = \{y \in X : (x, y) \in P(X, T)\}$ is called a *proximal cell* of x. A point x is said to be *distal* if whenever y is in the orbit closure of x and (x, y) is proximal, then $x = y$.

A t.d.s. (X, T) is called *distal* if (x, x') is distal whenever $x, x' \in X$ are distinct. A t.d.s. (X, T) is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies $d(T^n x_1, T^n x_2) < \varepsilon$ for every $n \in \mathbb{Z}_+$. As we assume that T is surjective, it is easy to see that each equicontinuous system is distal.

For a t.d.s. (X, T), $x \in X$ and $U \subset X$ let

$$N_T(x, U) = N(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\}.$$

A point $x \in X$ is said to be *recurrent* if for every neighbourhood U of x, $N_T(x, U)$ is infinite.

2.2. *Furstenberg families.* Let us recall some notions related to Furstenberg families (for details, see [**6**]). Let $\mathscr{P} = \mathscr{P}(\mathbb{Z}_+)$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathscr{F}$ of $\mathscr{P}$ is a *(Furstenberg) family* if it is hereditary upwards, that is, $F_1 \subset F_2$ and $F_1 \in \mathscr{F}$ imply $F_2 \in \mathscr{F}$. A family $\mathscr{F}$ is *proper* if it is a proper subset of $\mathscr{P}$, that is, neither empty nor all of $\mathscr{P}$. It is easy to see that $\mathscr{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathscr{F}$ and $\emptyset \notin \mathscr{F}$. Any subset $\mathscr{A}$ of $\mathscr{P}$ can generate a family $[\mathscr{A}] = \{F \in \mathscr{P} : F \supset A \text{ for some } A \in \mathscr{A}\}$. If a proper family $\mathscr{F}$ is closed under intersection, then $\mathscr{F}$ is called a *filter*. For a family $\mathscr{F}$, the *dual family* is

$$\mathscr{F}^* = \{F \in \mathscr{P} : \mathbb{Z}_+ \setminus F \notin \mathscr{F}\} = \{F \in \mathscr{P} : F \cap F' \neq \emptyset \ for\ all\ F' \in \mathscr{F}\}.$$

$\mathscr{F}^*$ is a family, proper if $\mathscr{F}$ is. Clearly, $(\mathscr{F}^*)^* = \mathscr{F}$ and $\mathscr{F}_1 \subset \mathscr{F}_2$ implies that $\mathscr{F}_2^* \subset \mathscr{F}_1^*$. Denote by $\mathscr{F}_{\mathrm{inf}}$ the family consisting of all infinite subsets of $\mathbb{Z}_+$.

2.3. *$\mathscr{F}$-recurrence and some important families.* Let $\mathscr{F}$ be a family and (X, T) be a t.d.s. We say that $x \in X$ is *$\mathscr{F}$-recurrent* if, for each neighbourhood U of x, $N_T(x, U) \in \mathscr{F}$. So the usual recurrent point is just the $\mathscr{F}_{\mathrm{inf}}$-recurrent one.

A subset S of $\mathbb{Z}_+$ is *syndetic* if it has a bounded gaps, that is, there is $N \in \mathbb{N}$ such that $\{i, i+1, \ldots, i+N\} \cap S \neq \emptyset$ for every $i \in \mathbb{Z}_+$. S is *thick* if it contains arbitrarily long

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

runs of positive integers, that is, there is a strictly increasing subsequence $\{n_i\}$ of $\mathbb{Z}_+$ such that $S \supset \bigcup_{i=1}^{\infty}\{n_i, n_i+1, \ldots, n_i+i\}$. The collection of all syndetic (respectively, thick) subsets is denoted by $\mathscr{F}_s$ (respectively, $\mathscr{F}_t$). Note that $\mathscr{F}_s^* = \mathscr{F}_t$ and $\mathscr{F}_t^* = \mathscr{F}_s$.

A classic result states that x is a minimal point if and only if $N_T(x, U) \in \mathscr{F}_s$ for any neighbourhood U of x [**9**]. And a t.d.s. (X, T) is weakly mixing if and only if $N_T(U, V) \in \mathscr{F}_t$ for any non-empty open subsets U, V of X [**5**, **6**].

A subset S of $\mathbb{Z}_+$ is *piecewise syndetic* if it is an intersection of a syndetic set with a thick set. Denote the set of all piecewise syndetic sets by $\mathscr{F}_{ps}$. It is known that a t.d.s. (X, T) is an *M-system* if and only if there is a transitive point x such that $N_T(x, U) \in \mathscr{F}_{ps}$ for any neighbourhood U of x (see, for example, [**11**, Lemma 2.1]).

Let $\{p_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{N}$. One defines

$$FS(\{p_i\}_{i=1}^{\infty}) = \Big\{\sum_{i\in\alpha} p_i : \alpha \text{ is a non-empty finite subset of } \mathbb{N}\Big\}.$$

F is an *IP-set* if it contains some $FS(\{p_i\}_{i=1}^{\infty})$, where $p_i \in \mathbb{N}$. The collection of all IP-sets is denoted by $\mathscr{F}_{ip}$. A subset of $\mathbb{N}$ is called an *IP*-set* if it has non-empty intersection with any IP-set. It is known that a point x is a recurrent point if and only if $N_T(x, U) \in \mathscr{F}_{ip}$ for any neighbourhood U of x, and x is distal if and only if x is *IP**-recurrent [**6**].

2.4. *Systems* $(K(X), T_K)$ *and* $(M(X), T_M)$. Let (X, T) be a topological dynamics and let $\mathscr{B}(X)$ be the Borel σ-algebra of X. Let $M(X)$ be the collection of all Borel probability measures on X with the weak* topology. Then T induces a map T_M on $M(X)$ naturally by sending $\mu \in M(X)$ to $T\mu$, where $T\mu$ is defined by $T\mu(A) = \mu(T^{-1}A)$ for all $A \in \mathscr{B}(X)$.

Let $K(X)$ be the space of all non-empty closed subsets of X endowed with Hausdorff metric, and $T_K : K(X) \to K(X)$ be the induced map defined by $T_K(A) = T(A) = \{Tx : x \in A\}$ for any $A \in K(X)$.

2.5. *m-sets.* The notion of m-sets was introduced in [**11**].

Definition 2.1. A subset A of $\mathbb{Z}_+$ is called an *m-set* if there exist a minimal system (Y, S), $y \in Y$ and a non-empty open subset V of Y such that $A \supset N_S(y, V)$.

Recall that a subset A of $\mathbb{Z}_+$ is a dynamical syndetic set if there exist a minimal system (Y, S), $y \in Y$ and an open neighbourhood V_y of y such that $A \supset N_S(y, V_y)$. Let A be a m-set. Then there exist a minimal system (Y, S), $y \in Y$ and a non-empty open subset V of Y such that $A \supset N_S(y, V)$. Since (Y, S) is minimal, there exists some k such that $S^k y \in V$. Thus $S^{-k}V$ is an open neighbourhood of y and $N_S(y, S^{-k}V)$ is a dynamical syndetic set. Note that $N_S(y, V) \supset N_S(y, S^{-k}V) + k$. It follows that each m-set is a translation of some dynamical syndetic set.

For a transitive system, whether it is in $\mathscr{M}^{\perp}$ can be checked through m-sets, as the following proposition shows. For a minimal dynamical system (Y, S), we define

$$\mathscr{F}_Y = \{A \subset \mathbb{Z}_+ : A \supset N_S(y, V) \text{ for some } y \in Y \text{ and non-empty open subset } V\},$$
$$\mathscr{F}_Y^* = \{B \subset \mathbb{Z}_+ : B \cap A \neq \emptyset \text{ for each } A \in \mathscr{F}_Y\}.$$

PROPOSITION 2.2. *Let* (X, T) *be a transitive system and* $x \in Trans_T$.

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

(1) $(X, T) \in \mathscr{M}^{\perp}$ *if and only if* $N_T(x, U) \cap A \neq \emptyset$ *for any neighbourhood* U *of* x *and any m-set* A.

(2) $(X, T) \perp (Y, S)$ *if and only if, for any open neighbourhood* U *of* x, *one has* $N_T(x, U) \in \mathscr{F}_Y^*$.

We say that (X, T) is *strongly disjoint from all minimal systems* if $(X^n, T^{(n)})$ is disjoint from all minimal systems for any $n \in \mathbb{N}$, where $T^{(n)} = T \times \cdots \times T$ (n times). Then we have the following result.

PROPOSITION 2.3. [**12**] *Let* (X, T) *be a t.d.s.*

(1) *If* $(K(X), T_K)$ *is weakly mixing and is disjoint from all minimal systems, then* (X, T) *is weakly mixing and is disjoint from all minimal systems.*
(2) *If* (X, T) *is strongly disjoint from all minimal systems, then both* $(K(X), T_K)$ *and* $(M(X), T_M)$ *are disjoint from all minimal systems.*

3. *Main results*

3.1. *Basic lemmas.* To show the main result we need the following two lemma. The first was obtained in [**11**, Proposition 2.2]. The second was proved in [**1**].

LEMMA 3.1. *If a transitive system is disjoint from all minimal systems, then it is weakly mixing and has a dense set of minimal points, that is, it is a weakly mixing* M*-system.*

LEMMA 3.2. *Let* (X, T) *be a weakly mixing system. Then each proximal cell is residual, that is, for each* $x \in X$, $P[x]$ *is residual in* X.

3.2. *Main results.* We are now ready to show the main results of the paper.

THEOREM 3.3. *Let* (X, T) *be a transitive t.d.s. Then the following statements are equivalent.*

(1) (X, T) *is disjoint from all minimal systems.*
(2) (X, T) *is weakly mixing and there is a countable dense subset* D *of* X *such that, for any minimal system* (Y, S), *any point* $y \in Y$ *and any open neighbourhood* V *of* y, *and for any non-empty open subset* $U \subset X$, *there is* $x \in D \cap U$ *such that that* $N_{T\times S}((x, y), U \times V)$ *is syndetic.*
(3) (X, T) *is weakly mixing and there is a countable dense subset* D *of* X *consisting of minimal points such that, for any minimal system* (Y, S), *any point* $y \in Y$ *and any open neighbourhood* V *of* y, *and for any non-empty open subset* $U \subset X$, *there is* $x \in D \cap U$ *such that* $N_{T\times S}((x, y), U \times V)$ *is syndetic.*

Proof. (3) $\Longrightarrow$ (2). This is clear.

(2) $\Longrightarrow$ (1). We follow the proofs in [**3**, **15**]. By the weak mixing property of (X, T) and countability of D, there is a point $x_0 \in X$ such that (x_0, x) is proximal for any $x \in D$ by Lemma 3.2.

Assume now that $J \subset X \times Y$ is a joining. Then there is $y \in Y$ such that $(x_0, y) \in J$. Let V be an open neighbourhood of y and U be any non-empty open set of X. Take a non-empty open subset U' of X and $\varepsilon > 0$ such that $B_\varepsilon(\overline{U'}) \subseteq U$. Then by the assumption,

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

there is $x \in D \cap U'$ such that $N_{T\times S}((x, y), U' \times V)$ is syndetic. At the same time, since (x, x_0) is proximal, we have that

$$\{k \in \mathbb{Z}_+ : d(T^k(x), T^k(x_0)) < \varepsilon\}$$

is thick. This implies that there is $k \in \mathbb{Z}_+$ such that

$$S^k(y) \in V, \quad T^k(x) \in U', \quad \text{and} \quad d(T^k(x_0), T^k(x)) < \varepsilon.$$

So $T^k(x_0) \in U$. We have $(T^k(x_0), S^k(y)) \in J \cap \overline{U} \times \overline{V}$. Since U is an arbitrary open set of X and V is an arbitrary open neighbourhood of y, it implies that $X \times \{y\} \subset J$. Thus $J = X \times Y$ as y has a dense orbit in Y.

(1) $\Longrightarrow$ (3). Suppose now that (X, T) is disjoint from all minimal systems. By Proposition 2.2, (X, T) is an M-system. Let D be any countable dense subset of X consisting of minimal points. We show that it is what we need.

Assume to the contrary that the condition in (3) does not hold for D. Then there exist a minimal system (Y, S), a point $y \in Y$ and its open neighbourhood V, and there is a non-empty open set $U \subset X$ such that $N_{T\times S}((x, y), U \times V)$ is not syndetic for any point $x \in D \cap U$.

CLAIM. *For each $x \in D \cap U$, there is some $T \times S$-invariant subset $J_{(x,y)} \subseteq (U \times V)^c$ such that $x \in p_1(J_{(x,y)})$, where p_1 is the projection of $X \times Y$ to the first coordinate.*

Proof of Claim. Since $N_{T\times S}((x, y), U \times V)$ is not syndetic, $A = N_{T\times S}((x, y), (U \times V)^c)$ is a thick set. As x is a minimal point, for each $\varepsilon > 0$, $N_T(x, B_\varepsilon(x))$ is a syndetic set. Thus, for any fixed $\varepsilon > 0$ and $L \in \mathbb{N}$, there are infinitely many $n \in N_T(x, B_\varepsilon(x))$ such that

$$n, n + 1, \ldots, n + L \in A.$$

From this fact we choose an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that $n_k \in N_T(x, B_{1/k}(x))$ and $n_k, n_k + 1, \ldots, n_k + k \in A$. Without loss of generality assume that

$$\lim_{k\to\infty} (T \times S)^{n_k}(x, y) = (x_0, y_0).$$

Since $n_k \in N_T(x, B_{1/k}(x))$, it is clear that $x_0 = x$. Now, for each m, we have that

$$(T \times S)^m(x_0, y_0) = \lim_{k\to\infty} (T \times S)^{n_k+m}(x, y) \in (U \times V)^c.$$

Now let

$$J_{(x,y)} = \overline{\text{orb}((x_0, y_0), T \times S)}.$$

Then $J_{(x,y)} \subseteq (U \times V)^c$, and $x = x_0 \in p_1(J_{(x,y)})$. This concludes the proof of the claim. □

Now let

$$J = \overline{\bigcup_{x\in D\cap U} J_{(x,y)}}.$$

Then it is clear that J is $T \times S$-invariant and $J \subseteq (U \times V)^c$. By the above claim, we have that $D \cap U \subseteq p_1(J)$. Since D is a dense set of X, it follows that

$$U \subseteq p_1(J).$$

Since (X, T) is transitive and $p_1(J)$ is T-invariant and closed, it follows that $p_1(J) = X$. Moreover, as (Y, S) is minimal, the projection of J to the second coordinate is Y. We conclude that J is a joining of X and Y. But $J \subset (U \times V)^c$, a contradiction. □

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

We also have the following theorem which is easier to handle in some situations.

THEOREM 3.4. *Let (X, T) be a transitive t.d.s. Then the following statements are equivalent.*

(1) *(X, T) is disjoint from all minimal systems.*

(2) *(X, T) is weakly mixing and, for any non-empty open subsets $U \subset X$ and any non-empty open subset $V \subset Y$ of any minimal system (Y, S), there is $x = x(U, V) \in U$ such that $N_{T\times S}((x, y), U \times V)$ is syndetic for any $y \in V$.*

(3) *(X, T) is weakly mixing and, for any non-empty open subset $U \subset X$ and any non-empty open subset $V \subset Y$ of any minimal system (Y, S), there is a minimal point $x = x(U, V) \in U$ such that $N_{T\times S}((x, y), U \times V)$ is syndetic for any $y \in V$.*

Proof. The proof is only slightly different from that of Theorem 3.3. For the sake of completeness, we give the proof. Obviously, (3) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1). We follow the proofs in [**3**, **15**]. Assume that $\{U_n\}_{n=1}^{\infty}$ and $\{V_m\}_{m=1}^{\infty}$ are bases for the topologies of X and Y, respectively. Then by (2) for given U_n and V_m, there is $x_{n,m} \in U_n$ such that $N_{T\times S}((x_{n,m}, y), U_n \times V_m)$ is syndetic for any $y \in V_m$. By the weak mixing property of (X, T), there is a point $x \in X$ such that $(x, x_{n,m})$ is proximal for any $n, m \in \mathbb{N}$ by Lemma 3.2.

Assume now that $J \subset X \times Y$ is a joining. Then there is $y \in Y$ such that $(x, y) \in J$. Let V be an open neighbourhood of y. Then there is $m_0 \in \mathbb{N}$ such that $y \in V_{m_0} \subset V$. We will show that $(x_{n,m_0}, y) \in J$ for any $n \in \mathbb{N}$.

To do this, let U be an open neighbourhood of x_{n,m_0}. Then there is $\varepsilon > 0$ such that $B_{2\varepsilon}(x_{n,m_0}) \subset U$ and put $W = B_{\varepsilon}(x_{n,m_0})$. Then there is $x_{n',m_0} \subset U_{n'} \subset W$. Then we have that $N_{T\times S}((x_{n',m_0}, y), U_{n'} \times V_{m_0})$ is syndetic. At the same time, since (x, x_{n',m_0}) is proximal, we have that

$$\{k \in \mathbb{Z}_+ : d(T^k(x), T^k(x_{n',m_0})) < \varepsilon\}$$

is thick. This implies that there is $k \in \mathbb{Z}_+$ such that

$$S^k(y) \in V_{m_0} \subset V, \quad T^k(x_{n',m_0}) \in U_{n'}, \quad \text{and} \quad d(T^k(x), T^k(x_{n',m_0})) < \varepsilon.$$

So $T^k(x) \in U$. We have $J \cap \overline{U} \times \overline{V} \neq \emptyset$. It follows that $(x_{n,m_0}, y) \in J$ for any $n \in \mathbb{N}$. This implies that $X \times \{y\} \subset J$ and hence we get $J = X \times Y$ as y has a dense orbit.

(1) $\Longrightarrow$ (3). Suppose now that (X, T) is disjoint from (Y, S). By Lemma 2.2, (X, T) is an M-system. Assume to the contrary that the condition does not hold. Then there are a non-empty open sets $U \subset X$ and $V \subset Y$ such that for any minimal point $x \in U$ there is $y = y(x) \in V$ such that $N_{T\times S}((x, y), U \times V)$ is not syndetic.

Let $x \in U$ be a transitive point. As (X, T) is an M-system, we can choose minimal points $x_n \in U$ such that $\lim_{n\to\infty} x_n = x$. Then for each $n \in \mathbb{N}$ there is $y_n = y_n(x_n) \in V$ such that $N_{T\times S}((x_n, y_n), U \times V)$ is not syndetic. That is, $N_{T\times S}((x_n, y_n), (U \times V)^c)$ is thick.

We now show that there is $(x'_n, y'_n) \in (U \times V)^c$ and $(x'_n, y'_n) \in \overline{\text{orb}((x_n, y_n), T \times S)}$ such that its orbit is outside $U \times V$. In fact there is a strictly increasing sequence $\{k_i\}_{i=1}^{\infty}$ of $\mathbb{N}$ such that $(T \times S)^{k_i+j}(x_n, y_n) \in (U \times V)^c$ for each $i \in \mathbb{N}$ and $1 \leq j \leq i$. Without

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

loss of generality assume that $\lim_{i\to\infty}(T\times S)^{k_i}(x_n, y_n) = (x'_n, y'_n)$. Then (x'_n, y'_n) is the point we want. Let

$$J = \overline{\bigcup_{n=1}^{\infty} \mathrm{orb}((x'_n, y'_n), T\times S)}.$$

It is clear that J is closed and $T\times S$-invariant. Moreover, as (Y, S) is minimal, the projection of J to the second coordinate is Y. We now show that $p_1(J) = X$, where p_1 is the projection of J to the first coordinate. We note that $p_1(J)$ is T-invariant and closed.

To do so, fix $n\in\mathbb{N}$. It is clear that there is a sequence $\{k_i\}$ such that $\lim_{i\to\infty} T^{k_i}x_n = x'_n$ by the construction of x'_n. Choose $0<\varepsilon_n<1/n$. Since x_n is a minimal point, $N_T(x_n, B_{\varepsilon_n}(x_n))$ is syndetic. Assume that that l_n is the gap of this subset of $\mathbb{N}$. By the continuity of T, there is $\delta_n>0$ such that if $z_1, z_2\in X$ and $d(z_1, z_2)<\delta_n$ then $d(T^i(z_1), T^i(z_2))<\varepsilon_n$ for each $i=1,\dots,l_n$. Since $\lim_{i\to\infty} T^{k_i}x_n = x'_n$, there is $j\in\mathbb{N}$ such that $d(T^{k_j}x_n, x'_n)<\delta_n$. This implies that

$$d(T^{k_j+i}x_n, T^i x'_n)<\varepsilon_n,\quad i=1,\dots,\ell_n.$$

There is $1\le i_n\le l_n$ such that $T^{k_j+i_n}x_n\in B_{\varepsilon_n}(x_n)$. This implies that $d(T^{i_n}(x'_n), x_n)<2\varepsilon_n$ for each $n\in\mathbb{N}$.

Hence $x=\lim_{n\to\infty}x_n=\lim_{n\to\infty}T^{i_n}(x'_n)\in p_1(J)$, since $x'_n\in p_1(J)$, and $p_1(J)$ is T-invariant and closed. It follows that $p_1(J)=X$, as x is a transitive point. We conclude that J is a joining of X and Y. It is clear $J\subset(U\times V)^c$, a contradiction. □

By the same proof we have the following theorem.

THEOREM 3.5. *Let (X, T) be a weakly mixing M-system and (Y, S) is a minimal t.d.s. Then the following statements are equivalent.*

(1) $(X, T)\perp(Y, S)$.
(2) *For any non-empty open subsets $U\subset X$ and $V\subset Y$, there is $x=x(U, V)\in U$ such that $N_{T\times S}((x, y), U\times V)$ is syndetic for any $y\in V$.*
(3) *For any non-empty open subsets $U\subset X$ and $V\subset Y$, there is a minimal point $x=x(U, V)\in U$ such that $N_{T\times S}((x, y), U\times V)$ is syndetic for any $y\in V$.*

3.3. *Central sets and another form of the main result.* First we recall the following form of the Auslander–Ellis theorem (see [**6**, Theorem 8.7], for example).

THEOREM 3.6. (Auslander–Ellis) *Let (X, T) be a compact metric t.d.s. Then, for any $x\in X$ and any T-invariant closed subset Z of $\overline{\mathrm{orb}(x, T)}$, there is some minimal point $x'\in Z$ such that (x, x') is proximal.*

Using the Auslander–Ellis theorem, Furstenberg introduced a notion called the central set. A subset $S\subseteq\mathbb{Z}_+$ is a *central set* if there exist a t.d.s. (X, T), a point $x\in X$ and a minimal point y proximal to x, and a neighbourhood U_y of y such that $N_T(x, U_y)\subset S$. It is known that any central set is an IP-set [**6**, Proposition 8.10]. Denote the set of all central sets by $\mathscr{F}_C$. A set from $\mathscr{F}_C^*$ is called a C^*-set. Note that all IP*-sets are C^*-sets, but there are some C^* sets which are not IP*. Moreover, we know that a point of a dynamical system is C^*-recurrent if and only if it is IP*-recurrent if and only if it is distal [**6**, Proposition 9.17].

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

Recall that a subset A of $\mathbb{Z}_+$ is called a *dynamical syndetic set* if there exist a minimal system (Y, S), $y \in Y$ and an open neighbourhood V_y of y such that $A \supset N_S(y, V_y)$. Denote the set of all dynamical syndetic sets by $\mathscr{F}_{ds}$. Let

$$\mathscr{F}_{dps} = \mathscr{F}_t \cap \mathscr{F}_{ds} = \{A \cap B : A \in F_t,\ B \in \mathscr{F}_{ds}\}.$$

Each element of $\mathscr{F}_{dps}$ is called a *dynamical piecewise syndetic set*.

THEOREM 3.7. $\mathscr{F}_{dps} = \mathscr{F}_C$.

Proof. First we show that $\mathscr{F}_C \subseteq \mathscr{F}_{dps}$. Let $Q \in \mathscr{F}_C$. Then, by definition, there there exist a system (X, T), a point $x \in X$ and a minimal point y proximal to x, and a neighbourhood U_y of y such that $N_T(x, U_y) \subset Q$. Without loss of generality, we may assume that $U_y = B_{2\varepsilon}(y) = \{z : d(z, y) < 2\varepsilon\}$ for some $\varepsilon > 0$. Let

$$A = \{n \in \mathbb{Z}_+ : d(T^n x, T^n y) < \varepsilon\}.$$

Then A is a thick set since (x, y) is proximal [**6**, Lemma 8.1]. It is easy to verify that

$$A \cap N_T(y, B_\varepsilon(y)) \subseteq N_T(x, B_{2\varepsilon}(y)) \subseteq Q.$$

Hence $\mathscr{F}_C \subseteq \mathscr{F}_{dps}$.

We now show the converse. Let $Q \in \mathscr{F}_{dps}$. Then there exist a thick set A and a dynamical syndetic set B such that $Q = A \cap B$. Let (Y, S) be a minimal system, $y \in Y$ and an open neighbourhood V_y of y such that $B \supset N_S(y, V_y)$.

Let $(\Sigma_2 = \{0, 1\}^{\mathbb{Z}}_+, \sigma)$ be the shift system. Let $\widetilde{X} = \Sigma_2 \times Y$ and $\widetilde{T} = \sigma \times S$. Let $x_0 = (1_A, y) \in \widetilde{X}$, $X = \overline{\mathrm{orb}(x_0, \widetilde{T})}$, and $T = \widetilde{T}|_X$. Then (X, T) is a t.d.s.†

We now show that $y_0 = (1, y) \in X$ is a minimal point which is proximal to x_0. First we show $y_0 \in X$. Since A is thick, there is an increasing sequence $\{n_i\}_{i=1}^\infty$ such that $\{n_i, n_i + 1, \ldots, n_i + i\} \subseteq A$ for all $i \in \mathbb{N}$. Without loss of generality, we assume that

$$x' = \lim_{i\to\infty} T^{n_i} x_0 \in X.$$

By the construction of $\{n_i\}_{i=1}^\infty$, x' has the form $(1, y')$ for some $y' \in Y$. As (Y, S) is minimal, there is some sequence $\{m_i\}_{i=1}^\infty$ such that $y = \lim_{i\to\infty} S^{m_i} y'$. Thus

$$y_0 = (1, y) = \lim_{i\to\infty} T^{m_i} x' \in X.$$

Since 1 is a fixed point of σ and y is a minimal point of (Y, S), y_0 is a minimal point of (X, T).

Note that $\{n_i, n_i + 1, \ldots, n_i + i\} \subseteq A$ for all $i \in \mathbb{N}$, and it follows that

$$\lim_{i\to\infty} d_X(T^{n_i} x_0, T^{n_i} y_0) = 0,$$

where d_X is the metric of X. That is, x_0 and y_0 are proximal.

† If T is not surjective, one may embed (X, T) into some surjective system. Let $Z = X \times D$, where $D = \{1/n\}_{n\in\mathbb{N}} \cup \{0\}$. Define $R : Z \to Z$ satisfying $R(x, 1/(n + 1)) = (x, 1/n)$, $n \in \mathbb{N}$, and $R(x, 1) = (Tx, 1)$ and $R(x, 0) = (x, 0)$ for all $x \in X$. Then (Z, R) is a t.d.s. and R is surjective. Identifying x with $(x, 1)$ for all $x \in X$, X can be viewed as a closed subset of Z and $T = R|_X$. We cite this approach from the proof of [**4**, Lemma 3.13].

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

Let $[1] = \{\xi \in \Sigma_2 : \xi_0 = 1\}$. Then $V = ([1] \times V_y) \cap X$ is an open neighbourhood of y_0 in X. Thus

$$Q = A \cap B \supseteq A \cap N_S(y, V_y) = N_T(x_0, V)$$

is a central set, that is, $Q \in \mathscr{F}_C$. Hence $\mathscr{F}_{dps} \subseteq \mathscr{F}_C$. The proof is complete. □

To show another form of the main result we need the following lemma.

LEMMA 3.8. *Let (X, T) be a transitive t.d.s. If (X, T) is disjoint from all minimal systems, then (X, T) is weakly mixing and there is a countable dense subset D of X such that, for each non-empty open subset U of X and each central set $S = A \cap B$ (where A is thick and B is dynamical syndetic), we can find $x = x(B) \in D \cap U$ independent of A such that $N_T(x, U) \cap S \neq \emptyset$.*

Proof. Assume that (X, T) is disjoint from all minimal systems. By Lemma 3.1, (X, T) is a weakly mixing M-system. Then, by Theorem 3.3, there is a countable dense subset D of X satisfying the condition in Theorem 3.3 (2).

Let U be any non-empty open set of X. By Theorem 3.7, let $S = A \cap B$ be a central set, where A is thick and B is a dynamical syndetic set. Let (Y, S) be a minimal system, $y \in Y$, and V be a non-empty open neighbourhood of Y such that $B \supseteq N_S(y, V)$. For y and V, by the choice of D there is some $x = x(B) \in D \cap U$ such that $N_{T\times S}((x, y), U \times V)$ is syndetic. Hence

$$\begin{aligned} N_T(x, U) \cap S &= N_T(x, U) \cap A \cap B \supseteq N_T(x, U) \cap (A \cap N_S(y, V)) \\ &= A \cap N_{T\times S}((x, y), U \times V) \neq \emptyset. \end{aligned}$$

The proof is complete. □

We are now ready to give the other form of the main result.

THEOREM 3.9. *Let (X, T) be a transitive t.d.s. Then $(X, T) \perp \mathscr{M}$ if and only if (X, T) is weakly mixing and there is a countable dense subset D of X such that, for each non-empty open subset U of X and each central set $S = A \cap B$, we can find $x = x(B) \in D \cap U$ with $N_T(x, U) \cap S \neq \emptyset$, where A is thick and B is dynamical syndetic.*

Proof. Assume that (X, T) is weakly mixing and there is a countable dense subset D of X such that, for each non-empty open subset U of X and each central set $S = A \cap B$ (where A is thick and B is dynamical syndetic), we can find $x = x(B) \in D \cap U$ independent of A such that $N_T(x, U) \cap S \neq \emptyset$. We show that (X, T) is disjoint from all minimal systems.

Let (Y, S) be a minimal system. We will show that for any non-empty open subsets $U \subset X$ and $V \subset Y$, $y \in V$, there is $x \in D \cap U$ such that $N_{T\times S}((x, y), U \times V)$ is syndetic. And hence by Theorem 3.3, (X, T) is disjoint from all minimal systems.

Let $A \in \mathscr{F}_t$. Then by Theorem 3.7, $A \cap N_S(y, V) \in \mathscr{F}_{dps} = \mathscr{F}_C$. By assumption, there is some $x \in D \cap U$ independent of A such that

$$N_T(x, U) \cap (A \cap N_S(y, V)) \neq \emptyset.$$

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

That is,

$$A \cap N_{T\times S}((x, y), U \times V) = N_T(x, U) \cap (A \cap N_S(y, V)) \neq \emptyset.$$

As A is an arbitrary thick set, $N_{T\times S}((x, y), U \times V)$ is syndetic.

The converse follows from the proof of Lemma 3.8. The proof is complete. □

3.4. *The general case.* We now discuss Furstenberg's problem without the transitivity assumption. It was proved in [**11**, Theorem 4.3] that if $(X, T) \perp \mathscr{M}$, then the set of minimal points of (X, T) is dense in X. Moreover, the following proposition was proved in [**11**, Corollary 4.6].

PROPOSITION 3.10. *Let (X, T) be a t.d.s. Then $(X, T) \perp \mathscr{M}$ if and only if, for any minimal system (Y, S), there exist countably many transitive subsystems of (X, T) such that their union is dense in X and each of them is disjoint from Y.*

It is natural to make the following conjecture. *If $(X, T) \perp \mathscr{M}$, then there are countably many transitive subsystems of (X, T) such that their union is dense in X and each of them is in $\mathscr{M}^{\perp}$.* We remark that this conjecture is not true, since there is a distal system disjoint from all minimal systems; see [**11**, Example 4.10].

Together with Theorem 3.5 and Proposition 3.10, we get a description of a dynamical system disjoint from all minimal systems. Since the characterization is not easy to handle, it is a natural to look for some other intrinsic characterizations.

4. *Applications*

In this section we give several applications of the main theorem.

4.1. *Some sufficient conditions.* We say that a t.d.s. (X, T) has *dense distal sets* if, for each non-empty open subset U of X, there is a distal point C of $(K(X), T_K)$ such that $C \subset U$. It is shown that a system (X, T) is a weakly mixing system with dense distal sets if and only if $(K(X), T_K)$ is a weakly mixing system with dense distal points [**12**].

Recall that a point is distal if and only if, for any neighbourhood U of x and any open neighbourhood V of a minimal system (Y, S), $N_{T\times S}((x, y), U \times V)$ is syndetic for any $y \in V$ if and only if x is IP*-recurrent [**6**, Theorem 9.11]. Hence by Theorem 3.4, we easily obtain the following corollary.

COROLLARY 4.1. *The following classes are subsets of $\mathscr{M}^{\perp}$.*

(1) *(X, T) is weakly mixing with dense distal sets.*

(2) *(X, T) is weakly mixing with dense distal points.*

COROLLARY 4.2. *If (X, T) is a weakly mixing t.d.s. such that, for any minimal system (Y, S) and any non-empty open set U of X, there are $x \in U$ and a non-empty open set V of Y such that (x, y) is minimal for any $y \in V$, then $(X, T) \in \mathscr{M}^{\perp}$. In particular, if, for any minimal system (Y, S) and each non-empty open subset U of X, there is a minimal subset M of (X, T) which is disjoint from Y and $U \cap M \neq \emptyset$, then $(X, T) \in \mathscr{M}^{\perp}$.*

Proof. We assume that, for any non-empty open set U of X, there are $x \in U$ and a non-empty open set V of Y such that (x, y) is minimal for any $y \in V$. Since (Y, S) is minimal,

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

there is $n \in \mathbb{N}$ such that $\bigcup_{i=1}^{n} S^n V = Y$. We note that if (x, y) is minimal then (x, Sy) is minimal. This follows from the fact that $\mathrm{id} \times S : (X \times Y, T \times S) \to (X \times Y, T \times S)$ is a factor map. Thus, (x, y) is minimal for any $y \in Y$. Applying Theorem 3.4, we get the proof of the first statement.

Note that when the assumption of the second statement holds, as M is disjoint from Y, we know that (x, y) is a minimal point of $T \times S$ for any $x \in U \cap M$ and any $y \in Y$. □

We remark that in fact Oprocha gave a very nice criterion which covers Corollaries 4.1 and 4.2 above.

PROPOSITION 4.3. [**15**] *Let (X, T) be a weakly mixing system. If for each non-empty open subset U of X there is $x \in U$ such that $N_T(x, U)$ is an IP*-set, then (X, T) is disjoint from all minimal systems, that is, $(X, T) \in \mathscr{M}^{\perp}$.*

Using results in the previous section, we have the following generalization.

PROPOSITION 4.4. *Let (X, T) be a weakly mixing system. If for each non-empty open subset U of X there is $x \in U$ such that $N_T(x, U)$ is a C^*-set, then (X, T) is disjoint from all minimal systems, that is, $(X, T) \in \mathscr{M}^{\perp}$.*

Proof. This is a direct application of Theorem 3.9. □

We note that the condition in Proposition 4.3 is not necessary, as the example in [**15**] shows. We now show that in fact the same example in [**15**] indicates that condition in Proposition 4.4 is also not necessary. One can verify this following the [**15**, proof of Theorem 1.4]. For the sake of completeness, we give a slightly different proof. To do so we need a lemma and a proposition.

LEMMA 4.5. *Let (X, T) be a t.d.s., $x \in X$ and M be a minimal weakly mixing subsystem. If x is proximal to some point in M then $P[x] \cap M$ is dense in M.*

Proof. We follow the argument in [**1**]. Assume that x is proximal to a point $z \in M$. Let G_k be the open balls (of X) of radius $1/k$ centred at z. Let U be a non-empty open subset of M.

Set $U_0 = U$ and define inductively open sets $U_1, U_2, \ldots$ of M and positive integers n_k as follows.

Since $N_T(x, G_k) \in \mathscr{F}_{ps}$ and $N_T(U_{k-1}, G_k \cap M) \in \mathscr{F}_{ts}$ ([**10**, Theorem 4.7]), we have that

$$N_T(x, G_k) \cap N_T(U_{k-1}, G_k \cap M)$$

is infinite. So we can choose a non-empty open set U_k of M with closure contained in U_{k-1} and an integer $n_k > k$ such that $T^{n_k} x \in G_k$ and $T^{n_k} \overline{U_k} \subset G_k$. If y is a point of the non-empty intersection $\bigcap_k \overline{U}_k = \bigcap_k U_k$ then $T^{n_k} x \in G_k$ and $T^{n_k}(y) \in G_k$ and so $d(T^{n_k} x, T^{n_k} y) \leq 2/k$. Thus y is in U and x, y are proximal. □

PROPOSITION 4.6. *Let (X, T) be a t.d.s. such that there are countably many non-trivial minimal subsystems M_i which are weakly mixing, $\bigcup_{i=1}^{\infty} M_i$ is dense in X and $\bigcup_{i=1}^{\infty} M_i$ is the set of minimal points of X. Then there is a non-empty open subset U of X such that, for any $z \in U$, $N_T(z, U)$ is not a C^*-set.*

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

Proof. First there is a non-empty open subset U such that, for each $i \in \mathbb{N}$, $U \cap M_i \neq \emptyset$ implies that $\mathrm{int}(U^c) \cap M_i \neq \emptyset$ (otherwise T has a fixed point), where $\mathrm{int}(A)$ is the interior of a subset A. It is clear that $U \neq X$ and we may assume that $\mathrm{int}(U^c) \neq \emptyset$. We will show that, for any $z \in U$, $N_T(z, U)$ is not a C^*-set.

(1) If there is $i \in \mathbb{N}$ such that $z \in M_i$ is a minimal point, then z is proximal to a point $y \in V$, where $y \in M_i$ and V is a non-empty open subset such that $U \cap V = \emptyset$ (since M_i is weakly mixing and $\mathrm{int}(U^c) \cap M_i \neq \emptyset$).

(2) If z is not a minimal point, then z is proximal to a minimal point $y_1 \in M_i$ for some $i \in \mathbb{N}$. By Lemma 4.5, z is proximal to a minimal point $y \in V$, where V is a non-empty open subset of X such that $U \cap V = \emptyset$ (as $\mathrm{int}(U^c) \cap M_i \neq \emptyset$).

In the above two cases we have that $N_T(z, V)$ is a C-set. This implies that $N_T(z, U)$ is not a C^*-set since $U \cap V = \emptyset$. □

4.2. $(X^n, T^{(n)})$ *and* (X, T^n). It is known that if (X, T) is weakly mixing, then so are $(X^n, T^{(n)})$ and (X, T^n) for any $n \in \mathbb{N}$.

THEOREM 4.7. *Assume that a transitive system (X, T) is disjoint from all minimal systems. Then $(X^n, T^{(n)})$ is also disjoint from all minimal systems for any $n \in \mathbb{N}$, that is, (X, T) is strongly disjoint from all minimal systems.*

Proof. Let (Y, S) be a minimal system. Assume that W is a non-empty open subset of X^n and V is a non-empty open subset of Y. We may assume that $W \supset W_1 \times \cdots \times W_n$, where W_i is a non-empty open subset of X. By the transitivity of (X, T), there is a non-empty open subset U of X such that, for each $1 \leq i \leq n$, there is $k_i \in \mathbb{N}$ with $T^{k_i} U \subset W_i$. By Theorem 3.4, there exists $x \in U$ such that $N_{T \times S}((x, y), U \times V)$ is syndetic for any $y \in V$. This implies that

$$N_{T^{(n)} \times S}(((T^{k_1}x, \ldots, T^{k_n}x), y), (T^{k_1}U \times \cdots \times T^{k_n}U) \times V) \supset N_{T \times S}((x, y), U \times V)$$

is syndetic for any $y \in Y$.

Observing that $x' = (T^{k_1}x, \ldots, T^{k_n}x) \subset T^{k_1}U \times \cdots \times T^{k_n}U \subset W_1 \times \cdots \times W_n \subset W$, we get that $N_{T^{(n)} \times S}((x', y), W \times V)$ is syndetic for any $y \in Y$. Again applying Theorem 3.4, we get the conclusion. □

We have the following corollary.

COROLLARY 4.8. *Let (X, T) be a transitive t.d.s. (X, T) is disjoint from all minimal systems if and only if $(K(X), T_K)$ is disjoint from all minimal systems.*

Proof. This follows from Theorem 4.7 and Proposition 2.3. □

THEOREM 4.9. *Assume that a transitive system (X, T) is disjoint from all minimal systems. Then so is (X, T^n) for any $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$ with $n \geq 2$ and let (Y, S) be a minimal system. Set $\tilde{Y} = \bigcup_{i=1}^{n} Y \times \{i\}$ and define $\tilde{S} : \tilde{Y} \to \tilde{Y}$ such that, for any $y \in Y$,

$$\tilde{S}(y, i) = (y, i + 1) \text{ for } i = 1, \ldots, n - 1 \quad \text{and} \quad \tilde{S}(y, n) = (Sy, 1).$$

It is clear that $(\tilde{Y}, \tilde{S})$ is also minimal and $\tilde{S}^{nk}(y, 1) = (S^k y, 1)$ for any $k \in \mathbb{N}$.

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press

Let U, V be open non-empty subsets of X and Y, respectively. Then, by Theorem 3.4, we know that there is $x \in U$ such that $N_{T\times\tilde{S}}((x, (y, 1)), U \times (V \times \{1\}))$ is syndetic for any $y \in Y$. Assume that $k \in N_{T\times\tilde{S}}((x, (y, 1)), U \times (V \times \{1\}))$; then there is some k_1 with $k = nk_1$ such that $(T^n)^{k_1}x \in U$ and $S^{k_1}(y) \in V$. So, $k_1 \in N_{T^n\times S}((x, y), U \times V)$, which implies that $N_{T^n\times S}((x, y), U \times V)$ is also syndetic for any $y \in Y$. Again applying Theorem 3.4, we conclude that (X, T^n) is disjoint from all minimal systems. □

Finally, we restate a question in [**3**].

QUESTION 4.10. *Let (X_1, T_1) and (X_2, T_2) be transitive and disjoint from all minimal systems. Is it true that $(X_1 \times X_2, T_1 \times T_2)$ is also disjoint from all minimal systems?*

We note that we do not know the answer even for the very simple case when $X_1 = X_2 = X$, $T_1 = T$ and $T_2 = T^2$.

Acknowledgements. The authors would like to thank P. Oprocha for sharing with us the early version of his recent paper. We also thank the referee for a very careful reading and many useful comments, which helped us to improve the paper. This research is supported by NNSF of China (11571335, 11431012, 11371339) and by 'the Fundamental Research Funds for the Central Universities'.

REFERENCES

[**1**] E. Akin and S. Kolyada. *Li-Yorke sensitivity. Nonlinearity* **16** (2003), 1421–1433.
[**2**] X. Dai. On disjointness and weak-mixing of $\mathscr{F}$-flows with discrete abelian phase groups. *Preprint*, 2017.
[**3**] P. Dong, S. Shao and X. Ye. Product recurrent properties, disjointness and weak disjointness. *Israel J. Math.* **188** (2012), 463–507.
[**4**] D. Feng and W. Huang. Variational principles for topological entropies of subsets. *J. Funct. Anal.* **263**(8) (2012), 2228–2254.
[**5**] H. Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory. Int. J. Math. Comput. Theory* **1** (1967), 1–49.
[**6**] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory (M. B. Porter Lectures)*. Princeton University Press, Princeton, NJ, 1981.
[**7**] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2003.
[**8**] E. Glasner and B. Weiss. On the disjointness property of groups and a conjecture of Furstenberg. *Preprint*, 2018, arXiv:1807.08493.
[**9**] W. Gottschalk and G. Hedlund. *Topological Dynamics (Colloquium Publications, 36)*. American Mathematical Society, Providence, RI, 1955.
[**10**] W. Huang and X. Ye. An explicit scattering, non-weakly mixing example and weak disjointness. *Nonlinearity* **15** (2002), 1–14.
[**11**] W. Huang and X. Ye. Dynamical systems disjoint from any minimal system. *Trans. Amer. Math. Soc.* **357**(2) (2005), 669–694.
[**12**] J. Li, K. Yan and X. Ye. Recurrence properties and disjointness on the induced spaces. *Discrete Contin. Dyn. Syst.* **35** (2015), 1059–1073.
[**13**] J. Li, P. Oprocha, X. Ye and R. Zhang. When are all closed subsets recurrent? *Ergod. Th. & Dynam. Sys.* **37**(7) (2017), 2223–2254.
[**14**] P. Oprocha. Weak mixing and product recurrence. *Ann. Inst. Fourier (Grenoble)* **60**(4) (2010), 1233–1257.
[**15**] P. Oprocha. Double minimality, entropy and disjointness with all minimal systems. *Discrete Contin. Dyn. Syst.* **39** (2019), 263–275.
[**16**] T. Yu. Dynamical systems disjoint from any minimal system under group actions. *Difference Equations, Discrete Dynamical Systems and Applications, 181–195 (Springer Proceedings in Mathematics and Statistics, 150)*. Springer, Cham, 2015.

https://doi.org/10.1017/etds.2019.23 Published online by Cambridge University Press