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Weighted Birkhoff ergodic theorem with oscillating weights

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Abstract. We consider sequences of Davenport type or Gelfond type and prove that sequences of Davenport exponent larger than $\frac{1}{2}$ are good sequences of weights for the ergodic theorem, and that the ergodic sums weighted by a sequence of strong Gelfond property are well controlled almost everywhere. We prove that for any q-multiplicative sequence, the Gelfond property implies the strong Gelfond property and that sequences realized by dynamical systems can be fully oscillating and have the Gelfond property.

1. *Introduction*

A sequence of complex numbers $(w_n)_{n\geq 0} \subset \mathbb{C}$ will be considered as weights for Birkhoff averages. Oscillating weights of higher order were defined in [**12**] and oscillating weights of order 1 appeared earlier in [**17**]. Recall that the sequence (w_n) is defined to be *oscillating of order d* ($d \geq 1$) if for any real polynomial P of degree less than or equal to d we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n e^{2\pi i P(n)} = 0.$$

A *fully oscillating sequence* is defined to be an oscillating sequence of all orders. It was proved in [**17**] that if $(w_n)_{n\geq 0}$ is oscillating (i.e. oscillating of order 1), then the following weighted ergodic limit:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0 \tag{1.1}$$

exists for any $f \in C(X)$ and any $x \in X$, where (X, T) is some topological dynamical system of zero entropy, for example, an arbitrary homeomorphism of the circle [**17**]. This is related to Sarnak's conjecture [**28**, **29**]. The following result on everywhere convergence

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of the multiple ergodic limit was proved in [**15**] under the assumption of full oscillation (only a higher order oscillation suffices for certain dynamics).

THEOREM 1. [**15**] *Let $\ell \geq 1$ be an integer. Suppose that $(w_n)_{n\geq 0}$ is fully oscillating and that $Tx = Ax + b$ is an affine linear map of zero entropy on a compact abelian group X. Then the following weighted multiple ergodic limit exists:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f_1(T^{q_1(n)}x) \cdots f_\ell(T^{q_\ell(n)}x) = 0 \tag{1.2}$$

for any $f_1, \ldots, f_\ell \in C(X)$, any polynomial $q_1, \ldots, q_\ell \in \mathbb{Z}[z]$ such that $q_j(\mathbb{N}) \subset \mathbb{N}$ $(1 \leq j \leq \ell)$ and any $x \in X$.

The same result in Theorem 1 was proved in [**12**] for topological dynamics systems of quasi-discrete spectrum in the sense of Hahn and Parry [**19**].

The above discussion is related to Sarnak's conjecture [**28**, **29**] concerning the everywhere convergence of weighted ergodic averages for topological dynamical systems of zero entropy. In this paper we discuss measure-preserving dynamical systems. We shall introduce some other properties of weights which are stronger than oscillation of order 1, but different from oscillations of higher order, and we shall discuss almost everywhere (a.e.) convergence of weighted ergodic averages.

For a measure-preserving dynamical system $(X, \mathcal{B}, \nu, T)$ and an integrable function $f \in L^1(\nu)$, we shall study the weighted ergodic limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x). \tag{1.3}$$

One of the possible assumptions that we shall make on the weights $(w_n)_{n\geq 0}$ is as follows. There exists a constant $h > 0$ such that

$$\max_{0\leq t\leq 1} \left| \sum_{n=0}^{N-1} w_n e^{2\pi i n t} \right| = O\left(\frac{N}{\log^h N}\right). \tag{1.4}$$

Then we say that $(w_n)_{n\geq 0}$ is of *Davenport type*. The largest h will be denoted by H and called the *Davenport exponent* of $(w_n)_{n\geq 0}$. Davenport [**8**] proved that the Möbius sequence μ has its Davenport exponent $H = +\infty$. Recall that $\mu(1) = 1$ and $\mu(n) = (-1)^k$ if n is square-free and has k prime factors. We shall remark that as a consequence of a well-known Davenport–Erdös–LeVeque theorem (see Lemma 1), the limit (1.3) exists and is equal to zero a.e. under the assumption $H > \frac{1}{2}$ (Theorem 2).

If the following stronger assumption is satisfied: there exists a constant $\frac{1}{2} \leq \alpha < 1$ such that

$$\max_{0\leq t\leq 1} \left| \sum_{n=0}^{N-1} w_n e^{2\pi i n t} \right| = O(N^\alpha), \tag{1.5}$$

we say that $(w_n)_{n\geq 0}$ is of *Gelfond type*. The smallest α will be denoted by Δ and called the *Gelfond exponent* of $(w_n)_{n\geq 0}$. The Thue–Morse sequence (t_n) is defined by $t_n = (-1)^{s_2(n)}$, where $s_2(n)$ is the sum of dyadic digits of n. Gelfond [**18**] proved that $\Delta = \log 3/\log 4$ for the Thue–Morse sequence. The Thue–Morse sequence is a 2-multiplicative arithmetic

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function (see §4 for the definition of q-multiplicative sequence). If $(w_n)_{n\geq 0}$ is a q-multiplicative sequence ($q \geq 2$ an integer) and satisfies the Gelfond property (1.5), we shall prove that it satisfies the following strong Gelfond property:

$$\text{for all } 0 \leq M < N, \quad \max_{0\leq t\leq 1} \left| \sum_{M}^{N-1} w_n e^{2\pi i n t} \right| = O((N-M)^{\alpha}) \tag{1.6}$$

(Theorem 4). Under the assumption (1.6) made on $(w_n)_{n\geq 0}$, we have the following estimate on the size of the weighted Birkhoff sum:

$$\nu\text{-a.e.} \quad \sum_{n=0}^{N-1} w_n f(T^n x) = O(N^{\alpha} \log^2 N \log^{1+\delta} \log N). \tag{1.7}$$

This holds for any $f \in L^2(\nu)$ and any $\delta > 0$ (Theorem 3).

We shall also study the oscillating properties of sequences of the form $h(S^n x)$, called realizations of dynamical systems (Theorem 6).

Here is the organization of the paper. In §2, we prove a weighted Birkhoff theorem under the condition $H > \frac{1}{2}$ on the weights of Davenport type (Theorem 2). In §3, for weights having the strong Gelfond property, we prove an upper bound for the size of weighted Birkhoff sums (Theorem 3). In §4, we prove that any q-multiplicative sequences having the Gelfond property must have the strong Gelfond property (Theorem 4). In §5, a random q-multiplicative sequence is considered and it is proved that it has the Gelfond property almost surely (Theorem 5). In §6, the fully oscillating property and the Gelfond property are proved for a class of sequences realized by measure-preserving dynamical systems (Theorem 6).

2. *Birkhoff theorem with Davenport weights*

Let $(w_n)_{n\geq 0} \subset \mathbb{C}$ be a sequence of complex numbers and $(u_n)_{n\geq 0} \subset \mathbb{N}$ be a sequence of integers. We say that the *DEL condition* is satisfied by $(w_n)_{n\geq 0}$ and $(u_n)_{n\geq 0}$ if

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \max_{0\leq t\leq 1} \left| \sum_{k=0}^{n-1} w_k e^{2\pi i u_k t} \right|^2 < \infty. \tag{2.1}$$

Here DEL refers to Davenport–Erdös–LeVeque. The reason for this terminology will soon become clear. We say that $(u_n)_{n\geq 0}$ is a *r-Bourgain sequence*, $1 \leq r < \infty$, if for any dynamical system $(X, \mathcal{B}, \nu, T)$ the following maximal inequality holds:

$$\|\max_{N\geq 1} |A_N f|\|_r \leq C \|f\|_r \quad (\text{for all } f \in L^r(\nu)), \tag{2.2}$$

where $A_N f = N^{-1} \sum_{n=0}^{N-1} f \circ T^{u_n}$ and $C > 1$ is a constant.

We deduce the following weighted Birkhoff theorem from a well-known Davenport–Erdös–LeVeque theorem (see Lemma 1).

THEOREM 2. *Let $(w_n)_{n\geq 0} \in \ell^{\infty}$ and $(u_n)_{n\geq 0} \subset \mathbb{N}$. Suppose that:*
(H1) *(u_n) is an r-Bourgain sequence for some $1 \leq r < \infty$; and*
(H2) *(w_n) and (u_n) satisfy the DEL condition.*

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Let $(X, \mathcal{B}, \nu, T)$ be a measure-preserving dynamical system. Then, for any $f \in L^r(\nu)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^{u_n} x) = 0 \tag{2.3}$$

holds for ν-almost every $x \in X$ and in L^r-norm.

Davenport, Erdös and LeVeque [**9**], in their study of uniform distribution based on Weyl's criterion, proved the following theorem. It was stated for a very special case, but the proof is valid for the general statement below.

LEMMA 1. (Davenport, Erdös and LeVeque [**9**]) *Let $(\xi_n)_{n\geq 0}$ be a sequence of bounded random variables defined on some probability space such that $\|\xi_n\|_\infty = O(1)$ as $n \to \infty$. Suppose that the following Davenport–Erdös–LeVeque condition is satisfied:*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}|\xi_0 + \xi_1 + \cdots + \xi_{n-1}|^2}{n^3} < \infty. \tag{2.4}$$

Then almost surely we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \xi_k = 0.$$

Proof of Theorem 2. First assume that $f \in L^\infty(\nu)$. Let $\xi_n = f(T^{u_n} x)$. The a.e. convergence follows directly from the Davenport–Erdös–LeVeque theorem, the hypothesis (H2) and the spectral lemma, which gives

$$\mathbb{E}|\xi_0 + \xi_1 + \cdots + \xi_{n-1}|^2 = \int \left| \sum_{k=0}^{n-1} w_k e^{2\pi i u_k t} \right|^2 d\sigma_f(t) \leq \max_{0\leq t\leq 1} \left| \sum_{k=0}^{n-1} w_k e^{2\pi i u_k t} \right|^2,$$

where σ_f is the spectral measure of f defined on the circle $\mathbb{R}/\mathbb{Z}$.

Now assume that $f \in L^r(\nu)$. For any $\epsilon > 0$, there exists $g \in L^\infty(\nu)$ such that $\|f - g\|_r < \epsilon$. By writing $f = g + (f - g)$, we get

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) \right| \leq \left| \frac{1}{N} \sum_{n=0}^{N-1} w_n g(T^n x) \right| + \|w\|_\infty \left| \frac{1}{N} \sum_{n=0}^{N-1} |f - g| \circ T^n(x) \right|.$$

Applying the above proved result to g, we get

$$\nu\text{-a.e.} \quad \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) \right| \leq \max_{N\geq 1} A_N |f - g|(x).$$

Then, by the hypothesis (H1), we have

$$\mathbb{E} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) \right|^r \leq C^r \|f - g\|_r^r < C^r \epsilon^r.$$

We have thus finished the a.e. convergence, for ϵ can be arbitrarily small.

After having proved the pointwise convergence, it is easy to deduce the L^r-convergence by Lebesgue's dominated convergence theorem. □

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Remark 1. If $(w_n)_{n\geq 0}$ is of Davenport type with Davenport exponent $H > \frac{1}{2}$, the series in the hypothesis (2.1) is bounded by $\sum (1/n \log^{2H} n)$, which is finite. On the other hand, the classical ergodic maximal inequality means that the set of natural numbers is a 1-Bourgain sequence. Therefore, we have the following corollary.

COROLLARY 1. *Suppose that* $(w_n)_{n\geq 0} \subset \ell^\infty$ *is of Davenport type with Davenport exponent* $H > \frac{1}{2}$. *Let* $(X, \mathcal{B}, \nu, T)$ *be a measure-preserving dynamical system. Then, for any* $f \in L^1(\nu)$, *the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0 \tag{2.5}$$

holds for ν*-almost every* $x \in X$ *and in* L^1*-norm.*

In [**1**, Proposition 3.1], El Abdalaoui *et al* proved Corollary 1 under the stronger condition $H > 1$ using a more elementary argument than that of Davenport–Erdös–LeVeque. Actually, the result in [**1**] was only stated for the Möbius function, but the argument works for all $(w_n)_{n\geq 0}$ with $H > 1$.

Bourgain [**4**] proved that the polynomial sequence $(u_n)_{n\geq 0} := (p(n))_{n\geq 0}$ with $p(x) \in \mathbb{Z}[x]$ is an r-Bourgain sequence for $r > 1$. So, we have the following corollary.

COROLLARY 2. *Let* $(X, \mathcal{B}, \nu, T)$ *be a measure-preserving dynamical system and let* $p(x) \in \mathbb{Z}[x]$ *taking values in* $\mathbb{N}$ *and* $(w_n)_{n\geq 0} \in \ell^\infty$. *Assume that* $r > 1$. *Suppose that*

$$\max_{0\leq t\leq 1} \left| \sum_{n=0}^{N-1} w_n e^{2\pi i p(n) t} \right| \leq C \frac{N}{\log^h N}$$

with $h > \frac{1}{2}$ *and* $C > 1$. *Then, for any* $f \in L^r(\nu)$, *the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^{p(n)} x) = 0 \tag{2.6}$$

holds for ν*-almost every* $x \in X$ *and in* L^r*-norm.*

Eisner [**11**] stated Corollary 2 with $(w_n) := (\mu(n))$ the Möbius function and observed that it can be proved using the argument in [**1**].

Remark 2. Assume that $f \in L^2(\nu)$ and $\mathbb{E}f = 0$. Cohen and Lin [**5**] proved that the condition

$$\sum_{n=1}^{\infty} \frac{\log n}{n^3} \| f + f \circ T + \cdots + f \circ T^{n-1} \|_2^2 < \infty \tag{2.7}$$

is necessary and sufficient for the ergodic Hilbert transform $\sum_{n=1}^{\infty} (f(T^n x)/n)$ to converge in L^2-norm. Cuny [**6**] proved that this condition implies the a.e. convergence of $\sum_{n=1}^{\infty} (f(T^n x)/n)$, a fortiori the a.e. convergence to zero of $(1/n) \sum_{k=0}^{n-1} f(T^k x)$. The Cohen–Lin condition (2.7) can be stated as follows:

$$\sum_{n=1}^{\infty} \frac{\log n}{n^3} \left\| \sum_{k=0}^{n-1} e^{2\pi i k t} \right\|_{L^2(\sigma_f)}^2 < \infty,$$

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which is to be compared with (2.1). The a.e. convergence of ergodic series of the form $\sum a_n f(T^n x)$ is studied in [**14**] and [**7**]. See the references therein on the subject.

Remark 3. If the limit (1.3) exists a.e. and in L^1-norm for every dynamical system $(X, \mathcal{B}, \nu, T)$ and for every $f \in L^1(\nu)$, we call $(w_n)_{n\geq 0}$ a good sequence of weights for the ergodic theorem. The conclusion of Theorem 2 is a little bit stronger because the limit is always zero. Lesigne and Mauduit [**24**, pp. 151–152] proved that, given a q-multiplicative sequence $(w_n)_{n\geq 0}$, a necessary and sufficient condition for all sequences $(\rho(w_n))_{n\geq 0}$, ρ being a continuous function defined on the circle, to be good is that for every real number t and every integer $d \geq 1$ the following limit exists:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n^d e^{2\pi i n t}. \tag{2.8}$$

That is the case when $(w_n)_{n\geq 0}$ is a q-multiplicative sequence taking only a finite number of values.

Remark 4. In [**25**], Lesigne, Mauduit and Mossé associated a skew product to a q-multiplicative sequence. They proved that if the cocycle is weakly mixing, all continuous images of every orbit of the skew product dynamics are good.

3. *Size of weighted Birkhoff sums*

Let $(w_n)_{n\geq 0}$ be a sequence of complex numbers and $(u_n)_{n\geq 0}$ be a strictly increasing sequence of positive integers. For any integer $N \geq 1$, denote

$$S_N^{w,u}(x) = \sum_{n=0}^{N-1} w_n e^{2\pi i u_n x}.$$

The following condition will be considered:

$$\max_{x\in\mathbb{R}} |S_N^{w,u}(x)| \leq C N^{\beta} \quad \text{(for all } N \geq 1) \tag{3.1}$$

for some $1/2 \leq \beta < 1$ and $C > 0$. Sometimes, the following stronger condition will also be considered:

$$\max_{x\in\mathbb{R}} |S_N^{w,u} - S_M^{w,u}| \leq D(N-M)^{\beta} \quad \text{(for all } N \geq M \geq 1) \tag{3.2}$$

for some $1/2 \leq \beta < 1$ and $D > 0$.

Give a measure-preserving dynamical system $(X, \mathscr{B}, \nu, T)$, we shall consider, for $f \in L^2(\mu)$, the weighted Birkhoff sums

$$\sigma_N^{w,u} f(x) := \sum_{n=0}^{N-1} w_n f(T^{u_n} x).$$

THEOREM 3. *Let $(w_n)_{n\geq 0}$ be a sequence of complex numbers and $(u_n)_{n\geq 0}$ be a strictly increasing sequence of positive integers. Suppose that $(w_n)_{n\geq 0}$ and $(u_n)_{n\geq 0}$ satisfy the*

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condition (3.2) with $\frac{1}{2} \le \beta < 1$ *and that* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing function such that* $\phi(x) \le C\phi(2x)$ *for some* $C > 1$ *and for all* $x > 0$*, and*

$$\sum_{n=1}^{\infty} \frac{1}{n\phi(n)} < \infty. \tag{3.3}$$

Then, for any dynamical system $(X, \mathcal{B}, \mu, T)$ *and any* $f \in L^2(\mu)$*, for* μ*-almost all* x*, we have*

$$\lim_{N\to\infty} \frac{1}{N^\beta \log^2 N\phi(\log N)} \sum_{n=0}^{N-1} w_n f(T^{u_n}x) = 0.$$

Proof. For integers $n \ge m \ge 0$, define $g_{m,n} = D^2(n-m)^{2\beta}$. By the spectral lemma (see [**22**, p. 94]), for all $0 \le a < b$, we have

$$\int \left| \sum_{n=a}^{b'-1} w_n f(T^n x) \right|^2 d\mu(x) \le g_{a,b}.$$

Since $2\beta \ge 1$, we have $(x+y)^{2\beta} \ge x^{2\beta} + y^{2\beta}$ for all $x \ge 0$ and $y \ge 0$, so that for $0 \le a \le b \le c$, we have the super-additivity

$$g_{a,c} \ge g_{a,b} + g_{b,c}.$$

Then, using the condition (3.2), we can apply the inequality (2.2) in [**27**] with $\gamma = 2$ to get

$$\int \max_{a\le b'\le b} \left| \sum_{n=a}^{b'-1} w_n f(T^n x) \right|^2 d\mu(x) \le D^2(b-a)^{2\beta} \log^2 2(b-a).$$

In particular, for a fixed $\rho > 1$, we have

$$\text{for all } m \ge 0, \quad \|M_m\|_{L^2(\mu)} = O(m\rho^{m\beta}),$$

where

$$M_m(x) = \max_{\rho^m \le b \le \rho^{m+1}} \left| \sum_{\rho^m \le n < b} w_n f(T^{u_n}x) \right|.$$

It follows that

$$\left\| \sum_{m=1}^{\infty} \frac{M_m}{m^2\phi(m)\rho^{m\beta}} \right\|_{L^2(\mu)} = O\left(\sum_{m=1}^{\infty} \frac{1}{m\phi(m)} \right) < +\infty.$$

So, for μ-almost all x, we have

$$M_m(x) = o(m^2\phi(m)\rho^{m\beta}).$$

For any $N \ge 1$, there exists m such that $\rho^m \le N < \rho^{m+1}$. So, we get

$$\left| \sum_{n=1}^{N-1} w_n f(T^n x) \right| \le \sum_{k=0}^{m} M_k = o\left(\sum_{k=1}^{m} k^2\phi(k)\rho^{k\beta} \right) = o(m^2\phi(m)\rho^{\beta m}).$$

Here we have used the monotonicity of ϕ. Since $\phi(2x) \le C\phi(x)$ and $\rho^m \le N$, $m^2\phi(m)\rho^{\beta m}$ is bounded by $N^\beta \log^2 N\phi(\log N)$, up to the multiplicative constant. We have thus completed the proof. □

A weaker result was obtained in [**10**].

For q-multiplicative sequences, we shall prove the strong Gelfond property (1.6), which will allow us to apply Theorem 3 to q-multiplicative sequences.

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4. *Strong Gelfond property of q-multiplicative sequences*

Let $q \geq 2$ be an integer. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be *q-multiplicative* if

$$f(aq^n + b) = f(aq^n)f(b) \tag{4.1}$$

for all integers $m \geq 1$, $a \geq 0$ and $0 \leq b < q^m$. A q-multiplicative function f is completely determined by its values $f(aq^m)$ for $m \geq 0$ and $0 \leq a < q$, which constitute the so-called skeleton of f. Indeed, for $n = \sum a_j q^j$, where $a_j \in \{0, 1, \ldots, q-1\}$, we have

$$f(n) = \prod f(a_j q^j).$$

It is also clear that $f(0) = 0$ or $f(0) = 1$. In the following, we assume that q-multiplicative sequences take values in $\mathbb{U} := \{\xi \in \mathbb{C} : |\xi| = 1\}$, the multiplicative group of complex numbers of modulus 1.

For a q-multiplicative function f, we consider the trigonometric polynomials

$$S_N^f(x) := S_N(x) := \sum_{n=0}^{N-1} f(n)e^{2\pi inx} \quad (x \in \mathbb{R}),$$

where $N \geq 1$. For simplicity, we will write $e(x)$ for $e^{2\pi ix}$. Notice that for any x, $n \mapsto e(nx)$ is q-multiplicative for any given q. Typical examples of q-multiplicative sequences are $e^{2\pi ics_q(n)}$, where c is a fixed real number and $s_q(n)$ is the sum of digits in the q-adic expansion of n.

THEOREM 4. *Let $f : \mathbb{N} \to \mathbb{U}$ be a q-multiplicative function. Suppose that there exist constants $C \geq 1$ and $\frac{1}{2} \leq \beta < 1$ such that*

$$\max_{x \in \mathbb{R}} |S_n^f(x)| \leq Cn^\beta \quad (\textit{for all } n \geq 1).$$

Then there exists a constant $D > 0$ such that

$$\max_{x \in \mathbb{R}} |S_n^f(x) - S_m^f(x)| \leq D(n-m)^\beta \quad (\textit{for all } n \geq m \geq 0).$$

Proof. We simply write S_n for S_n^f. For $a \in \mathbb{N}$ and $j \in \mathbb{N}$, the q-multiplicity implies that

$$S_{(a+1)q^j}(x) - S_{aq^j}(x) = \sum_{n=0}^{q^j-1} f(aq^j + n)e((aq^j + n)x) = f(aq^j)e(aq^j)S_{q^j}(x).$$

Then, by the hypothesis, we get

$$|S_{(a+1)q^j}(x) - S_{aq^j}(x)| \leq Cq^{j\beta}. \tag{4.2}$$

For any couple $(n, m) \in \mathbb{N}^2$ with $0 \leq m < n$, there exists a unique integer $j \in \mathbb{N}$ such that $q^j \leq n - m < q^{j+1}$. We are going to show, by induction on $j \geq 0$, that

$$|S_n(x) - S_m(x)| \leq Kq^{j\beta} \tag{4.3}$$

for all $(n, m) \in \mathbb{N}^2$ such that $q^j \leq n - m < q^{j+1}$, where $K \geq q^3$ is a constant to be determined. Since $K \geq q^3$, (4.3) is trivially true for $j = 0, 1, 2$.

First we prove a weaker version of (4.3) by induction on $j \geq 1$: the inequality (4.3) holds for all $(n, m) \in \mathbb{N}^2$ such that $0 \leq n - m < q^{j+1}$ and that one of n and m is of the

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form aq^{j-1} for some $a \in \mathbb{N}$. We have seen that (4.3) holds for $j = 1, 2$ (initiation of induction). Now suppose that (4.3) holds for $j = 1, 2, \ldots, k$ (with $k \geq 2$). Assume that $0 \leq n - m < q^{k+2}$ and $m = aq^k$ for some $a \in \mathbb{N}$ (the reasoning is the same if $n = aq^k$). Let $\tau = bq^k \leq n$ be the integer with the largest $b \in \mathbb{N}$. Then $0 \leq n - \tau < q^k$. Notice that $\tau = aq^{k-2}$ with $a = bq^2$. By the hypothesis of induction, we have

$$|S_n(x) - S_\tau(x)| \leq Kq^{(k-1)\beta}. \tag{4.4}$$

We now estimate

$$\begin{aligned} |S_n(x) - S_m(x)| &\leq |S_n(x) - S_\tau(x)| + |S_\tau(x) - S_m(x)| \\ &\leq Kq^{(k-1)\beta} + \sum_{\ell=a}^{b-1} |S_{(\ell+1)q^k}(x) - S_{\ell q^k}(x)| \\ &\leq Kq^{(k-1)\beta} + (b-a)Cq^{k\beta}, \end{aligned}$$

where the second inequality follows from (4.4) and the third one follows from (4.2). We remark that $(b-a)q^k \leq n - m < q^{k+2}$, which implies that $b - a \leq q^2$. Therefore, we get

$$|S_n(x) - S_m(x)| \leq (Kq^{-2\beta} + Cq^{2-\beta})q^{(k+1)\beta}.$$

Since $\beta \geq \frac{1}{2}$, we have $q^{2\beta} \geq q \geq 2$. We can take $K \geq q^3$ large enough so that $Kq^{-2\beta} + Cq^{2-\beta} \leq K$. Actually, we can take any K such that

$$K \geq q^3, \quad K \geq \frac{Cq^{1-\beta}}{1 - 2q^{-2\beta}} = \frac{Cq^{2+\beta}}{q^{2\beta} - 1}.$$

Now let us deduce the inequality (4.3) from its weak version that we have just proved. Assume that $q^j \leq n - m < q^{j+1}$ with $j \geq 2$. Let aq^{j-1} be an integer such that

$$\left| \frac{n+m}{2} - aq^{j-1} \right| < q^{j-1}.$$

Then, writing $n - aq^{j-1} = (n-m)/2 + ((n+m)/2 - aq^{j-1})$, we get

$$0 \leq n - aq^{j-1} < \tfrac{1}{2}q^{j+1} + q^j < q^{j+1}.$$

Similarly, we have

$$0 \leq aq^{j-1} - m < q^{j+1}.$$

Then, by the weak version of (4.3), we get

$$|S_n(x) - S_m(x)| \leq |S_n(x) - S_{aq^{j-1}}| + |S_{aq^{j-1}} - S_m(x)| \leq 2Kq^{j\beta}.$$

Finally, we can take

$$D = 2K = \max\left\{2q^3, \frac{2Cq^{2+\beta}}{q^{2\beta} - 1}\right\}. \qquad \square$$

The above proof can be simplified a little bit when $q \geq 3$.

The Thue–Morse sequence (t_n) is defined by $(-1)^{s_2(n)}$, where $s_2(n) = \sum a_i$ is the sum of dyadic digits of $n = \sum a_j 2^j$ ($a_j = 0, 1$). The sequences $(t_n^{(c)})$ defined by $t_n^{(c)} = e^{2\pi i c s_2(n)}$ ($0 < c < 1$) are also studied in the literature. They are 2-multiplicative. Recall that $t_n = t_n^{(1/2)}$.

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Gelfond [**18**] proved that

$$\left|\sum_{n=0}^{N-1} t_n e^{2\pi inx}\right| \le CN^{\log 3/\log 4} \quad \text{(for all } N \ge 1, \text{ for all } x \in \mathbb{R}),$$

where $C > 0$ is a constant.

For a general q-multiplicative sequence $f(n)$ taking values in $\mathbb{U}$, we have

$$\left|\sum_{n=0}^{N-1} f(n) e^{2\pi inx}\right| \le CN^{\beta} \quad \text{(for all } N \ge 1, \text{ for all } x \in \mathbb{R})$$

under the condition

$$\overline{\lim_{m\to\infty}} \max_{x\in\mathbb{R}} |F_m(x)F_{m+1}(x)\cdots F_{m+r-1}(x)| \le q^{r\beta} \tag{3}$$

for some $r \ge 2$, where

$$F_m(x) = \sum_{j=0}^{q-1} f(jq^m) e^{2\pi ijq^m x}.$$

Using this idea with $r = 2$, Mauduit, Rivat and Sárközy [**26**] proved that

$$\left|\sum_{n=0}^{N-1} t_n^{(c)} e^{2\pi inx}\right| \le CN^{1-(\pi^2/(20\log 2))\|c\|^2} \quad \text{(for all } N \ge 1, \text{ for all } x \in \mathbb{R}),$$

where $\|c\| = \inf_{n\in\mathbb{Z}} |c - n|$. This estimate is not optimal, but it gives an interesting upper bound.

In [**21**], Konieczny proved that for the Thue–Morse sequence t and for any positive integer s, we have $\|t\|_{U^s[N]} = O(N^{-c})$ for some $c := c(s) > 0$, where $\|\cdot\|_{U^s[N]}$ denotes the sth Gowers uniformity norm. The Gowers uniform norm would be a good tool to study the properties of a given sequence which are examined in this note. In fact, there is a bound like

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n e^{2\pi ip(n)}\right| \le \|a\|_{U^{s+1}[N]},$$

where p is a real polynomial of degree s. It is called the polynomial phase principle (see [**30**, Exercise 1.3.21]). A slightly more general fact is true: the above polynomial phase can be replaced by a nilsequence of an appropriate step. In fact, morally speaking, the inverse theorem for Gowers norms asserts that sequences with small Gowers norms are precisely the ones which do not correlate with nilsequences.

5. *Random q-multiplicative sequences*

Here we construct some random q-multiplicative sequences and prove that they have almost surely the Gelfond property. A q-multiplicative sequence $(f(n))_{n\ge 0}$ is determined by its values $f(0) = 1$ and $(f(q^t), \ldots, f((q-1)q^t))$ for $t \ge 0$. Take a sequence of random vectors $X = (X_t)_{t\ge 0}$, where

$$X_t = (X_t^{(1)}, \ldots, X_t^{(q-1)}).$$

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Then we have a random q-multiplicative sequence $f_X(\cdot)$ defined by

$$f_X(n) = \prod X_{n_k}^{(\epsilon_k)} \quad \text{for } n = \sum_k \epsilon_k q^{n_k}.$$

Let

$$S_{X,N}(t) = \sum_{n=0}^{N-1} f_X(n) e^{2\pi i n t}.$$

Then

$$S_{X,q^n}(t) = \prod_{j=0}^{n-1} (1 + X_j^{(1)} e^{2\pi i q^j t} + \cdots + X_j^{(q-1)} e^{2\pi i (q-1) q^j t}).$$

The following theorem gives us an upper bound for the Gelfond exponent of the random q-multiplicative sequence.

THEOREM 5. *Suppose that all the random variables $X_t^{(k)}$ $(t \geq 0, 1 \leq k \leq q-1)$ are independent and uniformly distributed on the circle, which means that $X_t^{(k)} = e^{2\pi i \omega_t^{(k)}}$, where $\omega_t^{(k)}$ are independent and uniformly distributed on* $[0, 1]$. *Then there exists a constant* $1/2 \leq \alpha < 1$ *such that almost surely*

$$\|S_{X,N}(\cdot)\|_\infty = O(N^\alpha).$$

Proof. We follow the idea of Salem and Littlewood developed by Kahane (see [**20**]). Let $M_N = \|S_{X,N}(\cdot)\|_\infty$. Observe that S_{X,q^n} is a trigonometric polynomial of degree q^n. By Bernstein's inequality, we have

$$\max_{0 \leq t \leq 1} |S'_{X,q^n}(t)| \leq q^n \max_{0 \leq t \leq 1} |S_{X,q^n}(t)|.$$

It follows that $|S_{X,q^n}(t)| \geq \frac{1}{2} M_{q^n}$ over a random interval I of length at least q^{-n}. Therefore,

$$M_{q^n}^\lambda \leq 2^\lambda \int_I |S_{X,q^n}(t)|^\lambda \, dt \leq 2^\lambda q^n \int_0^1 |S_{X,q^n}(t)|^\lambda \, dt.$$

It follows that

$$\mathbb{E} M_{q^n}^\lambda \leq 2^\lambda q^n \int_0^1 \mathbb{E} |S_{X,q^n}(t)|^\lambda \, dt = 2^\lambda [q^{1+\lambda} \varphi(\lambda)]^n,$$

where

$$\varphi(\lambda) = q^{-\lambda} \int_0^1 \cdots \int_0^1 |1 + e^{2\pi i x_1} + \cdots + e^{2\pi i x_{q-1}}|^\lambda \, dx_1 \cdots dx_{q-1}.$$

Fix $\tau > 0$. By Markov's inequality, we have

$$P(M_{q^n} \geq q^{n\tau}) \leq 2^\lambda e^{n[(1+(1-\tau)\lambda) \log q - \Phi(\lambda)]},$$

where

$$\Phi(\lambda) = -\log \varphi(\lambda).$$

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Notice that Φ is increasing and concave, and $\Phi(0+)=0$ and $\Phi(+\infty)=+\infty$. Let τ^* be such that $\lambda \mapsto [1+(1-\tau^*)\lambda]\log q$ is a tangent of Φ. In other words, the following system has a solution (τ^*, λ^*):

$$(1+(1-\tau^*)\lambda^*)\log q - \Phi(\lambda^*) = 0, \tag{5.1}$$

$$(1-\tau^*)\log q = \frac{\varphi'(\lambda^*)}{\varphi(\lambda^*)}. \tag{5.2}$$

It is clear that $\tau^* < 1$. So, for any $\tau \in (\tau^*, 1)$, we have

$$P(M_{q^n} \geq q^{n\tau}) \leq 2^{\lambda^*} e^{-\delta n},$$

where $\delta = -[1+(1-\beta)\lambda^*]\log q + \Phi(\lambda^*) > 0$. Then, by the Borel–Cantelli lemma, we have

$$\text{a.s.} \quad M_{q^n} = O(q^{\tau n}).$$

This implies what we have to prove with $\alpha = \tau$. □

6. *Sequences realized by dynamical systems*

Let $(\Omega, \mathcal{A}, \nu, S)$ be a measure-preserving dynamical system. For a given function $h \in L^1(\nu)$, the sequence $(h(S^n\omega))$ is referred to as a random sequence realized by the dynamical system $(\Omega, \mathcal{A}, \nu, S)$. Under some condition on h, the sequence $(h(S^n\omega))$ is almost surely of Gelfond type with Gelfond exponent $\frac{1}{2}$ and is fully oscillating. More precisely, we have the following result.

THEOREM 6. *Let $(\Omega, \mathcal{A}, \nu, S)$ be a measure-preserving dynamical system. Suppose that $h \in L^\infty(\nu)$ with $\int h\, d\nu = 0$ and*

$$\sum_{n=1}^{\infty} \|\mathbb{E}(h|S^{-n}\mathcal{A})\|_\infty < \infty. \tag{6.1}$$

Then, for ν-almost every ω and for any $d \geq 1$, we have

$$\sup_{P\in\mathbb{R}_d[t]} \left| \sum_{n=0}^{N-1} h(S^n\omega) e^{2\pi i P(n)} \right| = O_{d,\omega}\big(\sqrt{N\log N}\big). \tag{6.2}$$

Proof. We can repeat the proof of Theorem 2 in [**12**]. But the key point that we should point out for the situation considered here is that under the condition (6.1), the sequence $(h(T^n\omega))$ shares the following subgaussian property: for any $(a_n)_{n\geq 0} \in \ell^2(\mathbb{N})$, we have

$$\mathbb{E}\exp\left(\lambda\left|\sum_{n=0}^{\infty} a_n h(S^n\omega)\right|\right) \leq \exp\left(c\lambda^2 \sum_{n=0}^{\infty} |a_n|^2\right), \tag{6.3}$$

where $c > 0$ is a constant depending on h. This inequality (6.3) was proved in [**14**] (see Lemma 2.5 and the proof of Theorem 2.6 in [**14**]). □

Recall that $\mathbb{E}(h|S^{-n}\mathcal{A}) = L^n h$, where L is the Perron–Frobenius operator of the dynamical system, which is defined by

$$\int f Lh\, d\nu = \int f \circ Sh\, d\nu \quad (\text{for all } f \in L^1(\nu), \text{ for all } h \in L^\infty(\nu)).$$

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The decay of $\|L^n h\|_\infty$ was well studied and we know that the condition (6.1) is satisfied by many dynamical systems and many 'regular' functions h. See, for example, [**3**] and [**16**].

The subgaussian property (6.3) was proved for more general random sequences in [**7**]. So, these random sequences also share the same property stated in Theorem 6.

Let us mention the following results due to Lesigne [**23**] describing conditions for $h(S^n\omega)$ to be almost surely oscillating of a given order. The conditions are expressed by the quasi-discrete spectrum of the system. Let $(\Omega, \mathcal{A}, \nu, S)$ be an ergodic measure-preserving dynamical system. The isometry $g \mapsto g \circ S$ on $L^2(\nu)$ is still denoted by S. Let $E_0(S)$ be the set of eigenvalues of S. For any $k \geq 1$, we inductively define

$$E_k(S) := \{g \in L^2(\nu) : |g| = 1 \ \nu\text{-a.e.}, \ g \circ S \cdot \overline{g} \in E_{k-1}(S)\}.$$

The set $E_1(S)$ is essentially the set of eigenfunctions of S (up to multiplicative constants). Functions in $E_k(S)$ are the kth generalized eigenfunctions of S. The sequence $E_k(S)$ is increasing. Let

$$E_\infty(S) = \bigcup_{k\geq 0} E_k(S).$$

For $h \in L^1(\nu)$, by $f \in E_k(S)^\perp$ we mean $\int hg \, d\nu = 0$ for all $g \in E_k(S)$. The following results are proved by Lesigne in [**23**]. We state them in terms of the notion of oscillation:

(i) assume that $(\Omega, \mathcal{A}, \nu, S)$ is totally ergodic and $h \in L^1(\nu)$. The sequence $(h(S^n\omega))$ is almost surely oscillating of order d if and only if $f \in E_d(S)^\perp$;

(ii) assume that $(\Omega, \mathcal{A}, \nu, S)$ is weakly mixing (i.e. 1 is the only eigenvalue). For any $h \in L^1(\nu)$ with $\int h \, d\nu = 0$, the sequence $(h(S^n\omega))$ is almost surely fully oscillating. Moreover, almost surely for any $d \geq 1$,

$$\lim_{N\to\infty} \sup_{P\in\mathbb{R}_d[t]} \left| \frac{1}{N} \sum_{n=1}^{N} h(S^n\omega) e^{2\pi i P(n)} \right| = 0. \tag{6.4}$$

The weak mixing condition means that $E_1(S) = \mathbb{U}$, the set of complex numbers of modulus 1. In other words, $E_\infty(S) = \mathbb{U}$. In this case, the condition $\int h \, d\nu = 0$ means that $h \in E_\infty(S)^\perp$.

According to Abramov [**2**], we say that $(\Omega, \mathcal{A}, \nu, S)$ has *quasi-discrete spectrum* if $E_\infty(S)^\perp = \{0\}$. From (i), we deduce that:

(iii) if $(\Omega, \mathcal{A}, \nu, S)$ is totally ergodic and has quasi-discrete spectrum, then there is no $h \in L^1(\nu)$ with $h \neq 0$ such that $(h(S^n\omega))$ is almost surely fully oscillating;

(iv) assume that $(\Omega, \mathcal{A}, \nu, S)$ is totally ergodic and does not have quasi-discrete spectrum. For any $h \in E_\infty(S)^\perp$, the sequence $(h(S^n\omega))$ is almost surely fully oscillating;

(v) there are nilsequences which are fully oscillating. In fact, let us consider a Heisenberg nilrotation (X, S). Take a function $h \in L^2$ which is orthogonal to the quasi-discrete part; then, for almost every x, the nilsequence $h(S^n x)$ is fully oscillating.

Furthermore, we can take a continuous function h in (v) which is orthogonal to the quasi-discrete part. Then, for every x, the nilsequence $h(S^n x)$ is fully oscillating, by the topological polynomial Wiener–Wintner theorem proved in [**13**]. For example,

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if $1, \alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d$ are $\mathbb{Q}$-independent, then $n \mapsto \sum_{j=1}^{d} n\alpha_j[n\beta_j]$ is fully oscillating.

The property (6.4) is to be compared with (6.2). The weaker property (6.4) holds for all $h \in L^1(\nu)$ with $\int h \, d\nu = 0$, and the property (6.2) is stronger but requires more regularity on h.

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