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High frequency diffraction by a soft circular disc

I. The plane wave at normal incidence

By J. C. NEWBY
*University of Strathclyde, Glasgow**

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Abstract. The problem is governed by a Jones integral equation and the solution is shown to depend upon a single function which occurs naturally after a contour deformation has produced extensive cancellation in the work. The far-scattered field off the axis of symmetry is found in detail, yielding terms which are believed to be new.

Introduction. Diffraction by a circular disc has been considered by many authors with more recent interest being in high-frequency work. Levine (10) and Levine and Wu (11) modified the kernel of the standard integral equation in order to use Wiener–Hopf methods. Keller's Ray Theory (6–9), an approximation based upon physical optics, has also been applied to the circular disc. These methods involve various approximations.

An entirely new approach was made by Jones (1) and it is this approach which is used in the present work. The method involves the exact solution of an integral equation, the uniqueness of which has also been demonstrated. Jones found the high-frequency scattering coefficient for a normally incident plane harmonic scalar wave on both the soft (1) and hard (2) circular discs, and later considered the electromagnetic case (3). In the present work the far field off the axis of symmetry is found in detail yielding terms which are believed to be new. These results are compared with results obtained by Keller's Ray Theory.

In dealing with the known term in the integral equation a different approach is used to that employed by Jones. This term is expressed as a contour integral. Subsequent deformation of the contour yields a more amenable known function containing a single term fundamental to the problem. Extensive cancellation also occurs which considerably simplifies the analysis. The insight gained by this method leads to the solution of the obliquely incident plane wave problem, details of which are not included in the present work.

(1) *The integral equation.* For the normalized problem in a cylindrical polar coordinate system, with the soft disc occupying the region $0 \leqslant r \leqslant 1$, $z = 0$, Jones (1)

* The author is now at Brunel University, Uxbridge.

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has shown that the field on the disc, $f(w)$, satisfies

$$\int_0^1 wf(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v}+\frac{e^{-i\alpha(w-v)}}{w-v}\right\}dw = G(v), \tag{1}$$

where

$$G(v) = 2(2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}}\,x\frac{d}{dx}\int_0^x \frac{I_{-\frac{1}{2}}\{\alpha(x^2-r^2)^{\frac{1}{2}}\}}{(x^2-r^2)^{\frac{1}{4}}}\,ru_0(r,0)\,dr\,dx, \tag{2}$$

$u_0(r,z)$ being an axisymmetric incident field. Time dependence $e^{i\omega t}$ is understood and omitted throughout, α is the product of wave number and disc radius, and is large at high frequencies.

By letting

$$\int_0^1 wf(w)\frac{e^{-i\alpha(w-v)}}{w-v}\,dw = F(v), \tag{3}$$

Jones converted the integral equation of the first kind for $f(w)$ into a Fredholm integral equation of the second kind for $F(v)$,

$$F(v) = G(v)-\frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}}e^{-i\alpha v}\int_0^1\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{F(t)}{t+v}\,e^{-i\alpha t}\,dt. \tag{4}$$

Inversion of (3) and application of the condition that $f(w)$ should be bounded at the origin shows that

$$wf(w) = \frac{1}{\pi^2}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}e^{i\alpha w}\int_0^1\left(\frac{1-v}{v}\right)^{\frac{1}{2}}\frac{F(v)}{w-v}\,e^{-i\alpha v}\,dv. \tag{5}$$

(2) *The known function.* To proceed the known function $G(v)$, and hence the incident field $u_0(r,z)$, must be specified. For a normally incident plane wave from the negative z direction

$$u_0(r,z) = e^{-i\alpha z}, \tag{6}$$

and then

$$G(v) = 2(2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\int_0^v \frac{\cos\{\alpha(v^2-x^2)^{\frac{1}{2}}\}}{(v^2-x^2)^{\frac{1}{2}}}\,x^{\frac{3}{2}}I_{-\frac{1}{2}}(\alpha x)\,dx. \tag{7}$$

$G(v)$ does not have a uniformly valid asymptotic expansion. Such an expansion would be necessary to find a satisfactory solution to the integral equation (4). Some features of $G(v)$ will be discussed which give an insight into the problem and suggest a course of investigation.

Using certain identities for Bessel functions (e.g. Watson (12)) we may write

$$x^{\frac{3}{2}}I_{-\frac{1}{2}}(\alpha x) = e^{-\alpha x}\eta(\alpha x)-e^{\alpha x}\eta(-\alpha x) \tag{8}$$

where

$$\eta(\alpha x) = \frac{1}{\pi}x^{\frac{3}{2}}e^{\alpha x}K_{-\frac{1}{2}}(\alpha x) = \frac{x}{(2\pi\alpha)^{\frac{1}{2}}}. \tag{9}$$

Using this notation and the exponential form of the cosine $G(v)$ can be written

$$G(v) = (2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\left[\int_0^v \frac{e^{i\alpha\{(v^2-x^2)^{\frac{1}{2}}+ix\}}}{(v^2-x^2)^{\frac{1}{2}}}\eta(\alpha x)\,dx+\int_0^v \frac{e^{-i\alpha\{(v^2-x^2)^{\frac{1}{2}}-ix\}}}{(v^2-x^2)^{\frac{1}{2}}}\eta(\alpha x)\,dx\right.$$
$$\left.-\int_0^v \frac{e^{i\alpha\{(v^2-x^2)^{\frac{1}{2}}-ix\}}}{(v^2-x^2)^{\frac{1}{2}}}\eta(-\alpha x)\,dx-\int_0^v \frac{e^{-i\alpha\{(v^2-x^2)^{\frac{1}{2}}+ix\}}}{(v^2-x^2)^{\frac{1}{2}}}\eta(-\alpha x)\,dx\right].$$

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In the first integral make the substitution

$$(v^2-x^2)^{\frac{1}{2}}+ix=-y$$

to simplify the exponential. This integral is then given by

$$\int_0^v \frac{e^{i\alpha\{(v^2-x^2)^{\frac{1}{2}}+ix\}}}{(v^2-x^2)^{\frac{1}{2}}}\eta(\alpha x)\,dx=-i\int_{-v}^{-iv}\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\}dy.$$

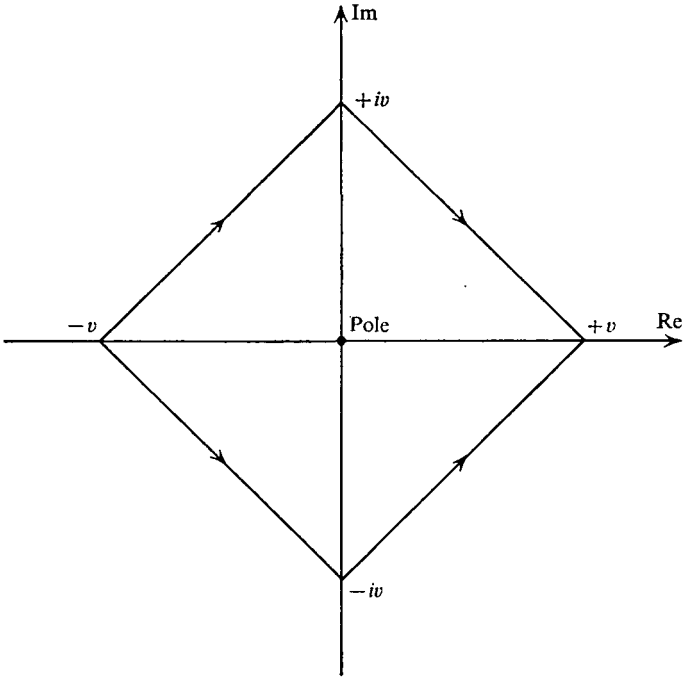


Fig. 1. Contour of $G(v)$ in the complex y-plane.

Similar changes in each of the other integrals are made so that all the exponentials are reduced to $e^{-i\alpha y}$. All the integrands are then the same and

$$G(v)=(2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\left[\left(-i\int_{-v}^{-iv}+i\int_{v}^{-iv}-i\int_{-v}^{iv}+i\int_{v}^{iv}\right)\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\}dy\right]. \quad (10)$$

The contour in the y-plane is illustrated in Fig. 1.

There are two paths of integration from $-v$ to $+v$, one being above the pole at the origin, and the other below it. The form of the contour and the exponential nature of the integrand suggest extending the contour from v to $-i\infty$. We therefore define a new function $\mathscr{G}(v)$ by

$$\begin{aligned}\mathscr{G}(v)&=-2i(2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\int_v^{-i\infty}\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\}dy\\&=-2(2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\int_0^{\infty}\frac{e^{-i\alpha(v^2-x^2)^{\frac{1}{2}}}}{(v^2-x^2)^{\frac{1}{2}}}\frac{1}{\pi}x^{\frac{1}{2}}K_{-\frac{1}{2}}(\alpha x)\,dx \quad (11)\end{aligned}$$

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in the original coordinates. $\mathscr{G}(v)$ is defined in such a way that when added to $G(v)$ it completes the contours above and below the pole to $-i\infty$. This enables a contour deformation to take place illustrated in Fig. 2.

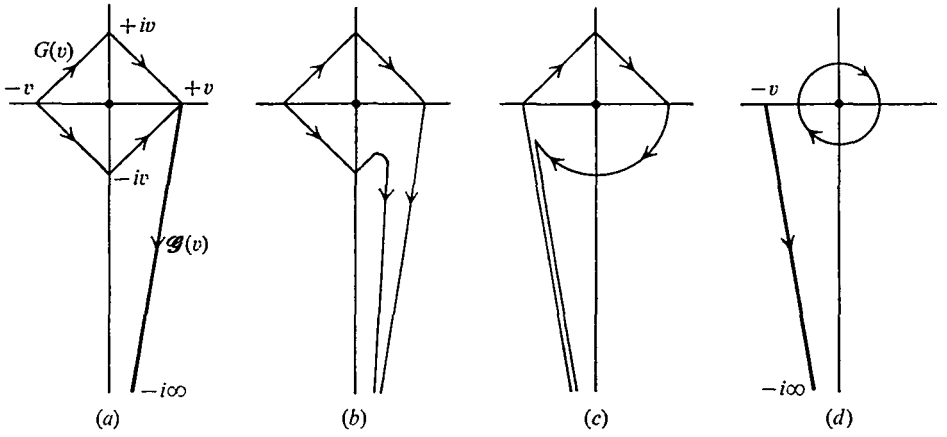


Fig. 2. The contours of $G(v)$ and $\mathscr{G}(v)$ in the complex y-plane, (a), and their subsequent deformation, (b), (c) (d).

Hence

$$G(v)+\mathscr{G}(v) = -i(2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\left[2\int_{-v}^{-i\infty}\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\}dy + \oint\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\}dy\right] \quad (12)$$

where $\oint$ is taken to mean a closed line integral round the pole in a clockwise direction.

If another known function, $\gamma(v)$, is defined by

$$\gamma(v) = -i(2\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\oint\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-v^2)\right\}dy \quad (13)$$

then

$$G(v)+\mathscr{G}(v) = \gamma(v)-\mathscr{G}(-v). \quad (14)$$

(3) *A new known function.* The known function $G(v)$ has rather complicated contours. The sum of $G(v)$ and $\mathscr{G}(v)$, however, has simpler contours and to introduce these into the integral equation (4) define a new unknown function $\mathscr{F}(v)$ by

$$\mathscr{F}(v) = F(v)-G(v)-\mathscr{G}(v). \quad (15)$$

The integral equation then becomes

$$\mathscr{F}(v) = g(v)-\frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}}e^{-i\alpha v}\int_0^1\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{\mathscr{F}(t)}{t+v}e^{-i\alpha t}dt \quad (16)$$

where

$$g(v) = -\mathscr{G}(v)-\frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}}e^{-i\alpha v}\int_0^1\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{e^{-i\alpha t}}{t+v}\{G(t)+\mathscr{G}(t)\}dt. \quad (17)$$

$g(v)$ is the new known function and can be expected to be more amenable than $G(v)$ since its contours are simpler.

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Using (14) in (17)

$$g(v) = -\mathscr{G}(v) - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v}\{\gamma(t) - \mathscr{G}(-t)\}dt. \tag{18}$$

We now consider the various functions involved. From (9) and (11)

$$\mathscr{G}(v) = -2ve^{-i\alpha v}\int_0^{-i\infty} \frac{e^{-i\alpha x}}{(x+v)^2}dx. \tag{19}$$

The integral involving $\mathscr{G}(-t)$ in (18) can then be written

$$\begin{aligned}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v}\mathscr{G}(-t)dt &= \int_0^{-i\infty} e^{-i\alpha x} 2\int_0^1 \frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{t+v}\frac{dt}{(x-t)^2}dx \\ &= \pi\left(\frac{1+v}{v}\right)^{\frac{1}{2}} e^{i\alpha v}\mathscr{G}(v) + \int_0^{-i\infty} e^{-i\alpha x}\oint \frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{t+v}\frac{dt}{(x-t)^2}dx \end{aligned} \tag{20}$$

using a result in the Appendix.

Now evaluation of (13) shows

$$\gamma(x) = 2\pi\alpha x \tag{21}$$

and evaluation of the inner integral in (20) gives

$$\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v}\mathscr{G}(-t)dt = \pi\left(\frac{1+v}{v}\right)^{\frac{1}{2}} e^{i\alpha v}\mathscr{G}(v) + \int_0^{-i\infty}\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{e^{-i\alpha t}}{t+v}\gamma(t)dt. \tag{22}$$

(22) in (18) then produces extensive cancellation resulting in a simpler expression

$$g(v) = \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v}\int_1^{1-i\infty}\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{e^{-i\alpha t}}{t+v}\gamma(t)dt. \tag{23}$$

$g(v)$ is therefore a straightforward integral with limits which yield a simple asymptotic expansion for large α. $\gamma(t)$ is an integral round a pole (evaluated in (21)), all other contour integrals having vanished. The cancellations are more extensive than might have been anticipated and the relatively simple form of $g(v)$ compared with $G(v)$ suggests that the contour deformations and the resulting function $\gamma(v)$ may be in some sense fundamental to the problem.

Essentially the same known function is derived by Jones (1) without the use of contour deformations. The present derivation is perhaps of more interest in that a deeper insight is given, particularly in respect of the way in which the various functions are associated with each other.

(4) *The field on the disc.* Using Jones (1) iteration scheme the solution of (16) is

$$\mathscr{F}(v) \sim \sum_{k=1}^{\infty}\int_0^{\infty}\Psi_k(w)\frac{e^{-i\alpha(w-v)}}{w-v}dw, \tag{24}$$

where

$$\Psi_0(w) = \frac{i}{\pi}\gamma(w), \tag{25}$$

$$\psi_{k+1}(v) = \int_1^{1-i\infty}\left(\frac{w-1}{w}\right)^{\frac{1}{2}}\frac{e^{-i\alpha w}}{w+v}\Psi_k(w)dw, \tag{26}$$

$$\Psi_{k+1}(w) = \frac{i\alpha}{4\pi}\int_0^{\infty}\left(\frac{v}{1+v}\right)^{\frac{1}{2}}\psi_{k+1}(v)e^{-i\alpha v}M(v,w)dv, \tag{27}$$

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and $$M(v,w) = \frac{4w}{v^2-w^2}\{vH_1^{(2)}(\alpha v)J_0(\alpha w) - wH_0^{(2)}(\alpha v)J_1(\alpha w)\}.$$

An expression for the field on the disc, $f(w)$, can be found from (5), (14), (15) and the equations above.

$$f(w) = f_0(w) + \sum_{k=1}^{\infty} f_k(w) \tag{28}$$

where $$f_0(w) = -\frac{e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{v-w}\{\gamma(v) - \mathscr{G}(-v)\}dv \tag{29}$$

and $$f_k(w) = -\frac{e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{v-w}\int_0^{\infty} \Psi_k(t)\frac{e^{-i\alpha(t-v)}}{t-v}dt\,dv. \tag{30}$$

The term involving $\mathscr{G}(-v)$ in (29) is similar to the left-hand side of (22). A similar analysis, allowing for the pole at $v = w$, then gives

$$\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{v-w}\mathscr{G}(-v)dv = \pi i\left(\frac{1-w}{w}\right)^{\frac{1}{2}} e^{-i\alpha w}\gamma(w) + \int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{v-w}\gamma(v)dv + \int_1^{1-i\infty} \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{v-w}\gamma(v)dv \tag{31}$$

where the integral on the left-hand side and the first integral on the right-hand side are principal values. Substitution of this result in (29) shows that

$$\begin{aligned} f_0(w) &= \frac{1}{w}\left\{\frac{i}{\pi}\gamma(w)\right\} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_1^{1-i\infty} \left(\frac{v-1}{v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{v-w}\left\{\frac{i}{\pi}\gamma(v)\right\}dv \\ &= \frac{1}{w}\Psi_0(w) + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\psi_1(-w). \end{aligned} \tag{32}$$

In considering the expression (30) for $f_k(w)$ change the order of integration. Since

$$\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{dv}{(v-w)(v-t)} = \pi^2\left(\frac{1-w}{w}\right)^{\frac{1}{2}}\delta(t-w) + \pi\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\frac{H(t-1)}{t-w}$$

$0 < w < 1$, $t > 0$, (Jones (1) appendix B), we can write

$$f_k(w) = \frac{1}{w}\Psi_k(w) + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\psi_{k+1}(-w). \tag{33}$$

The expressions for $f_0(w)$ and $f_k(w)$, $k \geqslant 1$, in (32) and (33) are of the same form and so

$$f(w) \sim \sum_{k=0}^{\infty} f_k(w) \tag{34}$$

and $$f_k(w) = \frac{1}{w}\Psi_k(w) + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\psi_{k+1}(-w). \tag{35}$$

(5) *The far-scattered field.* A direct application of Green's Theorem shows that the scattered field, $u_s(\mathbf{R})$, is given by

$$u_s(\mathbf{R}) = -\frac{1}{4\pi}\int_S f(r_1)\frac{e^{-i\alpha|\mathbf{R}-\mathbf{R}_1|}}{|\mathbf{R}-\mathbf{R}_1|}dS,$$

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where $\mathbf{R}_1$ is a point on the disc and the integral is over one side of the disc. In general the integral is complicated, but if $\mathbf{R}$ is assumed to be far from the disc a useful approximation can be made. Then, using spherical polar co-ordinates,

$$u_s(\mathbf{R}) \sim -\frac{1}{2}\frac{e^{-i\alpha R}}{R}\int_0^1 wf(w)J_0(\alpha w\sin\theta)\,dw + O\left(\frac{1}{R^2}\right). \tag{36}$$

Let

$$F(\theta) = \int_0^1 wf(w)J_0(\alpha w\sin\theta)\,dw \tag{37}$$

so that

$$u_s(\mathbf{R}) \sim -\frac{1}{2}\frac{e^{-i\alpha R}}{R}F(\theta). \tag{38}$$

Then, by (34),

$$F(\theta) \sim \sum_{k=0}^{\infty} F_k(\theta) \tag{39}$$

where

$$\begin{aligned} F_k(\theta) &= \int_0^1 wf_k(w)J_0(\alpha w\sin\theta)\,dw \\ &= \int_0^1 \Psi_k(w)J_0(\alpha w\sin\theta)\,dw + \frac{1}{\pi}\int_0^1 \left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_{k+1}(-w)\,e^{i\alpha w}J_0(\alpha w\sin\theta)\,dw \end{aligned} \tag{40}$$

after substituting for $f_k(w)$ from (35).

$F(\theta)$ can be written in a modified form which is more convenient to use. To do this it is necessary to return to the iteration scheme. (27) arises from the solution of an integral equation (Jones (1))

$$\int_0^\infty \Psi_k(w)\left\{\frac{e^{-i\alpha(w-v)}}{w-v} + \frac{e^{-i\alpha(w+v)}}{w+v}\right\}dw = \left(\frac{v}{1+v}\right)^{\frac{1}{2}}\psi_k(v)\,e^{-i\alpha v}.$$

This equation, among others, has been considered by Jones (4, 5). Multiply through by $J_0(\alpha v\sin\theta)$ and integrate from 0 to ∞. Then

$$\int_0^\infty \Psi_k(w)\,e^{-i\alpha w}\int_{-\infty}^{\infty}\frac{e^{i\alpha v}}{w-v}J_0(\alpha v\sin\theta)\,dv\,dw = \int_0^\infty \left(\frac{v}{1+v}\right)^{\frac{1}{2}}\psi_k(v)\,e^{-i\alpha v}J_0(\alpha v\sin\theta)\,dv$$

after changing the order of integration and then changing v to $-v$ in the second term of the integrand on the left-hand side. If the inner integral is now completed with an indentation above the pole at $v = w$ its contour may be deformed upwards without contribution, provided $\theta \neq \frac{1}{2}\pi$. Hence

$$\int_0^\infty \Psi_k(w)J_0(\alpha w\sin\theta)\,dw = \frac{i}{\pi}\int_0^\infty \left(\frac{v}{1+v}\right)^{\frac{1}{2}}\psi_k(v)\,e^{-i\alpha v}J_0(\alpha v\sin\theta)\,dv. \tag{41}$$

(The author is indebted to Professor W. E. Williams for a shortened proof of this result.)

Now consider the right-hand side of (40).

$$\begin{aligned} \int_0^1 \Psi_k(w)J_0(\alpha w\sin\theta)\,dw &= \int_0^\infty \Psi_k(w)J_0(\alpha w\sin\theta)\,dw - \int_1^\infty \Psi_k(w)J_0(\alpha w\sin\theta)\,dw \\ &= \frac{i}{\pi}\int_0^\infty \left(\frac{v}{1+v}\right)^{\frac{1}{2}}\psi_k(v)\,e^{-i\alpha v}J_0(\alpha v\sin\theta)\,dv \\ &\quad - \int_1^\infty \Psi_k(w)J_0(\alpha w\sin\theta)\,dw \end{aligned} \tag{42}$$

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after using (41)

$$\frac{1}{\pi}\int_0^1 \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw$$

$$= \frac{1}{\pi}\int_0^{i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw$$

$$-\frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw$$

$$= -\frac{i}{\pi}\int_0^{\infty} \left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_{k+1}(w)\, e^{-i\alpha w} J_0(\alpha w \sin\theta)\, dw$$

$$-\frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw \quad (43)$$

after deforming the contour of the first integral on the right-hand side onto the negative real axis and then changing the sign of w.

Comparing (42) with (43) shows that the first terms on the right-hand side are similar. Substitution of these two relationships back into (40), the expression for $F_k(\theta)$, shows that cancellation will occur down the iteration scheme. Hence

$$F(\theta) \sim \sum_{k=0}^{\infty} F_k(\theta)$$

$$\sim \int_0^1 \Psi_0(w) J_0(\alpha w \sin\theta)\, dw - \frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_1(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw$$

$$-\sum_{k=1}^{\infty}\left[\int_1^{\infty} \Psi_k(w) J_0(\alpha w \sin\theta)\, dw + \frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw\right]. \quad (44)$$

It is therefore apparent that there is no contribution to the far field from the centre of the disc, except perhaps from the first term. However

$$\int_0^1 \Psi_0(w) J_0(\alpha w \sin\theta)\, dw = \frac{2i}{\sin\theta} J_1(\alpha \sin\theta), \quad (45)$$

there being no contribution from the lower limit, and so the far field is determined entirely from conditions at the edge of the disc. The first two integrals in (44) will be called the zero-order contribution to $F(\theta)$ and subsequent pairs of integrals for each k will be called the kth order contributions.

The only restriction which applies to $F(\theta)$ so far is that $\theta \neq \frac{1}{2}\pi$ (cf. (41)), which means that the expression cannot be used to find the field in the plane of the disc.

In order to proceed to the asymptotic evaluation of second and subsequent integrals in (44) it is necessary to consider three regions in which the field may be required. The first region is that at or near the axis of symmetry, such that $\alpha \sin\theta \ll 1$. An important particular case of this is the far field along the axis of symmetry, which gives the scattering cross-section of the disc (Jones (1)). The field in this region may be determined by using the series forms of the Bessel functions. The second region is that in

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which $\alpha \sin\theta$ is neither very small nor very large. This requires special attention as a transition region and will not be considered here. The third region is that in which $\alpha \sin\theta \gg 1$, and at high frequencies this is by far the largest region. In this region the asymptotic forms of the Bessel function may be used in the integrals. This third region is the one considered in the present work.

(6) *The zero-order contribution to* $F(\theta)$. To find the far field it is necessary to calculate $F(\theta)$ in detail. The zero order contribution is given by

$$\int_0^1 \Psi_0(w) J_0(\alpha w \sin\theta)\, dw - \frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_1(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw. \tag{46}$$

At high frequencies, away from the axis of symmetry, $\alpha \sin\theta \gg 1$ and so the asymptotic form of the Bessel function,

$$J_\nu(z) \sim \frac{1}{\pi^{\frac{1}{2}}}\, e^{-i(\nu\frac{1}{2}\pi+\frac{1}{4}\pi)} \sum_{m=0}^{\infty} \frac{i^m(\nu+m-\frac{1}{2})!}{m!(\nu-m-\frac{1}{2})!} \frac{1}{(2z)^{m+\frac{1}{2}}} (e^{iz} + i(-1)^{m+\nu} e^{-iz}), \tag{47}$$

may be employed. The first integral of the zero-order field has been found in (45) and so

$$\begin{aligned}\int_0^1 \Psi_0(w) J_0(\alpha w \sin\theta)\, dw \\ \sim -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \alpha \left(\frac{i}{\alpha \sin\theta}\right)^{\frac{3}{2}} \Big\{ (e^{i\alpha\sin\theta} - i e^{-i\alpha\sin\theta}) + \frac{3i}{8\alpha\sin\theta} (e^{i\alpha\sin\theta} + i e^{-i\alpha\sin\theta}) \\ + \frac{15}{2^7\alpha^2\sin^2\theta} (e^{i\alpha\sin\theta} - i e^{-i\alpha\sin\theta}) \Big\} + O(\alpha^{-\frac{7}{2}}).\end{aligned} \tag{48}$$

The second integral of the zero-order field is

$$\begin{aligned}\frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_1(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw \\ = \frac{1}{\pi}\int_1^{1-i\infty} \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \Psi_0(t)\, e^{-i\alpha t} \int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \frac{J_0(\alpha w\sin\theta)}{t-w}\, e^{i\alpha w}\, dw\, dt.\end{aligned} \tag{49}$$

Using the asymptotic form of the Bessel function and some fairly straightforward analysis the inner integral can be written

$$\begin{aligned}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \frac{J_0(\alpha w\sin\theta)}{t-w}\, e^{i\alpha w}\, dw \\ \sim -\frac{1}{\pi^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{i^m(m-\frac{1}{2})!}{m!(-m-\frac{1}{2})!} \frac{1}{(2\alpha\sin\theta)^{m+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-m)!(n-\frac{1}{2})!\, i^n}{n!(-m-n)!} \\ \times \left[e^{i\alpha t(1+\sin\theta)} \int_{\alpha(1+\sin\theta)}^{\infty} \frac{e^{-i(t-1)x}}{x^{n+\frac{1}{2}}}\, dx + i(-1)^m e^{i\alpha t(1-\sin\theta)} \int_{\alpha(1-\sin\theta)}^{\infty} \frac{e^{-i(t-1)x}}{x^{n+\frac{1}{2}}}\, dx \right].\end{aligned} \tag{50}$$

Substitution in (49) and inversion of the order of integration yields integrals of the form

$$\int_{\alpha(1\pm\sin\theta)}^{\infty} \frac{e^{ix}}{x^{n+\frac{1}{2}}} \int_1^{1-i\infty} \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \Psi_0(t)\, e^{-it(x\mp\alpha\sin\theta)}\, dt\, dx.$$

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After substituting in the expression for $\Psi_0(t)$ from (25) the integrals may be asymptotically evaluated to give

$$\begin{aligned}
&\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_1(-w)\,e^{i\alpha w}J_0(\alpha w\sin\theta)\,dw\\
&\quad\sim-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\alpha\left(\frac{i}{\alpha\sin\theta}\right)^{\frac{3}{2}}\Big\{[\{1-(1+\sin\theta)^{\frac{1}{2}}\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta)^{\frac{1}{2}}\}e^{-i\alpha\sin\theta}]\\
&\qquad+\frac{i}{\alpha\sin\theta}\Big[\tfrac{5}{8}[\{1-(1+\sin\theta)^{\frac{1}{2}}\}e^{i\alpha\sin\theta}+i\{1-(1-\sin\theta)^{\frac{1}{2}}\}e^{-i\alpha\sin\theta}]\\
&\qquad-\tfrac{1}{4}[\{1-(1+\sin\theta)^{\frac{3}{2}}\}e^{i\alpha\sin\theta}+i\{1-(1-\sin\theta)^{\frac{3}{2}}\}e^{-i\alpha\sin\theta}]\Big]\\
&\qquad+\left(\frac{i}{\alpha\sin\theta}\right)^2\Big[\tfrac{1}{16}[\{1-(1+\sin\theta)^{-\frac{1}{2}}\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta)^{-\frac{1}{2}}\}e^{-i\alpha\sin\theta}]\\
&\qquad-\tfrac{55}{128}[\{1-(1+\sin\theta)^{\frac{1}{2}}\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta)^{\frac{1}{2}}\}e^{-i\alpha\sin\theta}]\\
&\qquad+\tfrac{11}{32}[\{1-(1+\sin\theta)^{\frac{3}{2}}\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta)^{\frac{3}{2}}\}e^{-i\alpha\sin\theta}]\\
&\qquad-\tfrac{3}{32}[\{1-(1+\sin\theta)^{\frac{5}{2}}\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta)^{\frac{5}{2}}\}e^{-i\alpha\sin\theta}]\Big]\Big\}+O(\alpha^{-\frac{7}{2}}).
\end{aligned}\tag{51}$$

It is noted that the terms of (48), the first integral in the zero-order contribution, correspond exactly with those terms of (51) coming from the first 1 of $\{1-(1\pm\sin\theta)^{r+\frac{1}{2}}\}$. In the zero-order contribution as a whole there is complete cancellation of the first integral with part of the second integral. The zero-order contribution to $F(\theta)$ is therefore

$$\begin{aligned}
&\int_0^1\Psi_0(w)J_0(\alpha w\sin\theta)\,dw-\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_1(-w)J_0(\alpha w\sin\theta)\,dw\\
&\qquad\sim\left(\frac{2}{\pi\alpha}\right)^{\frac{1}{2}}\frac{e^{-i\frac{1}{4}\pi}}{(\sin\theta)^{\frac{3}{2}}}\Big\{[(1+\sin\theta)^{\frac{1}{2}}e^{i\alpha\sin\theta}-i(1-\sin\theta)^{\frac{1}{2}}e^{-i\alpha\sin\theta}]\\
&\qquad\quad+\frac{i}{\alpha\sin\theta}\Big[\tfrac{5}{8}[(1+\sin\theta)^{\frac{1}{2}}e^{i\alpha\sin\theta}+i(1-\sin\theta)^{\frac{1}{2}}e^{-i\alpha\sin\theta}]\\
&\qquad\quad-\tfrac{1}{4}[(1+\sin\theta)^{\frac{3}{2}}e^{i\alpha\sin\theta}+i(1-\sin\theta)^{\frac{3}{2}}e^{-i\alpha\sin\theta}]\Big]\\
&\qquad\quad+\left(\frac{i}{\alpha\sin\theta}\right)^2\Big[\tfrac{1}{16}[(1+\sin\theta)^{-\frac{1}{2}}e^{i\alpha\sin\theta}-i(1-\sin\theta)^{-\frac{1}{2}}e^{-i\alpha\sin\theta}]\\
&\qquad\quad-\tfrac{55}{128}[(1+\sin\theta)^{\frac{1}{2}}e^{i\alpha\sin\theta}-i(1-\sin\theta)^{\frac{1}{2}}e^{-i\alpha\sin\theta}]\\
&\qquad\quad+\tfrac{11}{32}[(1+\sin\theta)^{\frac{3}{2}}e^{i\alpha\sin\theta}-i(1-\sin\theta)^{\frac{3}{2}}e^{-i\alpha\sin\theta}]\\
&\qquad\quad-\tfrac{3}{32}[(1+\sin\theta)^{\frac{5}{2}}e^{i\alpha\sin\theta}-i(1-\sin\theta)^{\frac{5}{2}}e^{-i\alpha\sin\theta}]\Big]\Big\}+O(\alpha^{-\frac{7}{2}}).
\end{aligned}\tag{52}$$

(7) *An asymptotic form of* $\Psi_k(w)$. In order to proceed with the kth order contribution to $F(\theta)$, given by

$$-\int_1^{\infty}\Psi_k(w)J_0(\alpha w\sin\theta)\,dw-\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_{k+1}(-w)\,e^{i\alpha w}J_0(\alpha w\sin\theta)\,dw,$$

it is necessary to consider a suitable form of $\Psi_k(w)$. It is noted that in the first integral $w\geqslant 1$. $\psi_{k+1}(-w)$ occurs in the second integral and may be expressed as an integral of Ψ_k from 1 to $1-i\infty$ by (26). In each case the modulus of the argument of Ψ_k is greater

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than or equal to unity. This knowledge may be used to find an asymptotic form of $\Psi_k(w)$.

From (27)
$$\Psi_k(w) = \frac{i\alpha}{4\pi}\int_0^\infty \left(\frac{v}{1+v}\right)^{\frac{1}{2}} \psi_k(v)\, e^{-i\alpha v} M(v,w)\, dw$$

where
$$M(v,w) = \frac{4w}{v^2-w^2}\{vH_1^{(2)}(\alpha v)J_0(\alpha w) - wH_0^{(2)}(\alpha v)J_1(\alpha w)\}.$$

$M(v,w)$ has poles at $v = \pm w$ and so $\Psi_k(w)$ is a principal value. It may be assumed (Jones(1)) that $\psi_k(v)$ is regular in the region $\mathrm{Re}\,(v) > -1$ of the complex v plane and is bounded as $|v| \to \infty$ in this region. Completion of the contour of integration with an indentation below the pole at $v = w$ gives

$$\Psi_k(w) = \frac{i\alpha}{4\pi}\int\!\!\!\!\!\!-_0^\infty \left(\frac{v}{1+v}\right)^{\frac{1}{2}} \psi_k(v)\, e^{-i\alpha v} M(v,w)\, dv + \frac{i}{\pi}\left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_k(w)\, e^{-i\alpha w}.$$

The contour can now be deformed onto the negative imaginary axis. On changing v to $-iv$ we obtain

$$\begin{aligned}\Psi_k(w) &= \frac{i}{\pi}\left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_k(w)\, e^{-i\alpha w} - \frac{2\alpha}{\pi^2} e^{i\frac{1}{4}\pi}\int_0^\infty \left(\frac{v}{1-iv}\right)^{\frac{1}{2}} \psi_k(-iv)\, e^{-\alpha v}\frac{w}{v^2+w^2} \\ &\quad \times \{vK_1(\alpha v)J_0(\alpha w) - wK_0(\alpha v)J_1(\alpha w)\}\, dv \\ &= \frac{i}{\pi}\left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_k(w)\, e^{-i\alpha w} - \frac{2\alpha}{\pi^{\frac{3}{2}}} e^{i\frac{1}{4}\pi} \sum_{m=0}^{\infty} \frac{(-\frac{1}{2})!(-i)^m}{m!(-\frac{1}{2}-m)!} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\psi_k^{(n)}(0) \sum_{p=0}^{\infty} \frac{(m+n+2p+\frac{1}{2})!(-1)^p}{(2\alpha)^{m+n+2p+\frac{3}{2}}} \\ &\quad \times \left\{\frac{(m+n+2p+\frac{5}{2})!}{2\alpha(m+n+2p-2)!}\frac{J_0(\alpha w)}{w^{2p+1}} - \frac{(m+n+2p+\frac{1}{2})!}{(m+n+2p+1)!}\frac{J_1(\alpha w)}{w^{2p}}\right\} \end{aligned} \tag{53}$$

after asymptotic evaluation of the integral. This is the asymptotic form of $\Psi_k(w)$ used for $w \geqslant 1$.

(8) *The kth order contribution to* $F(\theta)$. It is now possible to proceed with a detailed calculation of the kth order field given by

$$-\int_1^\infty \Psi_k(w) J_0(\alpha w \sin\theta)\, dw - \frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw. \tag{54}$$

In considering the first integral the asymptotic form of $\Psi_k(w)$ can now be employed since $w \geqslant 1$. The resulting integrals may then be evaluated asymptotically after some straightforward but tedious analysis to give

$$\begin{aligned}&\int_1^\infty \Psi_k(w) J_0(\alpha w \sin\theta)\, dw \\ &\sim \frac{e^{-i(\alpha+\frac{1}{4}\pi)}}{2(\pi a)^{\frac{3}{2}}(\sin\theta)^{\frac{1}{2}}}\left\{\psi_k(1)\left[\frac{e^{i\alpha\sin\theta}}{1-\sin\theta} + i\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right. \\ &\quad \left. -\frac{i}{\alpha}\left[(\psi_1'(1) - \tfrac{1}{4}\psi_k(1))\left[\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2} + i\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right] + \frac{\psi_k(1)}{8\sin\theta}\left[\frac{e^{i\alpha\sin\theta}}{1-\sin\theta} - i\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right]\right\} \\ &\quad - \frac{e^{i\frac{3}{4}\pi}}{2^{\frac{7}{2}}\pi^{\frac{3}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\psi_k(0)\left[\frac{e^{i\alpha(1+\sin\theta)} - e^{-i\alpha(1+\sin\theta)}}{1+\sin\theta} + i\frac{e^{i\alpha(1-\sin\theta)} + e^{-i\alpha(1-\sin\theta)}}{1-\sin\theta}\right] + O\left(\frac{\psi_k}{\alpha^{\frac{7}{2}}}\right). \end{aligned} \tag{55}$$

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The second integral of (54) can also be evaluated asymptotically though the labour involved is fairly extensive. The final result is

$$\begin{aligned}
&\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_{k+1}(-w)\,e^{i\alpha w}J_0(\alpha w\sin\theta)\,dw\\
&\quad\sim -\frac{e^{-i(\alpha+\frac{1}{4}\pi)}}{(\pi\alpha)^{\frac{3}{2}}(\sin\theta)^{\frac{1}{2}}}\left\{\tfrac{1}{2}\psi_k(1)\left[\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\left\{1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right.\\
&\qquad-\frac{i}{\alpha}\left[\tfrac{3}{4}(\psi_k'(1)-\tfrac{1}{4}\psi_k(1))\left(\left[\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}\right.\right.\right.\\
&\qquad\left.+i\left\{1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]\\
&\qquad-\frac{1}{3}\left[\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{3}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\left\{1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]\Big)\\
&\qquad\left.+\frac{\psi_k(1)}{2^4\sin\theta}\left[\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}-i\left\{1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right]\Big\}\\
&\qquad+\frac{\psi_k(0)e^{i\alpha}}{2^{\frac{7}{2}}\pi\alpha^2(\sin\theta)^{\frac{1}{2}}}\left[\frac{e^{i\alpha\sin\theta}}{(1+\sin\theta)^{\frac{1}{2}}}+i\frac{e^{-i\alpha\sin\theta}}{(1-\sin\theta)^{\frac{1}{2}}}\right]\\
&\qquad-\frac{\psi_k(0)e^{-i(\alpha-\frac{1}{4}\pi)}}{2^{\frac{7}{2}}\pi^{\frac{3}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\left[\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\left\{1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\\
&\qquad+\frac{\psi_k(0)e^{i(\alpha+\frac{3}{4}\pi)}}{2^{\frac{7}{2}}\pi^{\frac{3}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\left[\frac{e^{i\alpha\sin\theta}}{1+\sin\theta}+i\frac{e^{-i\alpha\sin\theta}}{1-\sin\theta}\right]+O\left(\frac{\psi_k}{\alpha^3}\right).
\end{aligned}\tag{56}$$

When the two terms of the kth order contribution are combined the first term cancels completely with part of the second term in a similar way to the cancellation which occurred in the zero-order contribution to $F(\theta)$. An asymptotic development of the kth order contribution is therefore given by

$$\begin{aligned}
&-\int_1^{\infty}\Psi_k(w)J_0(\alpha w\sin\theta)\,dw-\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_{k+1}(-w)\,e^{i\alpha w}J_0(\alpha w\sin\theta)\,dw\\
&\quad\sim\frac{e^{-i(\alpha+\frac{1}{4}\pi)}}{(\pi\alpha)^{\frac{3}{2}}(\sin\theta)^{\frac{1}{2}}}\left\{-\tfrac{1}{2}\psi_k(1)\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right.\\
&\qquad+\frac{i}{\alpha}\left[\tfrac{3}{4}(\psi_k'(1)-\tfrac{1}{4}\psi_k(1))\left(\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]\right.\right.\\
&\qquad-\frac{1}{3}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]\Big)\\
&\qquad\left.-\frac{\psi_k(1)}{2^4\sin\theta}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}-i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right]\Big\}\\
&\qquad-\frac{\psi_k(0)\,e^{i\alpha}}{\pi 2^{\frac{7}{2}}\alpha^2(\sin\theta)^{\frac{1}{2}}}\left[\frac{e^{i\alpha\sin\theta}}{(1+\sin\theta)^{\frac{1}{2}}}+i\frac{e^{-i\alpha\sin\theta}}{(1-\sin\theta)^{\frac{1}{2}}}\right]\\
&\qquad-\frac{\psi_k(0)\,e^{-(i\alpha-\frac{1}{4}\pi)}}{\pi^{\frac{3}{2}}2^{\frac{7}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]+O\left(\frac{\psi_k}{\alpha^3}\right).
\end{aligned}\tag{57}$$

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(9) *The first- and second-order contributions to* $F(\theta)$. An integral form of the far field has been derived in section 5 and the integrals involved considered in sections 6, 7 and 8. The zero-order contribution is given by (52) and the general kth order contribution by (57). This last equation, however, is in terms of ψ_k and its derivatives, and so expansions of these must be found.

From the iteration scheme (26)

$$\psi_{k+1}(v) = \int_1^{1-i\infty} \left(\frac{w-1}{w}\right)^{\frac{1}{2}} \frac{e^{-i\alpha w}}{w+v} \Psi_k(w)\, dw. \tag{58}$$

$\psi_1(v)$ therefore involves $\Psi_0(w)$ which is known. Asymptotic evaluation of the integral then gives

$$\psi_1(v) \sim \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i(\alpha+\frac{1}{4}\pi)} \left\{\frac{1}{1+v} + \frac{3i}{2\alpha}\left[\frac{1}{(1+v)^2} - \frac{1}{2(1+v)}\right] - \frac{15}{4\alpha^2}\left[\frac{1}{(1+v)^3} - \frac{1}{2(1+v)^2} - \frac{1}{8(1+v)}\right]\right\} + O(\alpha^{-\frac{7}{2}}). \tag{59}$$

Hence

$$\psi_1(1) \sim \frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i(\alpha+\frac{1}{4}\pi)}\left\{1 + \frac{15}{32\alpha^2}\right\} + O(\alpha^{-\frac{7}{2}}),$$

$$\psi_1'(1) \sim -\frac{1}{4}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i(\alpha+\frac{1}{4}\pi)}\left\{1 + \frac{3i}{4\alpha} - \frac{15}{32\alpha^2}\right\} + O(\alpha^{-\frac{7}{2}}),$$

$$\psi_1(0) \sim \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i(\alpha+\frac{1}{4}\pi)}\left\{1 + \frac{3i}{4\alpha} - \frac{45}{32\alpha^2}\right\} + O(\alpha^{-\frac{1}{2}}),$$

$$\psi_1'(0) \sim -\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i(\alpha+\frac{1}{4}\pi)}\left\{1 + \frac{9i}{4\alpha} - \frac{225}{32\alpha^2}\right\} + O(\alpha^{-\frac{7}{2}}).$$

From (58) it is noted that $\psi_{k+1}(v)$ is an integral of $\Psi_k(w)$ and the range of integration is such that $|w| \geqslant 1$. The asymptotic form of $\Psi_k(w)$ may therefore be used and so $\psi_{k+1}(v)$ becomes a function of ψ_k and its derivatives. Some fairly straightforward analysis shows that

$$\psi_{k+1}(v) \sim \frac{i\psi_k(0)}{8\alpha}\frac{1}{v}\{1-(1+v)^{\frac{1}{2}}\} + \frac{e^{-i(2\alpha+\frac{1}{4}\pi)}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\frac{\psi_k(1)}{1+v} + \frac{9}{2^7\alpha^2} \times \left[\{\psi_k'(0) - \tfrac{1}{2}\psi_k(0)\}\frac{1}{v}\{1-(1+v)^{\frac{1}{2}}\} - \psi_k(0)\frac{1}{v^2}\{1+\tfrac{1}{2}v-(1+v)^{\frac{1}{2}}\}\right] + O\left(\frac{\psi_k}{\alpha^{\frac{5}{2}}}\right). \tag{60}$$

Using this result, and the expressions for ψ_1 above, expressions for ψ_2 can be obtained.

$$\psi_2(1) \sim -\frac{2^{\frac{1}{2}}-1}{8\alpha^{\frac{3}{2}}}\pi^{\frac{1}{2}} e^{-i(\alpha-\frac{1}{4}\pi)} - \frac{i e^{-3i\alpha}}{2^5\alpha^2} + O(\alpha^{-\frac{5}{2}}),$$

$$\psi_2(0) \sim -\frac{\pi^{\frac{1}{2}}}{2^4\alpha^{\frac{3}{2}}} e^{-i(\alpha-\frac{1}{4}\pi)} - \frac{i e^{-3i\alpha}}{2^4\alpha^2} + O(\alpha^{-\frac{5}{2}}).$$

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If the expressions for ψ_1 and ψ_1' are now substituted in (57) we have, for the first-order contribution to $F(\theta)$,

$$-\int_1^\infty \Psi_1(w) J_0(\alpha w \sin\theta)\,dw - \int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_2(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\,dw$$

$$\sim \frac{i e^{-2i\alpha}}{\pi\alpha^2(\sin\theta)^{\frac{1}{2}}} \left\{ \frac{1}{4}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{i\alpha\sin\theta}}{1-\sin\theta} + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\sin\theta}}{1+\sin\theta} \right] \right.$$

$$+ \frac{i}{\alpha}\left[\frac{9}{32}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2} + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2} \right] \right.$$

$$- \frac{3}{32}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{3}{2}} \frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2} + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}} \frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2} \right]$$

$$\left.\left. + \frac{1}{32\sin\theta}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{i\alpha\sin\theta}}{1-\sin\theta} - i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\sin\theta}}{1+\sin\theta} \right] \right] \right\}$$

$$- \frac{e^{-i\frac{1}{4}\pi}}{\pi^{\frac{1}{2}} 2^{\frac{7}{2}} \alpha^{\frac{5}{2}} (\sin\theta)^{\frac{1}{2}}} \left[\frac{e^{i\alpha\sin\theta}}{(1+\sin\theta)^{\frac{1}{2}}} + i\frac{e^{-i\alpha\sin\theta}}{(1-\sin\theta)^{\frac{1}{2}}} \right]$$

$$- \frac{e^{-2i\alpha}}{\pi 2^{\frac{7}{2}} \alpha^3 (\sin\theta)^{\frac{1}{2}}} \left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{i\alpha\sin\theta}}{1-\sin\theta} + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\sin\theta}}{1+\sin\theta} \right] + O(\alpha^{-\frac{7}{2}}). \quad (61)$$

Similarly, the second-order contribution is given by

$$-\int_1^\infty \Psi_2(w) J_0(\alpha w \sin\theta)\,dw - \int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_3(-w)\, e^{i\alpha w} J_0(\alpha w \sin\theta)\,dw$$

$$\sim \frac{(2^{\frac{1}{2}}-1) e^{-2i\alpha}}{16\pi\alpha^3(\sin\theta)^{\frac{1}{2}}} \left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{i\alpha\sin\theta}}{1-\sin\theta} + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\sin\theta}}{1+\sin\theta} \right] + O(\alpha^{-\frac{7}{2}}). \quad (62)$$

Subsequent orders may be neglected if the degree of accuracy is to $O(\alpha^{-\frac{7}{2}})$.

(10) *An explicit form of the far-scattered field.* An integral form of the scattered field, $u_s(\mathbf{R})$, is given in (36). It has been shown that this can be written

$$u_s(\mathbf{R}) \sim -\frac{1}{2}\frac{e^{-i\alpha R}}{R} F(\theta) \quad (63)$$

where $F(\theta)$ is given by (44). The zero-, first- and second-order contributions to $F(\theta)$ are given by (52), (61) and (62) respectively. Collecting these expressions together shows that

$$F(\theta) \sim \left(\frac{2}{\pi\alpha}\right)^{\frac{1}{2}} \frac{e^{-i\frac{1}{4}\pi}}{(\sin\theta)^{\frac{3}{2}}} \left\{ [(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta}] \right.$$

$$+ \frac{i}{\alpha\sin\theta}\left[\tfrac{5}{8}[(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} + i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta}] \right.$$

$$\left. - \tfrac{1}{4}[(1+\sin\theta)^{\frac{3}{2}} e^{i\alpha\sin\theta} + i(1-\sin\theta)^{\frac{3}{2}} e^{-i\alpha\sin\theta}] \right]$$

$$- \frac{1}{(\alpha\sin\theta)^2}\left[\tfrac{1}{16}[(1+\sin\theta)^{-\frac{1}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{-\frac{1}{2}} e^{-i\alpha\sin\theta}] \right.$$

$$\left. - \tfrac{55}{128}[(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta}] \right.$$

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$$+\tfrac{11}{32}[(1+\sin\theta)^{\frac{3}{2}}e^{i\alpha\sin\theta}-i(1-\sin\theta)^{\frac{3}{2}}e^{-i\alpha\sin\theta}]$$

$$-\tfrac{3}{32}[(1+\sin\theta)^{\frac{5}{2}}e^{i\alpha\sin\theta}-i(1-\sin\theta)^{\frac{5}{2}}e^{-i\alpha\sin\theta}]]\Big\}$$

$$+\frac{ie^{-2i\alpha}}{\pi(\sin\theta)^{\frac{1}{2}}\alpha^2}\Big\{\frac{1}{4}\Big[\Big(\frac{1+\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\Big(\frac{1-\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\Big]$$

$$+\frac{i}{\alpha}\Big[\frac{9}{32}\Big[\Big(\frac{1+\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\Big(\frac{1-\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\Big]$$

$$-\frac{3}{32}\Big[\Big(\frac{1+\sin\theta}{2}\Big)^{\frac{3}{2}}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\Big(\frac{1-\sin\theta}{2}\Big)^{\frac{3}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\Big]$$

$$+\frac{1}{32\sin\theta}\Big[\Big(\frac{1+\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}-i\Big(\frac{1-\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\Big]\Big]\Big\}$$

$$-\frac{e^{-i\frac{1}{4}\pi}}{2^{\frac{7}{2}}(\pi\sin\theta)^{\frac{1}{2}}\alpha^{\frac{5}{2}}}[(1+\sin\theta)^{-\frac{1}{2}}e^{i\alpha\sin\theta}+i(1-\sin\theta)^{-\frac{1}{2}}e^{-i\alpha\sin\theta}]$$

$$-\frac{e^{-2i\alpha}}{16\pi(\sin\theta)^{\frac{1}{2}}\alpha^3}\Big[\Big(\frac{1+\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\Big(\frac{1-\sin\theta}{2}\Big)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\Big]+O(\alpha^{-\frac{7}{2}}). \tag{64}$$

(63) and (64) together give an expression for the scattered field which is independent of ϕ and is subject to the restrictions that $R \gg 1$ and $\theta \neq 0$ or $\frac{1}{2}\pi$. These are the conditions for a far field off the axis of symmetry and away from the plane of the disc.

Keller (7) and Karp and Keller (6), have given the leading term of the singly diffracted field and of the doubly diffracted field. The leading term of the singly diffracted field agrees with the first term of the zero-order field in the present work, after a change of notation. The leading term of the doubly diffracted field agrees with the first term of the first-order field. The results of Keller's approximate theory therefore agree with those of the present work, which are, however, more extensive and contain terms not included in Keller's theory.

It should be noted that Jones's iteration scheme gives iterates which correspond to fields produced by multiple diffractions. The kth iterate corresponds to the field produced after k diffractions across the disc, the zero-order field coming directly from a single diffraction of the incident field.

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APPENDIX

In this Appendix the integral

$$2\int_0^1 \frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(t+v)}\frac{dt}{(x-t)^2} \tag{A 1}$$

is considered where x is a point on the negative imaginary axis.

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On taking a Fourier transform with respect to v we have

$$\int_{-\infty}^{\infty} e^{-i\beta v} 2 \int_0^1 \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{t+v} \frac{dt}{(x-t)^2} dv = -2\pi i \operatorname{sgn}\beta \int_0^1 \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt. \tag{A 2}$$

Take branch lines from 1 to $+\infty$ for $(1-t)^{\frac{1}{2}}$ and from $-\infty$ to 0 for $t^{\frac{1}{2}}$. If $\beta > 0$ extend the contour of integration above the branch lines and then (A 2) may be written

$$-2\pi i \left\{ \int_{-\infty}^{\infty} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt - \int_{-\infty}^{0} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt - \int_1^{\infty} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt \right\}$$
$$= -2\pi \int_0^{\infty} \frac{(1+t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x+t)^2} e^{-i\beta t} dt + 2\pi \int_1^{\infty} \frac{(t-1)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt \tag{A 3}$$

after deforming the first integral upwards to $+i\infty$ without contribution. If $\beta < 0$ extend the contour below the branch lines and (A 2) gives

$$2\pi i \left\{ \int_{-\infty}^{\infty} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt - \int_{-\infty}^{0} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt - \int_1^{\infty} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt \right\}.$$

The first contour may be deformed downwards to $-i\infty$ and in doing so it loops round the singularity at $t = x$ on the negative imaginary axis. Hence we obtain

$$2\pi i \oint \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt - 2\pi \int_0^{\infty} \frac{(1+t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x+t)^2} e^{-i\beta t} dt + 2\pi \int_1^{\infty} \frac{(t-1)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt. \tag{A 4}$$

Combining (A 3) and (A 4) we have, for all β,

$$\int_{-\infty}^{\infty} e^{-i\beta v} 2 \int_0^1 \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{t+v} \frac{dt}{(x-t)^2} dv$$
$$= 2\pi \int_1^{\infty} \frac{(t-1)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt - 2\pi \int_0^{\infty} \frac{(1+t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x+t)^2} e^{-i\beta t} dt + H(-\beta) 2\pi i \oint \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t} dt. \tag{A 5}$$

Taking an inverse Fourier transform gives back the original integral. Hence

$$2 \int_0^1 \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{t+v} \frac{dt}{(x-t)^2} = -2\pi \frac{(1+v)^{\frac{1}{2}} v^{\frac{1}{2}}}{(x+v)^2} \{H(v) + H(-1-v)\} + \oint \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{t+v} \frac{dt}{(x-t)^2}. \tag{A 6}$$

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