Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 85 MAY 1979 PART 3

Math. Proc. Camb. Phil. Soc. (1979), 85, 387
With 5 text-figures
Printed in Great Britain

Affine cubic functions

I. The complex plane

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(*Received* 18 *July* 1978)

Although the classification of affine cubic curves was undertaken by Newton (4), in one of the first major exercises ever in coordinate geometry (see Cayley (2) for a fuller account), a parallel study of cubic functions seems not to have been contemplated till recently. The essential difference is that a function f defines a pencil of curves – its level curves – which have to be considered simultaneously. The author's interest in the subject arose from problems in singularity theory (concerning canonical stratifications), and a later paper in the series will have applications of this kind. Here we study the simplest case as an introduction. Our techniques are entirely classical, but the results are hard to find elsewhere. In later papers we intend to study cubic functions on $\mathbb{R}^2$, and on $\mathbb{C}^3$.

Enumeration of cases: conventions

We first fix notation. We study the affine function $f = f(x, y)$, throughout assumed to be polynomial, of degree $\leqslant 3$. With this we associate the homogeneous function $f^H(x, y, z) = z^3 f(x/z, y/z)$. Similarly the level curve Γ_c defined by $f(x, y) = c$ has projective completion $\overline{\Gamma}_c$ with equation $(f-c)^H = 0$. We have $\overline{\Gamma}_c \subset P_2(\mathbb{C})$, and identify $\mathbb{C}^2 \subset P_2(\mathbb{C})$ by $(x, y) \mapsto (x, y, 1)$. Its complement is the 'line at infinity' with equation $z = 0$, which we usually denote by z.

The intersection $\overline{\Gamma}_c \cap z$ is determined by the terms of degree 3 in f, so is independent of c. The condition for a point P of $\overline{\Gamma}_c \cap z$ to be singular on $\overline{\Gamma}_c$ is also independent of c. We shall frequently omit the subscript c when discussing properties common to all levels. We can thus give a preliminary classification of functions f into six divisions:

(I) z is a chord of $\overline{\Gamma}_c$. Each curve Γ_c has 3 asymptotes, and is called (by Newton) a cubical hyperbola.

(II) z is a tangent at a simple point P of $\overline{\Gamma}_c$. Γ_c has one asymptote, and a parabola meeting it 5 times at P: it is called a parabolic hyperbola.

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(III) z is an inflexional tangent at a simple point P of $\overline{\Gamma}_c$. There is a unique semi-cubical parabola meeting $\overline{\Gamma}_c$ 7 times at P. The curve Γ_c is a divergent parabola.

(IV) z meets $\overline{\Gamma}_c$ in a double point P and a further point. We have three asymptotes, of which two are parallel (and may coincide); Γ_c is a hyperbolism.

(V) z meets $\overline{\Gamma}_c$ in one point, a multiple point on $\overline{\Gamma}_c$.

(VI) z is a component of $\overline{\Gamma}_c$; equivalently, $\deg f \leqslant 2$.

The above describes the singularities of $\overline{\Gamma}_c$ (if any) at infinity; the rest depend on c. The basis of our further classification is a listing of singularities of Γ_c, for such values of c as occur. The singular points in question are just the critical points of f.

The types of critical point Q which f may possess are as follows. We can take coordinates such that Q is at the origin, $f(Q) = 0$ and Q is a singular point of f.

If the Hessian matrix of f at Q is nonsingular, Q has type A_1.

If the Hessian has rank 1, take

$$f = y^2 + ax^3 + bx^2y + cxy^2 + dy^3.$$

If $a \neq 0$, Q has type A_2;

if $a = 0$, $b \neq 0$, Q has type A_3;

and if $a = b = 0$, Q is a non-isolated singularity (type X, say).

If the Hessian is the zero matrix, f is homogeneous cubic. If f has distinct factors, Q has type D_4. Otherwise, say $f = cxy^2 + dy^3$. Q is a non-isolated singularity, type Y if $c \neq 0$, Z if $c = 0, d \neq 0$. (We exclude the case when f vanishes identically, or is constant.)

For a projective plane cubic curve, the possibilities of singular points are as follows.

For an irreducible curve: non-singular, nodal (A_1) or cuspidal (A_2). For a curve reducing to conic and line, the line may be a chord ($2A_1$) or tangent (A_3). For a curve formed of three lines we may have a triangle ($3A_1$), three concurrent lines (D_4), a repeated line with a line (the intersection has type Y, other points on the repeated line type X) or a threefold line (each point of which then has type Z).

Enumeration I. Since the roots of $f^H(x, y, 0)$ are distinct, we can change coordinates to put this binary cubic in the standard form $x^3 + y^3$. Then

$$f(x, y) = x^3 + y^3 + a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6.$$

The translation $x' = x + \frac{1}{3}a_1$, $y' = y + \frac{1}{3}a_3$ reduces a_1 and a_3 to 0. We take as our standard form

$$f(x, y) = x^3 + y^3 + 2axy + bx + cy (+ d).$$

Observe that this is the universal unfolding of the standard singularity of type D_4.

We can find singular points by solving

$$0 = f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

We obtain two conics,

$$0 = \frac{\partial f}{\partial x} = 3x^2 + 2ay + b, \tag{1·1}$$

$$0 = \frac{\partial f}{\partial y} = 3y^2 + 2ax + c, \tag{1·2}$$

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meeting in four points P_i (which may coincide, but cannot become infinite since

$$z = \frac{\partial f^H}{\partial x} = \frac{\partial f^H}{\partial y} = 0 \quad \text{imply} \quad x = y = 0),$$

and at these points, $f = 0$ if

$$0 = 3f - x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 2axy + 2bx + 2cy + 3d, \tag{1·3}$$

giving a unique value of d at each point.

In general we have 4 distinct points, and 4 distinct values of d. Two types of catastrophe may occur (separately or together):

(A) The pencil defined by (1·1) and (1·2) may be non-general and determine fewer than 4 distinct points P_i.

(B) Two or more distinct points P_i may correspond to the same value of d.

It is easy to determine when these cases arise.

(A) The pencil has discriminant

$$\begin{vmatrix} 3\lambda & 0 & a\mu \\ 0 & 3\mu & a\lambda \\ a\mu & a\lambda & b\lambda + c\mu \end{vmatrix} = 9\lambda\mu(b\lambda + c\mu) - 3a^2(\lambda^3 + \mu^3) = \delta(\lambda, \mu), \tag{1·4}$$

so the three degenerate members are roots of the equation $\delta = 0$ for $\lambda : \mu$. The pencil is non-general if and only if the roots of this are not all distinct, i.e. the discriminant of δ vanishes. Thus (A) occurs if and only if

$$0 = \Delta \equiv a^8 - 6a^4bc - 4a^2(b^3 + c^3) - 3b^2c^2. \tag{1·5}$$

(B) means that some cubic f constant, say $f = 0$, has at least two distinct singular points. This can only occur if it is reducible. Thus f has a linear factor, which may be taken as $x + \omega y + \lambda$, where $\omega^3 = 1$. Set

$$f = (x + \omega y + \lambda)(x^2 - \omega xy + \omega^2 y^2 + \alpha x + \beta y + \gamma).$$

The coefficients of x^2 and y^2 vanish: $0 = \lambda + \alpha = \lambda\omega^2 + \omega\beta$ so $\alpha = -\lambda$, $\beta = -\omega\lambda$. We then find

$$a = -\tfrac{3}{2}\omega\lambda, \quad b = \gamma - \lambda^2, \quad c = \omega(\gamma - \lambda^2), \quad d = \lambda\gamma.$$

This case occurs precisely when $c = \omega b$ with $\omega^3 = 1$, i.e. when $b^3 = c^3$.

We now analyse the cases occurring more fully. We start with (B), which we now allow to include all cases when the cubic $f = 0$ is reducible. We have the line

$$l : x + \omega y + \lambda = 0$$

and the conic

$$\Sigma : x^2 - \omega xy + \omega^2 y^2 - \lambda(x + \omega y) + \gamma = 0. \tag{1·6}$$

However, Σ degenerates to a line-pair (not a repeated line since the points of $\overline{\Sigma} \cap z$ are distinct) if

$$0 = \begin{vmatrix} 1 & -\frac{1}{2}\omega & -\frac{1}{2}\lambda \\ -\frac{1}{2}\omega & \omega^2 & -\frac{1}{2}\omega\lambda \\ -\frac{1}{2}\lambda & -\frac{1}{2}\omega\lambda & \gamma \end{vmatrix} = \tfrac{3}{4}\omega^2(\gamma - \lambda^2)$$

and l touches Σ (it cannot be a component of it since the point $\bar{l} \cap z$ does not lie on $\overline{\Sigma}$) if

$$3\omega^2 y^2 + 3\omega\lambda y + 2\lambda^2 + \gamma = 0$$

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has equal roots, i.e. $3\lambda^2 = 4(2\lambda^2+\gamma)$, $\gamma = -\frac{5}{4}\lambda^2$. Thus

if $\gamma = \lambda^2 \neq 0$ we have a triangle $(3A_1)$,

if $\gamma = -\frac{5}{4}\lambda^2 \neq 0$ a conic with tangent (A_3),

if $\gamma = \lambda = 0$ three concurrent lines (D_4);

otherwise a conic with chord $(2A_1)$.

We next determine from (1·3) all the corresponding values of d, and thus confirm the calculations above. Since

$$f = x^3+y^3-3\omega\lambda xy+(\gamma-\lambda^2)(x+\omega y)+d,$$

the conics (1·1), (1·2) reduce to

$$0 = 3x^2-3\omega\lambda y+(\gamma-\lambda^2), \tag{1·1}'$$

$$= 3y^2-3\omega\lambda x+\omega(\gamma-\lambda^2). \tag{1·2}'$$

Then $(1\cdot1)'-\omega^2(1\cdot2)'$ gives $0 = 3(x-\omega y)(x+\omega y+\lambda)$. The two points of intersection on $x+\omega y+\lambda = 0$ are the points $\Sigma \cap l$. We have f factorized as above, and $d = \gamma\lambda$. For the remaining two critical points, $x = \omega y$,

$$3y^2-3\omega^2\lambda y+\omega(\gamma-\lambda^2) = 0 \tag{1·7}$$

i.e.
$$3(y-\tfrac{1}{2}\omega^2\lambda)^2 = \tfrac{1}{4}\omega(7\lambda^2-4\gamma)$$

and d is given in terms of the roots y of (1·7) by

$$\begin{aligned} 0 &= -3\omega\lambda xy+2(\gamma-\lambda^2)(x+\omega y)+3d \\ &= -3\omega^2\lambda y^2+4\omega(\gamma-\lambda^2)y+3d \\ &= \omega(4\gamma-7\lambda^2)y+3d+\lambda(\gamma-\lambda^2) \quad (\text{add } \omega^2\lambda\times(1\cdot7)). \end{aligned}$$

These two values of d are equal if either $4\gamma = 7\lambda^2$ or the equation for y has equal roots; a condition which also reduces to $4\gamma = 7\lambda^2$. For $d = \gamma\lambda$ to appear again, we must have

$$\omega(4\gamma-7\lambda^2)y+\lambda(4\gamma-\lambda^2) = 0,$$

so
$$\begin{aligned} 0 &= 3\lambda^2(4\gamma-\lambda^2)^2+3\lambda^2(4\gamma-7\lambda^2)(4\gamma-\lambda^2)+(\gamma-\lambda^2)(4\gamma-7\lambda^2)^2 \\ &= 16\gamma^3+24\gamma^2\lambda^2-15\gamma\lambda^4-25\lambda^6 = (\gamma-\lambda^2)(4\gamma+5\lambda^2)^2 \end{aligned}$$

repeating cases previously considered.

The case $4\gamma = 7\lambda^2$ is however new: as well as $d = \gamma\lambda$ (multiplicity 2), yielding a conic + chord $(2A_1)$ we have $d = -\frac{1}{4}\lambda^3$ (multiplicity 2), yielding a cubic which must be irreducible, with a singular point of multiplicity 2, i.e. a cuspidal cubic (A_2).

We return to case (A), which breaks up naturally into two subcases

(A i) $a = 0$. The discriminant of the pencil is $9\lambda\mu(b\lambda+c\mu)$ so has a repeated factor if and only if $b = 0$ or $c = 0$ (the case when both vanish is trivial). If $a = b = 0$ but $c \neq 0$ the pencil is of type $[(1,1),1]$, defined by $3x^2$ and $3y^2+c$. There are two points P_i (each with multiplicity 2): $(0, \pm\sqrt{(-\frac{1}{3}c)}$, and the corresponding values of $d = -\frac{2}{3}cy$ are distinct. These two cubics (each irreducible) must be cuspidal.

(A ii) $a \neq 0$. If $a^2(\lambda^3+\mu^3)-3\lambda\mu(b\lambda+c\mu)$ has a repeated factor, it may be written as $f^{-2}(a\lambda-f\mu)^2(f^2\lambda+a^2\mu)$ for some $f \neq 0$. The factor has multiplicity 3 only when

$$a^3+f^3 = 0,$$

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which yields the case studied under (B) when $\gamma = -\frac{5}{4}\lambda^2$. In general, we have

$$-3f^2b = a^4 - 2af^3, \quad -3f^2c = f^4 - 2a^3f \tag{1·8}$$

and the degenerate conics of the pencil are

$$\Sigma_1 = f(3x^2 + 2ay + b) + a(3y^2 + 2ax + c), \tag{1·9}$$

$$\begin{aligned}\Sigma_2 &= a^2(3x^2 + 2ay + b) - f^2(3y^2 + 2ax + c) \\ &= \frac{1}{3f^2}(3afx + 3f^2y - a^3 - f^3)(3afx - 3f^2y + a^3 - f^3).\end{aligned} \tag{1·10}$$

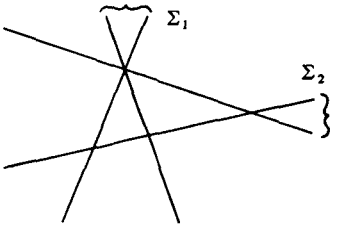


Fig. 1.

The vertex of Σ_1 is the point $(-a^2/3f, -f/3)$. This lies on the line represented by the second factor of Σ_2, but not (unless $a^3 + f^3 = 0$) on the first (Fig. 1). The pencil has just two degenerate members, and there are three distinct points P_i. These correspond to distinct values of d (by the above) unless $b^3 = c^3$ – i.e.

$$\begin{aligned}f^4 - 2a^3f &= \omega(a^4 - 2af^3) \quad \text{for some} \quad \omega \quad \text{with} \quad \omega^3 = 1, \\ 0 &= (f - \omega a)(f + \omega a)(f^2 + \omega^2a^2 + 2\omega af) \\ &= (f - \omega a)(f + \omega a)^3.\end{aligned}$$

The case $f + \omega a = 0$, hence $f^3 + a^3 = 0$, corresponds to the earlier case $\gamma = -\frac{5}{4}\lambda^2$ (as already noted);

$$\text{if} \quad f = \omega a, \quad b = \omega(a^2/3) \quad c = \omega^2(a^2/3) \quad \text{so we have} \quad \gamma = \tfrac{7}{4}\lambda^2.$$

In the remaining cases, the one value of d with multiplicity 2 corresponds to a cubic with one singular point of multiplicity 2: a cusp (A_2).

We summarize in Table 1. In the final column a solidus separates singularities occurring at different critical levels.

This list clarifies a point in a paper of Liaschko(3). This describes the strata in unfoldings of simple singularities as follows: take a diagram associated to the singularity (via intersections of a canonical basis in the Milnor fibre (1)) and a partition of the set of vertices of the diagram. Then there is a nearby function with one critical value for each part of the partition, and the singularities at that value are determined by the subgraph formed with those vertices. Liaschko also shows that for each nearby function, its singularities can be so described using some canonical diagram, and gives an algorithm which (for (D_4) yields the same list of species as Table 1. The standard graph of type D_4 admits 15 partitions, as shown in Fig. 2.

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Table 1. $f(x, y) = x^3 + y^3 + 2axy + bx + cy$

Species	Conditions	Parameters	Type of pencil	Singular levels
α	$\Delta \neq 0$, $b^3 \neq c^3$	a, b, c	$[1, 1, 1]$	$A_1/A_1/A_1/A_1$
β	$\Delta = 0$, $b^3 \neq c^3$, $a \neq 0$	a, f	$[2, 1]$	$A_2/A_1/A_1$
γ	$b^3 = c^3 \neq 0$, $-a^6, a^6/27$	λ, γ	$[1, 1, 1]$	$2A_1/A_1/A_1$
δ	$b^3 = c^3 = a^6/27 \neq 0$	$\gamma = \frac{7}{4}\lambda^2$ or $f^3 = a^3$	$[2, 1]$	$A_2/2A_1$
ϵ	$b^3 = c^3 = -a^6 \neq 0$	$\gamma = -\frac{5}{4}\lambda^2$ or $f^3 = -a^3$	$[3]$	A_3/A_1
ζ	$b = c = 0$, $a \neq 0$	$\gamma = \lambda^2$	$[1, 1, 1]$	$3A_1/A_1$
η	$a = b = 0$, $c \neq 0$	c	$[(1, 1), 1]$	A_2/A_2
	or $a = c = 0$, $b \neq 0$	b		
θ	$a = b = c = 0$		$[1, 1;;1]$	D_4

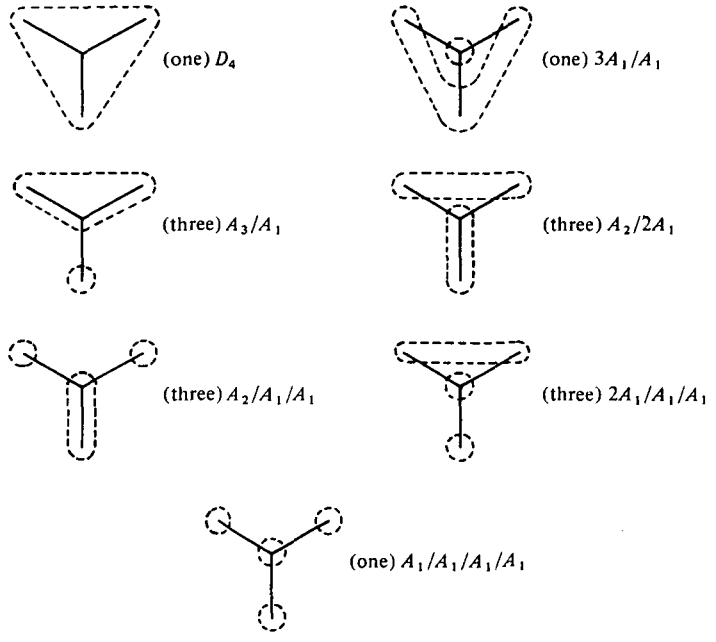


Fig. 2.

These are thus none corresponding to A_2/A_2, which can only be obtained from an alternative diagram, Fig. 3. It seems that there are phenomena here yet to be adequately explained: the above correctly predicts the 3 strata ($b = c$, $b = \omega c$, $b = \omega^2 c$) obtained for each of the species γ, δ, ϵ and the total number 15 of strata (we have only one of species γ, not three, but there are two of species η). Further, η is the only stratum

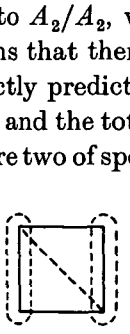


Fig. 3.

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which does not appear in the real unfolding of an elliptic umbilic (they all occur for a hyperbolic umbilic). These facts appear not to be accidental, but are not explained by present theory.

Enumeration II. Take the third degree terms of f as x^2y; then translate the coordinates to eliminate the terms in xy and x^2. Now

$$f(x,y) = x^2y + ay^2 + bx + cy + d$$

and since $(0,1,0)$ is a simple point, $a \neq 0$. We can replace a by 1 (multiply x, y by suitable constants.) We have the universal unfolding of a singular point of type A_3 (which occurs when $b = c = d = 0$).

If f is reducible, $f = l(x,y)\,m(x,y)$ with $\deg l = 1$, $\deg m = 2$. Then

$$x^2y = l^H(x,y,0)\,q^H(x,y,0),$$

so $l^H(x,y,0)$ can be taken as x or y: $l = x+k$ or $y+k$. The former is impossible (the second degree terms in q would be xy, and then we get a zero coefficient of y^2). In the latter case,

$$m(x,y) = x^2 + px + qy + r;$$

comparing coefficients of x^2 shows $k = 0$ and then setting $y = 0$ we find $b = d = 0$, so

$$f = y(x^2 + y + c).$$

This gives a conic and chord $(2A_1)$ if $c \neq 0$, a conic and tangent (A_3) if $c = 0$.

Otherwise, f is irreducible. The critical points are given by

$$0 = \frac{\partial f}{\partial x} = 2xy + b$$

$$= \frac{\partial f}{\partial y} = x^2 + 2y + c.$$

Eliminating y, $x^3 + cx - b = 0$. If this has distinct roots, we have three critical points which are distinct, non-degenerate, and (unless $b = 0$) at distinct critical values. As the cubic has discriminant $4c^3 + 27b^2$, we obtain the possibilities shown in Table 2.

Table 2. $f = x^2y + y^2 + bx + cy$

Species	Conditions	Singular levels
α	$b \neq 0, 4c^3 + 27b^2 \neq 0$	$A_1/A_1/A_1$
β	$b \neq 0, 4c^3 + 27b^2 = 0$	A_2/A_1
γ	$b = 0, \quad c \neq 0$	$2A_1/A_1$
δ	$b = c = 0$	A_3

Enumeration III. We can start by taking

$$f(x,y) = x^3 + a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6.$$

Since $(0,1,0)$ is simple on $\overline{\Gamma}$, $a_3 \neq 0$; replacing $y\sqrt{a_3}$ by y, we take $a_3 = 1$. Now write $y' = y + \frac{1}{2}(a_2x + a_5)$ to reduce a_2 and a_5 to zero, and $x' = x + \frac{1}{3}a_1$ to reduce a_1 to zero. So we have the form

$$f(x,y) = x^3 + y^2 + ax + b,$$

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which is the universal unfolding of a singularity of type A_2. All level curves are irreducible, since f^H is in Weierstrass normal form; we have a node (or cusp) if x^3+ax+b has a repeated (or threefold) root.

Table 3. $f = x^3+y^2+ax$

Species	Conditions	Singular levels
α	$a \neq 0$	A_1/A_1
β	$a = 0$	A_2

Enumeration IV. We may start as in II, but here $a = 0$ so

$$f(x,y) = x^2y+bx+cy+d.$$

Setting $c = -k^2$, we have

$$f = y(x^2-k^2)+bx+d,$$

which factorizes if $d = \pm bk$. As critical points satisfy

$$0 = \frac{\partial f}{\partial y} = x^2-k^2,$$

all lie on these reducible curves. We have four cases, as shown in Table 4.

Table 4. $f(x,y) = y(x^2-k^2)+bx$

Species	Conditions	Singularities of $\overline{\Gamma}$: General level	Singularities of $\overline{\Gamma}$: Special levels
α	$k \neq 0, b \neq 0$	A_1	$2A_1/2A_1$
β	$k = 0, b \neq 0$	A_2	A_3
γ	$k \neq 0, b = 0$	A_1	$3A_1$
δ	$k = 0, b = 0$	A_2	XY

In this table, as in the next two, the singularities of $\overline{\Gamma}_c$ all occur at infinity and are independent of c except for a finite number of special values of c.

Table 5.

Species	Normal form	Singularities of $\overline{\Gamma}$: General level	Singularities of $\overline{\Gamma}$: Special levels
α	x^3+xy	A_1	$2A_1$
β	x^3+y	A_2	—
γ	x^3+x	D_4	XY/XY
δ	x^3	D_4	Z

Enumeration V. We start as in III, but here $a_3 = 0$. First, suppose $a_2 \neq 0$: then we can reduce a_2 to 1. Setting $x' = x+b_1$, $y' = y+b_2x+b_3$ choose (in turn) b_1 to reduce a_5 to 0, b_2 to reduce a_1 to 0 and b_3 to reduce a_4 to 0. Thus

$$f(x,y) = x^3+xy+\text{constant}.$$

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If $a_2 = 0$, we can again reduce a_1 to 0 and have $f = x^3 + l$ with $\deg l \leqslant 1$. Unless l depends only on x, we can take $y' = l$.

The last two cases above effectively give the enumeration of affine cubic functions on $\mathbb{C}^1$. In the final table, we have the standard classification of affine functions of degree $\leqslant 2$ on $\mathbb{C}^2$.

Table 6.

Species	Normal form	Singularities of $\overline{\Gamma}$: General level	Singularities of $\overline{\Gamma}$: Special levels
α	xy	$2A_1$	$3A_1$
β	x^2+y	A_3	—
γ	x^2	D_4	XY
δ	x	XY	—

Geometric characterizations

The obvious classification of affine curves is by equisingularity of the pair $(\overline{\Gamma}, z)$. This, refined by consideration of symmetries and of inflexions, is the basis of Newton's and most subsequent classifications.

For $\overline{\Gamma}$ non-singular, we have division I if z is a chord, II if z is a tangent and III if z an inflexional tangent. For $\overline{\Gamma}$ nodal or cuspidal, we have the same list if z does not pass through the double point; for a nodal chord respectively tangent we have species IVα or IVγ respectively Vα and for a cuspidal chord respectively tangent we have species IVβ or IVδ respectively Vβ. Cases can be similarly enumerated for $\overline{\Gamma}$ reducible, but we see clearly that the classification of this paper involves more subtle geometry. This further geometrical study will be essential for the study of cubic functions on $\mathbb{C}^3$, where an enumeration on the above lines is not practicable.

We concentrate on division I. The non-general cases were those where either

(A) The pencil defined by (1·1), (1·2) is non-general, or

(B) Two or more critical points occur at the same critical level.

Now the pencil consists of the curves $0 = \lambda\,\partial f/\partial x + \mu\,\partial f/\partial y$, whose completions are just the polar curves of points $(\lambda, \mu, 0)$ at infinity with respect to $\overline{\Gamma}$. Now the Hessian curve $H = H(\overline{\Gamma})$ is defined as the locus of points whose polars with respect to $\overline{\Gamma}$ are singular. Thus the singular conics of the pencil correspond to the three points of intersection $z \cap H(\overline{\Gamma})$. The pencil is non-general if and only if there are fewer than three distinct singular members, i.e. intersection points, i.e. if and only if z touches $H(\overline{\Gamma})$.

As to possibility (B), this was characterized by $c = \omega b$ (for some ω with $\omega^3 = 1$) in the normal form. Now the tangent to $\overline{\Gamma}$ at $(1, -\omega^2, 0)$ is

$$3x + 3\omega y = 2a\omega^2$$

and meets the curve again where

$$\begin{aligned} 0 &= (\tfrac{2}{3}a\omega^2 - \omega y)^3 + y^3 + 2ay(\tfrac{2}{3}a\omega^2 - \omega y) + b(\tfrac{2}{3}a\omega^2 - \omega y) + cy + d \\ &= (c - \omega b)\,y + (d + \tfrac{2}{3}\omega^2 ab + \tfrac{8}{27}a^3). \end{aligned}$$

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So the tangent is inflexional precisely when $c = \omega b$. This possibility occurs when z passes through a point of inflexion of $\bar{\Gamma}$. Note that these are precisely the points of intersection $\bar{\Gamma} \cap H(\bar{\Gamma})$, also they are the points of inflexion of $H(\bar{\Gamma})$.

It follows that the species in division I can nearly all be discriminated by looking at the configuration of $\bar{\Gamma}_c \cap z$ and $H(\bar{\Gamma}_c) \cap z$ (both of which are independent of c); and the same goes for divisions II and III. Considering a non-singular curve $\bar{\Gamma} = \bar{\Gamma}_c$, the conditions are as follows.

First suppose z a chord of $\bar{\Gamma}$.

If z passes through 0 inflexion points of $\bar{\Gamma}$, it may be a chord of $H(\bar{\Gamma})$ (species $\mathrm{I}\alpha$) or tangent (species $\mathrm{I}\beta$). If z passes through 1 inflexion point P of $\bar{\Gamma}$, it may be a chord of $H(\bar{\Gamma})$ (species $\mathrm{I}\gamma$), tangent to $H(\bar{\Gamma})$ at another point (species $\mathrm{I}\delta$), or inflexional tangent at P (species $\mathrm{I}\epsilon$). If z passes through two inflexion points, it passes through 3 (species $\mathrm{I}\zeta$).

Further cases arise if $\bar{\Gamma}$ is equianharmonic, when $H(\bar{\Gamma})$ degenerates into a triangle. If z passes through a vertex, we have species $\mathrm{I}\eta$; if z is an edge, species $\mathrm{I}\theta$.

Next suppose z an ordinary tangent to $\bar{\Gamma}$. If z passes through 0 inflexions, then z may be a chord of $H(\bar{\Gamma})$ (species $\mathrm{II}\alpha$) or a tangent (species $\mathrm{II}\beta$). If z goes through 1 inflexion (it cannot pass through two), we have species $\mathrm{II}\gamma$ if z is a chord of $H(\bar{\Gamma})$; if z is a tangent, then $\bar{\Gamma}$ is harmonic and z an inflexional tangent to $H(\bar{\Gamma})$ (species $\mathrm{II}\delta$).

If z is an inflexional tangent to $\bar{\Gamma}$, then it is a chord of $H(\bar{\Gamma})$ and we have species $\mathrm{III}\alpha$ unless $\bar{\Gamma}$ is equianharmonic, when z passes through a node of $H(\bar{\Gamma})$ and we have species $\mathrm{III}\beta$.

These assertions may easily be verified by simple calculations. Note the behaviour of the modular invariant j as $\bar{\Gamma}$ varies in the pencil $\bar{\Gamma}_c$. If the pencil contains non-singular cubics and one of type A_1, $2A_1$ or $3A_1$ (for which $j = \infty$ is well defined), then $j(\bar{\Gamma}_c)$ takes at least two distinct values, and hence almost all values. The remaining cases are (from enumeration)

$$\begin{aligned} \mathrm{I}\eta,\ \mathrm{I}\theta,\ \mathrm{III}\beta \quad & j = 0 \quad \text{for almost all } c, \\ \mathrm{II}\delta \quad & j = 1 \quad \text{for almost all } c, \\ \mathrm{IV}\alpha,\ \mathrm{IV}\gamma,\ \mathrm{V}\alpha \quad & j = \infty \quad \text{for almost all } c, \end{aligned}$$

and for further species in divisions IV, V, VI j is everywhere undefined.

The above characterizations cover the cases where $\bar{\Gamma}$ is nonsingular. In view of the importance of $H(\bar{\Gamma})$, we recall its structure for singular $\bar{\Gamma}$

$\bar{\Gamma}$	A_1	$2A_1$	$3A_1$	$A_2(y^2z = x^3)$	$A_3(xy^2 - x^2z)$	other
$H(\bar{\Gamma})$	A_1	$2A_1$	$3A_1$	$XY(xy^2)$	$Z(x^3)$	$H \equiv 0$

The nodes of H coincide with those of $\bar{\Gamma}$ (also nodal tangents).

We can now extend the characterizations to singular curves $\bar{\Gamma}$. The assertions are mostly clear by inspection, but a few cases need calculations to verify them.

$\bar{\Gamma}$ nodal. First suppose z a chord not through the node. If z passes through 0 inflexions we have $\mathrm{I}\alpha$ in general, but $\mathrm{I}\beta$ if z touches H. If z contains 1 inflexion we have $\mathrm{I}\gamma$ in general, $\mathrm{I}\epsilon$ if it is tangent (necessarily inflexional) to H. If z contains 3 inflexions, we have $\mathrm{I}\zeta$.

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Now suppose z a tangent (not through the node). We have $\mathrm{III}\,\alpha$ if the tangent is inflexional, $\mathrm{II}\,\gamma$ if it passes through an inflexion, $\mathrm{II}\,\beta$ if it touches H, otherwise $\mathrm{II}\,\alpha$.

If z passes through the node, we have $\mathrm{IV}\,\gamma$ if it goes through an inflexion, $\mathrm{V}\,\alpha$ for a nodal tangent and $\mathrm{IV}\,\alpha$ otherwise.

$\bar{\Gamma}$ *cuspidal.* If z is a chord not through the cusp we have $\mathrm{I}\,\delta$ if z passes through the inflexion, $\mathrm{I}\,\eta$ if the three intersections from an equianharmonic range with the cusp, and $\mathrm{I}\,\beta$ otherwise. For a general tangent we have $\mathrm{II}\,\beta$, for an inflexional tangent $\mathrm{III}\,\beta$.

If z passes through the cusp, we have $\mathrm{IV}\,\delta$ if it goes through the inflexion, $\mathrm{V}\,\beta$ for the cuspidal tangent and $\mathrm{IV}\,\beta$ otherwise.

$\bar{\Gamma}$ *conic and chord.* If z is a chord through neither intersection, $\mathrm{I}\,\gamma$ unless z touches H (if and only if the cross-ratio λ of intersections of the conic with the chord and with z satisfies $\lambda+\lambda^{-1}=-14$), when $\mathrm{I}\,\delta$. If z a general tangent, $\mathrm{II}\,\gamma$.

For z a chord through an intersection, $\mathrm{IV}\,\alpha$, for a tangent there, $\mathrm{V}\,\alpha$, and for z coinciding with the given chord, $\mathrm{VI}\,\alpha$.

$\bar{\Gamma}$ *conic and tangent.* If z does not pass through the intersection, we have $\mathrm{I}\,\epsilon$ for z a chord, $\mathrm{II}\,\delta$ for z a tangent. If it does pass through it, we have $\mathrm{IV}\,\beta$ for z a chord, $\mathrm{VI}\,\beta$ for z the given tangent.

$\bar{\Gamma}$ *a triangle* If z passes through no vertex, $\mathrm{I}\,\zeta$; through one vertex, $\mathrm{IV}\,\gamma$; if z is an edge of the triangle, $\mathrm{VI}\,\alpha$.

$\bar{\Gamma}$ *three concurrent lines.* If z does not pass through the vertex, $\mathrm{I}\,\theta$. If it does, then we have $\mathrm{V}\,\gamma$ in general, but $\mathrm{V}\,\delta$ if z is one of the Hessian lines of $\bar{\Gamma}$ (considered as binary cubic) and $\mathrm{VI}\,\gamma$ if z is one of the lines of $\bar{\Gamma}$.

$\bar{\Gamma}$ *of type* x^2y. If z does not pass through the intersection, $\mathrm{IV}\,\delta$. If it does, we have $\mathrm{V}\,\gamma$ except when $z=y$ $\mathrm{VI}\,\gamma$ or $z=x$ $\mathrm{VI}\,\delta$.

$\bar{\Gamma}$ *of type* x^3. Then $z\neq x$ (by non-triviality assumption) and we have species $\mathrm{V}\,\delta$.

This concludes our geometric characterizations. It would be natural to proceed to obtain a list of specializations between the species. Many such are evident from the above characterizations, but the list is long and it would be a tedious matter to ensure its completeness.

Symmetries and moduli

The coordinate changes which preserve $z=0$ are those belonging to the affine group $\mathrm{Aff}_2(\mathbb{C})$

$$\left.\begin{aligned} Ax &= ax+by+c \\ Ay &= a'x+b'y+c' \end{aligned}\right\} ab'\neq ba'.$$

However, since we are mainly concerned with the level curves of f, and each function $\alpha f+\beta$ $(\alpha\neq 0)$ has the same levels, we introduce also the affine group $\mathrm{Aff}_1(\mathbb{C})$. We make the product $G=\mathrm{Aff}_2(\mathbb{C})\times\mathrm{Aff}_1(\mathbb{C})$ act on functions by

$$f'(x,y)=\alpha^{-1}(f(Ax,Ay)-\beta).$$

We write G_0 for the normal subgroup $\alpha=1$ of G, with quotient $\mathbb{C}^\times$, and

$$G_1=\mathrm{Aff}_2(\mathbb{C})\times\{1\},$$

so that $G_1\lhd G_0$ with quotient $\mathbb{C}^+$.

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Since f is assumed non-constant, $\alpha f+\beta = f$ implies $\alpha = 1$, $\beta = 0$. Thus the isotropy group G_f of f does not meet $\mathrm{Aff}_1(\mathbb{C})$ non-trivially, and projects injectively into $\mathrm{Aff}_2(\mathbb{C})$. We have a homomorphism $\alpha_f: G_f \to \mathbb{C}^\times$ with kernel G_{0f}, and a homomorphism

$$\beta_f: G_{0f} \to \mathbb{C}^+$$

with kernel G_{1f}. If β_f is surjective, we have symmetries interchanging all the different levels of f, so there are no special levels. We now turn to our enumeration.

f has no special levels: species $\mathrm{V}\beta$, $\mathrm{VI}\beta$ and $\mathrm{VI}\delta$.

In these cases, coordinates can be chosen so that f takes the respective forms

$$x^3+y, x^2+y, y$$

so in each of them,

$$f(x, y+1) = f(x, y)+1,$$

and β_f is indeed surjective. By further inspection, we see that these symmetry groups are as follows:

$\mathrm{V}\beta$	$x' = ax,$	$y' = a^3y+b,$	$\alpha = a^3,\ \beta = b$	$(a \neq 0)$
$\mathrm{VI}\beta$	$x' = ax+b,$	$y' = -2abx+a^2y+c,$	$\alpha = a^2,\ \beta = b^2+c$	$(a \neq 0)$
$\mathrm{VI}\delta$	$x' = ax+by+c,$	$y' = dy+e,$	$\alpha = d,\ \beta = e$	$(a \neq 0,\ d \neq 0).$

In each case, G_f projects onto $\mathrm{Aff}_1(\mathbb{C})$.

In general, the projection of G_f is an algebraic subgroup S of $\mathrm{Aff}_1(\mathbb{C})$. Since $\mathbb{C}^+$ has no proper algebraic subgroup, if S does not contain all the translations (case β_f surjective), it contains none. In this case, S has a fixed point in $\mathbb{C}$. (This standard result may be proved by taking an invariant mean $\overline{w}$ of the translates of any $w \in \mathbb{C}$ by a maximal compact subgroup K in S: as K is Zariski-dense, $\overline{w}$ is then fixed by S). Adding a suitable constant to f, we can conjugate G_f so that this fixed point is zero. We then have $\beta = 0$ for all $(A, \alpha, \beta) \in G_f$. All elements of G_f induce symmetries of the level surface $f(x, y) = 0$, but each other level surface only has the symmetries of G_{0f}.

f has just one special level: species $\mathrm{I}\theta$, $\mathrm{II}\delta$, $\mathrm{III}\beta$, $\mathrm{IV}\beta$, $\mathrm{IV}\gamma$, $\mathrm{IV}\delta$, $\mathrm{V}\alpha$, $\mathrm{V}\delta$, $\mathrm{VI}\alpha$, $\mathrm{VI}\gamma$.

We have corresponding normal forms

$$x^3+y^3, \quad x^2y+y^2, \quad x^3+y^2, \quad x^2y+x, \quad x^2y+y,$$
$$x^2y, \quad x^3+xy, \quad x^3, \quad xy, \quad x^2.$$

Each of these has type x^py^q $(p \neq 0)$ or $x^py^q+x^ry^s$ $(ps \neq qr)$ and the symmetry

$$(x, y) \to (ax, y) \quad \text{or} \quad (a^{s-q}x, a^{p-r}y)$$

shows that in each case α_f is surjective. So in these cases $S \cong \mathbb{C}^\times$ and all the non-zero levels of f are equivalent under $\mathrm{Aff}_2(\mathbb{C})$.

In other cases, S permutes the non-zero special levels of f, so must be finite.

f has two special levels of the same type: species $\mathrm{I}\eta$, $\mathrm{III}\alpha$, $\mathrm{V}\gamma$, $\mathrm{IV}\alpha$.

We have corresponding normal forms

$$x^3-3x+y^3, \quad x^3-3x+y^2, \quad x^3-3x, \quad x^2y+x-y.$$

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In the first three, we have exceptional levels ± 2 (since $x^3 - 3x \mp 2 = (x \pm 1)^2(x \mp 2)$) which are interchanged by $x' = -x$, $y' = -y$ respectively iy respectively y. In the last, the exceptional levels ± 1 are interchanged by $x' = -x$, $y' = -y$. Since an element of $\mathrm{Aff}_1(\mathbb{C})$ with two fixed points is trivial, we see that in these cases S has order 2.

f has two special levels of different types: species $\mathrm{I}\delta$, $\mathrm{I}\epsilon$, $\mathrm{I}\zeta$, $\mathrm{II}\beta$, $\mathrm{II}\gamma$. Here the elements of S must preserve both levels, so S is trivial.

We can take as normal forms

$$x^3+y^3+6xy+3x+3y, \quad x^3+y^3+2xy-x-y$$
$$x^3+y^3+xy$$
$$x^2y+y^2+2x-3y, \quad x^2y+y^2-y.$$

f has more than two special levels: species $\mathrm{I}\alpha$, $\mathrm{I}\beta$, $\mathrm{I}\gamma$, $\mathrm{II}\alpha$.

The remaining species are just those where any normal form for f will involve a parameter. In general, G_f may depend on this parameter.

We pause here in our discussion of symmetries to count the moduli. The space of functions f is a 10 dimensional vector space F, and the group G has dimension 8. For f in a species defining an n-dimensional stratum S in F, one would expect in general that if $n \geqslant 8$, $\dim G_f = 0$; it takes $(n-8)$ moduli to describe f or a special level curve and $(n-7)$ for a general level curve.

If $n \leqslant 7$, $\dim G_f = 8-f$, $\dim G_{0f} = 7-f$. S is a single orbit under G, and the level curves fall into finitely many orbits under $\mathrm{Aff}_2(\mathbb{C})$.

n	10	9	8
Species	$\mathrm{I}\alpha$	$\mathrm{I}\beta$, $\mathrm{I}\gamma$, $\mathrm{II}\alpha$	$\mathrm{I}\delta$, $\mathrm{I}\epsilon$, $\mathrm{I}\zeta$, $\mathrm{I}\eta$, $\mathrm{II}\beta$, $\mathrm{II}\gamma$, $\mathrm{III}\alpha$, $\mathrm{IV}\alpha$

n	7	6	5	4	3
Species	$\mathrm{I}\theta$, $\mathrm{II}\delta$, $\mathrm{III}\beta$, $\mathrm{IV}\beta$, $\mathrm{IV}\gamma$, $\mathrm{V}\alpha$	$\mathrm{IV}\delta$, $\mathrm{V}\beta$, $\mathrm{VI}\alpha$	$\mathrm{V}\gamma$, $\mathrm{VI}\beta$	$\mathrm{V}\delta$, $\mathrm{VI}\gamma$	$\mathrm{VI}\delta$

Fig. 4.

In fact the cases occurring are as shown in Fig. 4. Thus the only case where our expectations fail is $\mathrm{V}\gamma$, where a modulus is needed to describe the level curves. Each level is a set of three parallel lines; a suitable parameter is the ratio of the intercepts. If P, Q, R have coordinates p, q, r on the line, take $\lambda = (r-q)/(q-p)$ or a symmetrized version thereof (see Fig. 5).

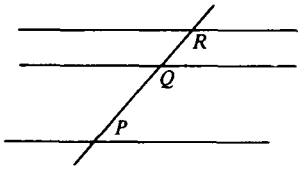


Fig. 5.

We now describe G_{1f} in more detail. Divisions V and VI are relatively uninteresting. In $\mathrm{V}\gamma$, $\mathrm{V}\delta$, $\mathrm{VI}\gamma$, $\mathrm{VI}\delta$ f does not involve y; $\mathrm{V}\beta$ and $\mathrm{VI}\beta$ were discussed above; for $\mathrm{VI}\alpha$ we have $(x', y') = (ax, a^{-1}y)$ or $(ay, a^{-1}x)$.

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For divisions III, V, $f-x^3$ is a sum of terms of degree $\leqslant 2$ not involving x^2. Hence we must have $x' = \omega x$ with $\omega^3 = 1$. For Vα (resp. Vβ), since G_{1f} leaves x^3 fixed, it fixes $f-x^3 = xy$ (resp. y) so we have $y' = \omega^{-1}y$ (resp. y): in each case, G_{1f} has order 3. For III, $f = x^3+y^2+$ linear terms, so $y' = \pm y + a$; as the linear terms do not involve y, $a = 0$. We now see that for IIIα, $\omega = 1$ so $|G_{1f}| = 2$; for IIIβ, G_{1f} is cyclic of order 6.

For divisions II, IV, $f = x^2y +$ lower terms, so we must have $x' = ux+v$,

$$y' = u^{-2}y + w.$$

Due to the absence of terms in xy respectively y^2, we have $v = 0$ respectively $w = 0$. The possible values for u depend on the case:

$$\begin{array}{cccccccc} \text{IV}\alpha & \{1\} & \text{IV}\beta & \{1\} & \text{IV}\gamma & \{\pm 1\} & \text{IV}\delta & \mathbb{C}^\times \\ \text{II}\alpha & \{1\} & \text{II}\beta & \{1\} & \text{II}\gamma & \{\pm 1\} & \text{II}\delta & \{\pm 1, \pm i\}. \end{array}$$

The same argument applied to G_f shows that we must have

$$x' = \lambda x, \quad y' = \mu y$$

where (for division II) $\mu = \lambda^2$ and $\alpha = \lambda^4$. Since

$$f = x^2y + y^2 + bx + cy$$

we have also $\alpha = \lambda$ if $b \neq 0$, $\alpha = \mu = \lambda^2$ if $c \neq 0$. So for IIδ, λ is arbitrary; for IIγ, $\lambda = \{\pm 1\}$; for IIβ, $\lambda = 1$ (confirming the above) and for IIα $\lambda = 1$ if $c \neq 0$ but if $c = 0$, $\lambda \in \{1, \omega, \omega^2\}$ and we have symmetries permuting the three critical levels.

It remains to discuss automorphisms in division I, where

$$f = x^3 + y^3 + 2axy + bx + cy. \quad (*)$$

Any automorphism leaves invariant the 3 points at infinity on $f = 0$, hence their Hessian points $xy = 0$, so is of the form

$$\begin{array}{lll} \text{(A)} & x' = px+q & \text{or} \quad \text{(B)} \quad x' = py+q \\ & y' = ry+s & \qquad\quad y' = rx+s. \end{array}$$

Checking coefficients of x^2 and y^2, we see that $q = s = 0$ in either case. Since

$$\alpha = p^3 = r^3,$$

we can set $r = p\omega$ with $\omega^3 = 1$. The conditions for these to yield symmetries are

$$\begin{array}{lll} \text{(A)} & a(p-\omega) = 0, & b(p^2-1) = 0, \quad c(p^2-\omega) = 0, \\ \text{(B)} & a(p-\omega) = 0, & b = p^2c, \qquad\quad c\omega = p^2b. \end{array}$$

Now if $a \neq 0$, $p = \omega$ so $\alpha = p^3 = 1$: thus $G_f = G_{0f}$. If $a = 0$ but b or c is non-zero then in case (A) p^2 is a cube root of unity, so $\alpha^2 = p^6 = 1$; in case (B), b and c must both be nonzero, so $\omega = p^4$ and $\alpha^4 = 1$. In fact, $|G_f : G_{0f}| = 4$ is achieved only here, and the curve is easily reduced to the form

$$f = x^3 + y^3 + 3(x-y), \quad \text{which is of type I}\alpha.$$

Otherwise, $G_f/G_{0f} \cong \mathbb{C}^\times$ for type Iθ (already noted), and for all other cases where $a = 0$, $|G_f : G_{0f}| = 2$ (take $x' = -x$, $y' = -y$).

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As to G_{0f}, we now have the group of 18 operations

$$(x' = px, \quad y' = ry \quad \text{or} \quad x' = py, \quad y' = rx \quad \text{with} \quad p^3 = r^3 = 1),$$

which can be considered as acting on the set of normal forms (*) by

$$(a, b, c) \mapsto (apr, bp, cr) \quad \text{(A)}$$
$$\text{or } (apr, cr, bp) \quad \text{(B)}.$$

This action is a good example of the general theory of Slodowy (5). The stabilizers are as follows:

Order			Species
18	(whole group)	$a = b = c = 0$	$\mathrm{I}\theta$
6	($pr = 1$, non-abelian)	$a \neq 0, b = c = 0$	$\mathrm{I}\zeta$
3	(type (A) only, $r = 1$)	$a = 0, b = 0, c \neq 0$	$\mathrm{I}\eta$
2	(just one of type (B))	$b^3 = c^3 \neq 0$	$\mathrm{I}\gamma$, $\mathrm{I}\delta$, $\mathrm{I}\epsilon$
1		otherwise	$\mathrm{I}\alpha$, $\mathrm{I}\beta$.

Thus G_{0f} depends only on the species of f – though G_f does not.

In fact $a = 0$ for all f of species $\mathrm{I}\eta$, $\mathrm{I}\theta$; none of $\mathrm{I}\beta$, $\mathrm{I}\delta$, $\mathrm{I}\epsilon$, $\mathrm{I}\zeta$ and a proper subset of $\mathrm{I}\alpha$, $\mathrm{I}\zeta$. Geometrically $a = 0$ is the condition for the asymptotes to concur, i.e. for z to touch the Cayleyan of $f = 0$. The extra element of S in this case in species $\mathrm{I}\alpha$ must interchange two pairs of critical levels; in $\mathrm{I}\gamma$ it fixes the $2A_1$ level and interchanges the A_1 levels. For $f = x^3 + y^3 + 3(x - y)$, the critical levels are $2(\pm i, \pm 1)$, so are permuted cyclically on multiplication by $i (x' = iy, y' = ix)$.

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