1

Physical Foundations

1.1 Introduction

This chapter presents an overview of the foundations of classical electromagnetism through descriptions of the fundamental quantities encountered in the subject, and the relationships between them embodied in the equations of motion of the electromagnetic fields and the relevant matter fields. The goal is to lay out the whole structure of electromagnetism in one place with a minimum of mathematical manipulation. In this book we will largely focus on nonrelativistic and nonquantum aspects of electromagnetism, occasionally pointing out workarounds that make the subject intellectually profitable and of practical utility. This approach certainly has shortcomings in that it does not reveal the deeper relativistic and quantum mechanical structure of the electromagnetism or the origin of some of the workarounds we employ. Nevertheless, the material presented here will be an excellent springboard from which relativistic and quantum versions of the subject can be developed while retaining important intuition and insights provided in this book.

Maxwell's equations are formulated in the language of differential vector calculus using compact symbolic notation introduced by Gibbs and Heaviside around 1900. The notation identifies mathematical objects and operations that have important connections with physics and geometry in three spatial dimensions. Elaboration on the mathematical language and methods of the vector calculus are presented in Chapter 2.

1.2 Excitations and Interactions

The electromagnetic field describes a temporally and spatially dependent excitation in space-time that can store and transport energy, forces, and information over both large and small distances. The electromagnetic field only makes its presence

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known by influencing the motion of charged matter. Part of the symmetry of nature is that charged matter is responsible, in turn, for creating electromagnetic fields. Charged matter thus plays a dual role in the structure of electromagnetism. When creating or exciting electromagnetic fields, charged matter is called a source. In this role it can transfer properties we commonly associate with matter, such as energy and momentum, to the electromagnetic field. Additionally, charged matter can receive energy and momentum from electromagnetic fields. We describe this dual role by saying the charged matter and electromagnetic fields are coupled or equivalently interacting with each other and, consequently, couple charged matter to itself via the electromagnetic interaction.

Sometimes the interactions induced are so strong and nearly instantaneous that the motions induced may be described by an instantaneous action at a distance. The electromagnetic fields in this case are essentially bound to the sources and move around with them. We discuss this case, called the near-field approximation, in Chapter 5. By contrast, for charged matter interacting over large distances, the interaction can be observed to be significantly delayed. Empty space then more clearly supports electromagnetic fields (radiation fields) that can decouple from their sources and propagate at the speed of light. This topic, the far-field approximation, will be covered in Chapter 6.

A full mathematical description of the evolving state of interacting charged matter via the electromagnetic field can be formulated in terms of two sets of coupled differential equations of motion: one set that describes the time and spatial evolution of the electromagnetic field called Maxwell's equations and a second set that describes the mechanical motion of the matter incorporating the forces due to the electromagnetic fields. Each set of dynamical equations requires information from the other, making the whole set a self-consistent unity that describes an important part of nature.

1.3 Maxwell's Field Equations

The dynamical equations of the electromagnetic fields driven by charged matter sources can be cast in many equivalent forms. The most common are Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{r}, t) &= \frac{\rho(\mathbf{r}, t)}{\epsilon_0}, \qquad & \nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t); \\ \nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0, & \nabla \times \mathbf{B}(\mathbf{r}, t) &= \mu_0 \mathbf{J}(\mathbf{r}, t) + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r}, t). \end{aligned} \tag{1.1}$$

These partial differential equations contain charged matter sources, represented by charge density ρ and charge current density $\mathbf{J}$, vector electric and magnetic

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fields $\mathbf{E}$ and $\mathbf{B}$, the free-space electric permittivity ϵ_0, and free-space magnetic permeability μ_0, a linear spatial vector differential operator ∇, and partial derivatives with respect to time. The equations describe both the excitation and propagation of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ through space and time, induced by sources governed by their own equation of motion.

The individual equations that make up Maxwell's equations have names given for historical reasons: the first for $\nabla \cdot \mathbf{E}$ is called Gauss' law, the second for $\nabla \times \mathbf{E}$ is called Faraday's law, and the fourth for $\nabla \times \mathbf{B}$ is called the Ampère–Maxwell law. It is convenient to have descriptive names for these equations as well. Only the first and the fourth equation directly involve the sources. We might therefore call the first the charge law and the fourth the current law. The third equation for $\nabla \cdot \mathbf{B}$ should probably be called the solenoidal law for reasons that will be made clear in Chapter 2, but it is also called the no-magnetic-monopole law. Often, simply saying or writing $\nabla \cdot \mathbf{B} = 0$ is fastest.

The sources' classical equations of motion depend on $\mathbf{E}$ and $\mathbf{B}$ and thus require integrating Maxwell's equations over time and three-dimensional space. This involves a working familiarity with the calculus and vector calculus, which is reviewed in Chapter 2.

Due to the linearity of the differential operators in Maxwell's equations, the classical fields obey the superposition principle. This means that if the sources are partitioned into additive components, the electromagnetic fields due to each component may be calculated in isolation and then summed to find the total. The resulting summed electromagnetic fields can either reinforce one another or cancel out one another at different points in space and time. This phenomenon is called interference. As a consequence, a particular source component can either increase or decrease the total electromagnetic field at a particular position and instant of time.

As mentioned earlier, the reason the $\mathbf{E}$ and $\mathbf{B}$ fields are important to us is that they play a simple and direct role in determining the dynamical evolution of ρ and $\mathbf{J}$. We will generally focus on nonrelativistic classical motion in this book, until we reach Chapter 15. The electromagnetic fields are then coupled to motional degrees of freedom of ρ and $\mathbf{J}$ through an electromagnetic force density,

$$\mathbf{f}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t). \tag{1.2}$$

Detailed discussions of the charge and current densities will follow in Sections 1.4–1.6, during which the SI units for all quantities in Equation (1.2) will be introduced and appear natural. Important things to keep in mind here are as follows:

1. There are two coupled systems of differential equations, one for the electromagnetic field and one for the charged matter motional degrees of freedom which contains $\mathbf{f}(\mathbf{r}, t)$. Both sets contain the electromagnetic fields and charge

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and current densities. The material equation of motion will also contain inertial forces and perhaps frictional or contact forces as well. Both sets of dynamical equations must be integrated self-consistently to find a complete description of the system.

2. ρ and $\mathbf{J}$ generally serve as both sources and sinks of momentum, energy, angular momentum, and other information that can flow between the charged matter system and the electromagnetic fields.
3. In Equation (1.1), ρ and $\mathbf{J}$ may represent only a localized subset of the charged matter present, and the electromagnetic fields in the Equation (1.1) only then refer to those belonging to this charged matter.
4. The force density in Equation (1.2) may also contain contributions from other electromagnetic fields not directly associated with ρ and $\mathbf{J}$ in Equation (1.1).
5. Other forces not necessarily of an electromagnetic nature may also contribute to the equation of motion for ρ and $\mathbf{J}$. As mentioned earlier, these may include frictional or contact forces. From these considerations, it follows that Equation (1.1) may always be used to calculate the fields belonging to ρ and $\mathbf{J}$, but not necessarily vice versa. The long range of electromagnetic fields relative to the extent of a localized ρ and $\mathbf{J}$ is one reason that considerations like these become important in formulating and solving some problems in electromagnetism.
6. A finite time, $t - t'$, is required for the effect of a change in the sources at location $\mathbf{r}'$ to arrive at an observation point, $\mathbf{r}$, where the fields are measured. This causal relationship is baked into Maxwell's equations but is not explicitly visible until the integral forms of the equations are written out. For example, Faraday's law indicates that a change in ρ and $\mathbf{J}$ at an earlier time, t', at some distance away, $\mathbf{r}'$, causes the curl of $\mathbf{E}$ and time rate of change of $\mathbf{B}$ at $(\mathbf{r}, t)$.

An overview of the way the mechanical and electromagnetic equations of motion couple and interact with one another is shown in Figure 1.1.

We have not yet quoted any actual equations of motion for ρ and $\mathbf{J}$ given by Newtonian mechanics. These equations can be quite complex, are often

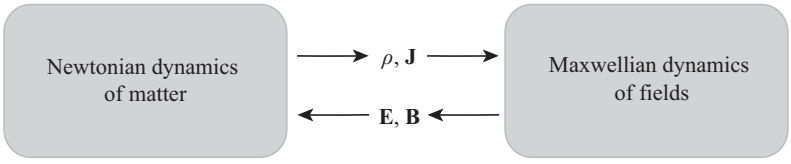


Figure 1.1 Coupling between the mechanical and electromagnetic equations of motion.

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nonlinear, and are often simplified to enable us to find approximate solutions either analytically or by encoding into complex computer programs to find specific numerical approximations. We will discuss various analytic approaches to formulating equations of motion for the charged matter at different points in Sections 1.6 and 1.7.

In Section 1.4, we present quantitative mathematical definitions of ρ and $\mathbf{J}$ from which various formulations of their equations of motion can be effected. The force on moving charged particles that affects their motion is ultimately how we measure the presence of the electromagnetic fields. You may therefore view Equation (1.2) as a definition of the electromagnetic fields once ρ and $\mathbf{J}$ are defined. The job of Maxwell's equations in Equation (1.1) is to get the fields right with Equation (1.2) as one of the forces driving the motion of matter.

1.4 Continuum View of Sources, ρ and J

To understand and define the sources ρ and $\mathbf{J}$ in Maxwell's equations, we must first discuss the concept of charge. There are two types of charges: positive and negative. Different subatomic particles, such as electrons and nuclei, have different discrete charge values. The concept of charge was invented before Maxwell's equations to explain forces appearing between macroscopic objects. The discovery of discrete charges explained that macroscopic objects hold net imbalances of discrete charged particles. The accumulation of one charge species (negative or positive) on a macroscopic object is referred to as charging that object. Charging objects can be achieved by a variety of procedures, like rubbing dissimilar materials together, connecting batteries to them, heating them, exposing them to charged particle beams, or shining short wavelength light on them. All of these processes involve adding energy to the objects and involve the addition, or removal, of atomic particles to or from the object.

The forces between charged objects can be repulsive or attractive, leading to the notion of two opposite kinds of charges. Two objects charged with the same kind of charge are found to repel one another while two objects charged with opposite kinds of charges attract one another. The charge on an object is also found to be an additive property that is proportional to the electromagnetic force it experiences in a fixed environment. Two equal and opposite kinds of charges merged together on an object result in the object experiencing no net electromagnetic force. The connection between charge and attractive or repulsive forces suggests that each of the two kinds of charge may be assigned either a positive or negative quantity.

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It is most common to find macroscopic objects to be uncharged (neutral) resulting in no net electromagnetic force between them. This suggests that the attractive forces between opposite charges have been canceled out due to the balanced distributions of positive and negative charges on the objects. Charging such objects requires separating and transporting charge to or from the objects. The neutral uncharged state is usually assumed for a macroscopic object or a macroscopic part of one, unless other information is supplied. This is sometimes described as the principle of charge neutrality. Care must be used in applying this principle because charge separation can often exist in inhomogeneous materials, and metastable states with charge separation can sometimes exist for very long times in a good insulator. Perhaps the best example of charge neutrality exists inside homogeneous conducting media like metals, plasmas, or electrolytes.

All apparently continuous distributions of charges and currents are ultimately made up of elementary discrete charged particles. The most important charged particles for us will be electrons and atomic nuclei. The smallest charge an atomic particle is observed to carry is the fundamental charge e; this has a value approximately equal to 1.6×10^{-19} Coulomb (C), where Coulomb is the SI unit for charge. One Coulomb is therefore approximately equal to 6×10^{18} elementary charges. Electrons have charge $-e$ and atomic nuclei have charge Ze, where Z is the positive integer atomic number of the nucleus being considered. Neutral atoms that have extra electrons attached to or removed from them are called ions and are often considered elementary charged particles when they appear in plasmas, electrolytes, and charged particle beams. Avogadro's number of positive charges is called a Faraday, which is approximately equal to 96, 485 C/mol, as determined by Faraday using electrochemistry.

When charging a macroscopic object, the excess charge often resides on the surface and rarely exceeds $\pm 1 \times 10^{12}$ e/cm^2 there. By comparison, the number of atoms on a solid or liquid surface is $\sim 5 \times 10^{14}$ atoms/cm^2. Very large capacitors exist that can store several C. These devices often have very large internal surface areas that can be charged by external batteries or other electrical power sources. The most extreme example of a charged object is the nucleus of a heavy atom, which can be $\sim 100e$ all packed into the incredibly small volume of $\sim 1.8 \times 10^{-36}$ cm^3. It is impressive to consider the amount of energy needed to push all those charges so close together in the first place. This indicates the extraordinary energies available in cosmic environments in which these nuclei were fused and hints at the prospect of releasing this energy in nuclear reactions.

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The first source in Maxwell's equations is $\rho(\mathbf{r}, t)$. On length scales larger than the distance between elementary charged particles, the spatially and time continuous scalar function $\rho(\mathbf{r}, t)$ is used to characterize the potentially changing charge state distribution on a finer scale than the total charge q of a macroscopic object. It is $\rho(\mathbf{r}, t)$ that appears in Maxwell's equations as well as the expression for the electromagnetic force density. Charge density in electromagnetism is defined in analogy to mass density $\rho_M(\mathbf{r}, t)$ used in classical continuum mechanics. In a small volume δV centered at $\mathbf{r}$, in which a net charge δq resides,

$$\rho(\mathbf{r}, t) = \lim_{\delta V \to 0} \frac{\delta q}{\delta V}, \tag{1.3}$$

which has SI units of C/m^3. Note that for $\rho(\mathbf{r}, t)$ to be a smooth continuous function of $\mathbf{r}$, there can be no finite charge strictly localized at any location in this continuum description. Equation (1.3) is an approximation to reality because the charged matter is always composed of localized point-like atomic and elementary particles, each of which is endowed with a distinct localized constant charge of its own as depicted in Figure 1.2. Strictly speaking, the limit in Equation (1.3) does not physically exist. The limit can be modified to provide an essentially continuous ρ by stopping the approach of δV to zero when there is only the order of one charged particle in it. We must then not ask questions of our continuum electromagnetism at length scales $\sim< (e/\rho)^{1/3}$. If a particular species, s, of charged particles is characterized by a macroscopic number density of $n_s(\mathbf{r}, t)$ and a charge q_s, then in a coarse-grained continuum description, the charge density is given by

$$\rho(\mathbf{r}, t) = \sum_s q_s n_s(\mathbf{r}, t). \tag{1.4}$$

Charge is treated as an additive quantity in Maxwell's equations, even for different charge species in the calculation of ρ. Examples of different species in

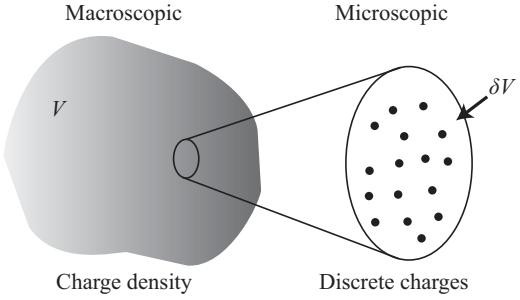


Figure 1.2 Continuum approximation.

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Equation (1.4) include Na^+ and Cl^- in an electrolyte solution, electrons and holes in a semiconductor, electrons and immobile ions in a metal, and electrons and mobile ions in a plasma. A collection of microscopic particles having opposite charges can and often does lead to a continuum charge density that vanishes when δV is large enough. Also, when $\rho(\mathbf{r}, t)$ changes with time, it is always because net charge is flowing into or out of δV.

The current density, $\mathbf{J}$, is the source that appears in the Ampère–Maxwell law. It is related to both the charges and their velocities in δV. Experiments show this quantity to be the electric charge current density $\mathbf{J}(\mathbf{r}, t)$ defined by

$$\mathbf{J}(\mathbf{r}, t) = \sum_s \rho_s(\mathbf{r}, t)\mathbf{v}_s(\mathbf{r}, t), \tag{1.5}$$

where $\mathbf{v}_s(\mathbf{r}, t)$ is the velocity of the charged matter of species s at $(\mathbf{r}, t)$. Metals can have very large current densities because their copious electrons are quite mobile, while their equally copious positive ionic cores are immobile. It is important to realize that both $\mathbf{r}$ and t are treated as independent variables in $\mathbf{v}$ as well as in ρ. Note that the current density at a point can change because the velocity as well as the charge density is changing in time. Finally, the mass or mass density does not appear directly in the sources; its role only comes into play in the material equations of motion via the inertial force.

The definitions of ρ and $\mathbf{J}$ are far from arbitrary. Without these definitions, Maxwell's equations would not predict the observed electromagnetic fields in nature or even be consistent. The SI units for the sources also provide the SI units for $\mathbf{E}$ and $\mathbf{B}$ from Equation (1.2).

1.5 Continuity Equations

The conservation of charge and definitions of ρ and $\mathbf{J}$ guarantee that these quantities satisfy an equation called the charge continuity equation which reads

$$-\frac{\partial}{\partial t}\rho(\mathbf{r}, t) = \nabla \cdot \mathbf{J}(\mathbf{r}, t). \tag{1.6}$$

This field equation can be interpreted as saying that the only reason the charge density can decrease at a point is because it flows away from that point. Thus, $\nabla \cdot \mathbf{J}$ characterizes the flow rate of charge density at $\mathbf{r}$.

Other quantities like mass density obey an equation of a similar form because of the conservation of mass. A modified form of Equation (1.6) also applies to quantities such as kinetic energy or potential energy that are not conserved. The right-hand side of Equation (1.6) is augmented by a term that accounts for additional loss or source mechanisms. Many examples of this occur for systems that

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are not closed. These considerations will be explained and derived in Sections 1.11 and 1.12.

1.6 Charged Matter: Continuum Equations of Motion

To complete the physical description, dynamical equations of motion for the charged matter sources in Maxwell's equations are required. They describe the time evolution of ρ and $\mathbf{J}$. Four scalar equations are needed for each charge species. These can be obtained from the conservation of mass and Newton's laws for the momentum changes in a charged continuum fluid caused by forces it experiences. There are many different physical situations where very different approximations to the Newtonian equations of the charged fluid are relevant. Some of these for a charged matter fluid are presented at the end of this chapter.

For an electron fluid in a conductor, collisions with lattice vibrations and impurities can give rise to strong frictional forces which dominate inertial forces and limit the velocity of the fluid to small values. Empirically, the fluid velocity is observed to be proportional to the driving force rather than the fluid acceleration as may be expected by Newton's second law. For this important case, the four scalar equations of motion for the electron charge and current densities ρ and $\mathbf{J}$ are

$$\begin{aligned} \frac{\partial}{\partial t}\rho(\mathbf{r}, t) + \nabla \cdot \mathbf{J}(\mathbf{r}, t) &= 0, \\ \mathbf{J} &= \sigma \mathbf{E}, \end{aligned} \tag{1.7}$$

where $\sigma = \rho^2\tau/\rho_M$ is called the electrical conductivity of the electron fluid, with ρ_M the mass density of the electron fluid and τ the average time between collisions experienced by the electrons flowing through the conductor. The dependence of σ on these temporally and spatially varying quantities makes σ a function of $(\mathbf{r}, t)$. The equation $\mathbf{J} = \sigma\mathbf{E}$ is referred to as Ohm's law and relates the empirical relationship between force and velocity. Equations in (1.7) also apply to other physical systems containing different charged particle species, for example, electrons and holes in a semiconductor, or positive and negative ions in an electrolyte, which require accounting for multiple charged species in the formulation of conductivity.

Sometimes there can be only external constraints or forces that essentially determine ρ and $\mathbf{J}$, regardless of the electric fields they create. For example, a wire or coil may be connected to an externally programmable, current-regulated power supply to generate a prescribed magnetic field, or a voltage-regulated power supply may be connected to some conducting objects to establish charge distributions that result in desired electric fields. Then there are no equations of motion for ρ and $\mathbf{J}$ because they are simply given functions from which Maxwell's equations

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can be solved. Problems based on this assumption are often a focus of introductory presentations of electromagnetism. These may be referred to as problems with controlled sources.

The nonrelativistic, dynamical equations of motion for ρ and $\mathbf{J}$ have their origins in macroscopic continuum mechanics laws of motion. These laws are based on Newtonian mechanics applied to continuous mass and momentum distributions,

$$\rho_M(\mathbf{r}, t) = Mn(\mathbf{r}, t), \quad J_M(\mathbf{r}, t) = \rho_M(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t), \tag{1.8}$$

where $\rho_M(\mathbf{r}, t)$ is the mass density, M is the total mass, $\mathbf{J}_M(\mathbf{r}, t)$ is the mass flux, $n(\mathbf{r}, t)$ is the number density, and $\mathbf{v}(\mathbf{r}, t)$ is the velocity of a given particle species at $(\mathbf{r}, t)$. By virtue of the conservation of mass M, ρ_M, and $\mathbf{J}_M$ satisfy their own continuity equation,

$$-\frac{\partial}{\partial t}\rho_M(\mathbf{r}, t) = \nabla \cdot \mathbf{J}_M(\mathbf{r}, t), \tag{1.9}$$

for each species. There is a close connection between the mass and charge densities and current given by

$$\rho_{Ms} = \frac{M_s}{q_s}\rho_s, \quad J_{Ms} = \frac{M_s}{q_s}\mathbf{J}_s = \frac{M_s}{q_s}\rho_s\mathbf{v}_s, \tag{1.10}$$

again for each species, s. In the language of continuum mechanics, the electromagnetic forces are called body forces. The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in the force density both control the motion of ρ and $\mathbf{J}$ via the material equations of motion but are also determined by ρ and $\mathbf{J}$ via Maxwell's equations. It is quite legitimate to claim that a continuous charge distribution interacts with itself via its own electromagnetic fields. Other forces like gravity, contact forces, and friction forces may play a role in determining the motion of the matter fields.

The macroscopic approach to describing ρ and $\mathbf{J}$ is often used or assumed in presenting classical electromagnetism. Maxwell was well versed in continuum mechanics and considered the flow of charge as that of a deformable continuous fluid. It should be no surprise that many of the mathematical methods for electromagnetism were adopted from continuum mechanics. The presentation and derivation of detailed continuum equations of motion are presented in Section 1.12.

1.7 Charged Matter: Particle Equations of Motion

As an alternative to the macroscopic, continuum view, one may adopt a microscopic view, where one chooses to study and present classical electromagnetism in terms of moving point-like particles whose charge is localized at the particle location in space and time. In this approach, Maxwell's equations remain the same

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except that the charged matter sources are represented with the aid of highly singular density functions called Dirac delta functions. Consider a single charged particle with constant charge q_i that is completely localized at a time-dependent location $\mathbf{r}_i(t)$. This position is not an independent variable like the position variables that appear in the arguments of the electromagnetic fields of Maxwell's equations. It is a dependent variable representing the trajectory of the ith charged particle: $\mathbf{r}_i(t)$ is obtained by solving the Newtonian equations of motion for the single particle.

This situation has an analytic solution that involves the introduction of the three-dimensional Dirac delta function $\delta(\mathbf{r} - \mathbf{r}_i(t))$, which is a highly singular function localized at $\mathbf{r} = \mathbf{r}_i(t)$. The properties of the Dirac delta are defined only when it appears inside a volume integral. For example, if the charge and current densities are taken to be

$$\rho(\mathbf{r}, t) = q_i \delta(\mathbf{r} - \mathbf{r}_i(t)), \quad J(\mathbf{r}, t) = q_i \mathbf{v}_i(t) \delta(\mathbf{r} - \mathbf{r}_i(t)), \tag{1.11}$$

then the force on a particle with trajectory $\mathbf{r}_i(t)$ will be

$$\begin{aligned} \mathbf{F}(\mathbf{r}_i(t)) = \int_V \mathrm{d}V \mathbf{f}(\mathbf{r}, t) &= \int_V \mathrm{d}V \left(\rho(\mathbf{r}, t)\mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)\right) \\ &= q_i \mathbf{E}(\mathbf{r}_i(t), t) + q_i \mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i(t), t), \end{aligned} \tag{1.12}$$

with $\mathrm{d}V = \mathrm{d}r^3$ defining the volume element. This expression is the well-known Lorentz force on a moving charged particle. It follows from the defining property of the three-dimensional delta function,

$$g(\mathbf{r}_i(t), t) = \begin{cases} \displaystyle\int_V \mathrm{d}V \delta(\mathbf{r} - \mathbf{r}_i(t)) g(\mathbf{r}, t)\,, & \mathbf{r}_i(t) \in V, \\ 0\,, & \mathbf{r}_i(t) \notin V, \end{cases} \tag{1.13}$$

for any field function g. More will be said about delta functions in Chapter 2. For a single isolated point charge at rest at $\mathbf{r}_i = 0$, Maxwell's equations can be integrated to yield an electric field,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r}|^2} \mathbf{e}_\mathbf{r}, \tag{1.14}$$

which is called the Coulomb field at distance $|\mathbf{r}|$ from the charge and $\mathbf{e}_\mathbf{r}$ is the unit vector equal to $\mathbf{r}/|\mathbf{r}|$. The mathematical method for obtaining this will be presented in Chapter 2.

When more than one charged particle is involved in the source,

$$\rho(\mathbf{r}, t) \to \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i(t)) \tag{1.15}$$

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and

$$\mathbf{J}(\mathbf{r}, t) \to \sum_i q_i \mathbf{v}_i(t) \delta(\mathbf{r} - \mathbf{r}_i(t)), \tag{1.16}$$

where q_i, $\mathbf{r}_i(t)$, and $\mathbf{v}_i(t)$ are the charge, position, and velocity of the ith particle at time t. The ith point-like particle might be an electron, proton, positron, ion, nucleus, muon, tau, and so on. The charge of these particles only takes on discrete quantized values that are independent of their motion and that of the observer. $\mathbf{r}_i(t)$ and $\mathbf{v}_i(t)$ are governed by the many-particle Newtonian equations of motion in which the forces depend, among other things, on the particle charge, the velocity, and the electromagnetic fields evaluated at the changing positions of the particles. For this purpose, the relevant electromagnetic fields are given by the Lorentz force defined earlier so that for each particle,

$$M_i \frac{\mathrm{d}\mathbf{v}_i(t)}{\mathrm{d}t} = \sum_{j \neq i} \left[q_i \mathbf{E}_j(\mathbf{r}_i(t), t) + q_i \mathbf{v}_i \times \mathbf{B}_j(\mathbf{r}_i(t), t) \right], \text{ with } \mathbf{v}_i(t) = \frac{\mathrm{d}\mathbf{r}_i(t)}{\mathrm{d}t}. \tag{1.17}$$

This equation predicts the observed motion when other non-electromagnetic forces are absent. The term on the left accounts for the inertial force. On the right-hand side of Equation (1.17) is the electromagnetic force on the ith particle due to the fields, $\mathbf{E}_j$ and $\mathbf{B}_j$, from the jth particle. To determine this electromagnetic force, we must find $\mathbf{E}_j$ and $\mathbf{B}_j$ from all j particles which behave as sources in Maxwell's equations, in which ρ and $\mathbf{J}$ from Equations (1.15) and (1.16) are used.

Use of Equations (1.15) and (1.16) raises the question how the delta functions behave in Maxwell's equations. We will discuss how to find the fields by integrating Maxwell's equations in Chapter 4. When this is done, the fields may always be expressed by volume integrals that contain the delta function from the sources, which makes the evaluation of these integrals straightforward. The only new and nontrivial problem one encounters is that the field values $\mathbf{E}(\mathbf{r}_i(t), t)$ and $\mathbf{B}(\mathbf{r}_i(t), t)$ excited by the particle at $\mathbf{r}_i(t)$ are indeterminate. This problem means we cannot evaluate the force that a particle experiences due to its own field.

Several fixes have been proposed. The simplest might be called the electromagnetic exclusion principle. It tells us to exclude the contribution to the electromagnetic fields at $\mathbf{r}_i(t)$ due to the ith particle itself. This procedure has been followed in Equations (1.15) and (1.16). Essentially, this procedure is equivalent to saying that a charged particle does not interact with its own self-fields. This argument is consistent with the notion that, although this self-field is highly singular at the particle's position, it is also highly symmetric, leading to no net electromagnetic force. From a practical standpoint, this procedure works rather well and is often used; however, it is conceptually ugly because, in principle, we now need

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different electromagnetic fields for each particle's equation of motion. In addition, it is known that highly accelerated charged particles do experience forces, called radiation reaction forces, that are not accounted for in the Lorentz force law. This effect is mostly important for electrons, which, due to their low mass, can experience large accelerations in high fields like those in an atom. Further discussion of radiation reaction forces and their associated formulas will be presented in Chapter 6.

The mathematical problem of these infinite self-forces does not appear in the macroscopic continuum version of the many-particle theories, where a dense collection of charged particles is approximated by a continuous charge density and resultant charge density field functions. However, this approximation gets the physics of very small distances like the self-field problem wrong.

1.8 The Wave Equation Form of Maxwell's Equations

We return to the dynamical Maxwell's equations of Equation (1.1). These equations are believed to be true no matter in what medium the electromagnetic fields exist, for example, vacuum, water, glass, metals, semiconductors, gases, and plasmas. In all media, Maxwell's equations will provide solutions for the fields, assuming all are known. The constants ϵ_0 and μ_0 are called the permittivity and permeability of the vacuum, and they have the values in SI units of

$$\begin{aligned} \epsilon_0 &= 8.85 \times 10^{-12} \text{ Farad/m}, \\ \mu_0 &= 1.26 \times 10^{-6} \text{ Henry/m}. \end{aligned} \tag{1.18}$$

The Farad is a measure of capacitance and the Henry is for inductance, stressing the fundamental connection between ϵ_0 and electric fields and μ_0 with magnetic fields. This connection will appear naturally in connection with the near-field approximation to Maxwell's equations and its connection with circuit theory presented in Chapter 5.

Further insight is gained when the four first-order Maxwell's equations are combined (using the rules of vector calculus reviewed in Chapter 2) into two inhomogeneous second-order equations known as the wave equations. These wave equations are given by

$$\begin{aligned} \nabla^2 \mathbf{E} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t}, \\ \nabla^2 \mathbf{B} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\mu_0 \nabla \times \mathbf{J}. \end{aligned} \tag{1.19}$$

In these equations, the dynamics of each electromagnetic field are isolated on the left-hand side and the sources that drive the dynamics are on the right-hand side.

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From the left-hand side, it is seen that the efficacy with which a charge drives the electric field in a vacuum is $\sim 1/\epsilon_0$, whereas the efficacy with which a current drives both electromagnetic fields is $\sim \mu_0$.

The left-hand side of Equation (1.19) contains ϵ_0 and μ_0 only as a product $\epsilon_0\mu_0$, which has the same dimensions as an inverse squared velocity. In Chapter 2, we demonstrate how the mathematics works in one dimension to show that $1/\sqrt{\epsilon_0\mu_0}$ is the velocity of propagation, c, of an electromagnetic signal in free space. This method is due to d'Alembert in his study of sound propagation. Later, in Chapter 4, we will show how this result is obtained by the integration of Equation (1.19). Thus, it is quite generally true that

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}. \tag{1.20}$$

In Chapter 4, we will develop a method to integrate Equation (1.19) to obtain the full spatial and time dependence of the electromagnetic field for specified sources in a form called Jefimenko's equations.

1.9 Point-Like Dipole Moment Sources

Our discussion of delta function localized sources characterized by a scalar quantity q is incomplete. There is much experimental evidence that shows that particles can have an additional complex internal structure associated with internal charge separation and internal currents. This internal structure can be electromagnetically characterized by a series of multipole moments, each of which responds to and creates electromagnetic fields. The charge q is called the monopole. Usually the next most important multipole is called a dipole (and the next the quadrupole). Dipoles result from vector-like properties of the particle. There are two types of dipoles: electric and magnetic. The former is characterized by an electric dipole moment, $\mathbf{p}$, and the latter by a magnetic dipole moment, $\mathbf{m}$. The SI units for $\mathbf{p}$ are Coulomb-meter (C m), and for $\mathbf{m}$ Ampère-meters-squared (A m^2). In external electromagnetic fields, particles with both monopole and dipole moments will experience both a net force,

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} + \nabla(\mathbf{p} \cdot \mathbf{E}) + \nabla(\mathbf{m} \cdot \mathbf{B}), \tag{1.21}$$

which determines the particle motion in space, and a net torque,

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} + \mathbf{m} \times \mathbf{B}, \tag{1.22}$$

around the particle position, determining the orientation of moment directions. The magnitude and direction of these moments are closely related to an internal charge separation or angular momentum of the charged constituents of the particle.

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One is often particularly interested in dipole moments when the particle net charge is zero because then the very strong Coulomb force does not dominate the dipole forces. The electromagnetic fields excited by an isolated electric dipole at rest at $\mathbf{r} = 0$ are given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\mathbf{e_r})\mathbf{e_r} - \mathbf{p}}{|\mathbf{r}|^3}, \tag{1.23}$$

and for a magnetic dipole,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{e_r})\mathbf{e_r} - \mathbf{m}}{|\mathbf{r}|^3}, \tag{1.24}$$

which are both obtained from Maxwell's equations by methods discussed in Chapter 5.

The dipole moments may be intrinsic to the particles. For example, electrons, protons, and nuclei have intrinsic magnetic dipole moments, and some molecules like water have intrinsic electric dipole moments. Other particles only have dipole moments when they are immersed in electromagnetic fields. Neutral atoms and molecules usually have such induced moments and are called polarizable or magnetizable.

1.10 The Hidden Sources and Maxwell's Equations in Dense Media

Neutral macroscopic objects can generate electromagnetic fields either spontaneously or in response to external charges or currents. This effect arises from macroscopic numbers of internal dipoles that in aggregate can produce macroscopic charge or currents in the macroscopic object. Either intrinsic or induced dipoles can play a role. We call these internal charge and current densities. Maxwell's equations are still valid, assuming we know all the charges and currents because they belong to sources belonging to the object. In the early days of electromagnetism, the origin and quantification of these internal sources were unknown or poorly understood, but their experimental influence on the behavior of electromagnetic phenomena was often quite dramatic and disparate for different kinds of materials. Many of these phenomena can be explained by introducing new auxiliary fields $\mathbf{P}$ and $\mathbf{M}$, called the polarization and magnetization fields, respectively. $\mathbf{P}$ is the density of microscopic electric dipole moments in the object. $\mathbf{M}$ is the density of magnetic dipole moments. In Chapters 8 and 9, these new fields give rise to internal sources ρ_{int} and $\mathbf{J}_{int}$ obtained via

$$\begin{aligned} \rho_{int} &= -\nabla\cdot\mathbf{P}, \\ \mathbf{J}_{int} &= \frac{\partial}{\partial t}\mathbf{P} + \nabla\times\mathbf{M}. \end{aligned} \tag{1.25}$$

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Both results depend on $\mathbf{P}$ and $\mathbf{M}$ being defined as dipole densities with the field due to the dipoles being given by Equations (1.23) and (1.24). Because these sources are bound to the medium, they are sometimes referred to as bound charge and current density, where $\rho_{bound} = \rho_{int} = -\nabla \cdot \mathbf{P}$ and $\mathbf{J}_{bound} = \nabla \times \mathbf{M}$; the contribution $\mathbf{J}_{pol} = \partial \mathbf{P}/\partial t$ to $\mathbf{J}_{int}$ is referred to as the polarization current. Note that these internal sources satisfy the continuity equation. $\mathbf{P}$ and $\mathbf{M}$ must of course be guessed or modeled appropriately to give results that account for many experiments. Maxwell's equations can now be recast by renaming the charge and current densities that are not ρ_{int} and $\mathbf{J}_{int}$ that arise from macroscopic aggregates of dipoles as ρ_{free} and $\mathbf{J}_{free}$,

$$\begin{aligned} \nabla \cdot \mathbf{D}(\mathbf{r}, t) &= \rho_{free}(\mathbf{r}, t), & \nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \\ \nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0, & \nabla \times \mathbf{H}(\mathbf{r}, t) &= \mathbf{J}_{free}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}, \end{aligned} \tag{1.26}$$

where two new fields,

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P}, \\ \mathbf{H} &= \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \end{aligned} \tag{1.27}$$

have been introduced. Note that in Equation (1.26), $\mathbf{E}$ and $\mathbf{B}$ are still the electromagnetic fields due to all charges and currents, which are now $\rho_{free} + \rho_{int}$ and $\mathbf{J}_{free} + \mathbf{J}_{int}$.

Often, well-founded physical models for $\mathbf{P}$ and $\mathbf{M}$ can be converted via Equation (1.27) to the form

$$\mathbf{D} = \mathbf{D}(\mathbf{E}), \quad \mathbf{H} = \mathbf{H}(\mathbf{B}). \tag{1.28}$$

These kinds of equations are called constitutive relations and are needed to solve Maxwell's equations in the form of Equation (1.26). In empty space, they take the form

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \tag{1.29}$$

from which the original Maxwell's equations are recovered.

An example of constitutive equations for slow, time-dependent fields in what is called a linear medium is obtained by assuming

$$\mathbf{D}(\mathbf{r}, t) = \epsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}, t), \quad \mathbf{B}(\mathbf{r}, t) = \mu(\mathbf{r}) \mathbf{H}(\mathbf{r}, t) \tag{1.30}$$

inside the medium and

$$\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \mathbf{E}(\mathbf{r}, t), \quad \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t) \tag{1.31}$$

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outside it. $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ are called the dielectric and permeability functions, respectively. More complicated constitutive relations can be envisioned and indeed are commonly used to explain electromagnetic phenomena in the optical regime. The connection between the constitutive relations and the medium's charge and current densities and their equations of motion will be discussed later in Chapters 8, 9, and 13.

When $\mathbf{E}$ or $\mathbf{B}$ become extremely large, the constitutive relations can become nonlinear. This destroys the linearity of Maxwell's equations and leads to a wide range of new electromagnetic phenomena.

1.11 Energy, Momentum, and Angular Momentum

Consider the material charge sources and the electromagnetic fields as a closed system. Momentum, energy, and angular momentum may be viewed as belonging jointly to, and capable of moving between, the material and electromagnetic fields, as suggested by the conservation principles of these quantities and the electromagnetic interactions between sources and fields. From a continuum perspective, a material system has

$$\begin{aligned} \text{momentum density:} \quad & \rho_M(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t), \\ \text{energy density:} \quad & \frac{1}{2}\rho_M(\mathbf{r}, t)\mathbf{v}^2(\mathbf{r}, t), \\ \text{and angular momentum density:} \quad & \rho_M(\mathbf{r}, t)\mathbf{r} \times \mathbf{v}(\mathbf{r}, t). \end{aligned} \tag{1.32}$$

The interaction between charged matter and the electromagnetic fields requires an extension of the local conservation of momentum, energy, and angular momentum density to the larger system of matter plus fields. The rate at which these quantities change is given by

$$\begin{aligned} \mathbf{f} &= \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \quad \text{for momentum density,} \\ \mathbf{v} \cdot \mathbf{f} &= \mathbf{J} \cdot \mathbf{E} \quad \text{for energy density,} \\ \mathbf{r} \times \mathbf{f} &= \rho\mathbf{r} \times \mathbf{E} + \mathbf{r} \times (\mathbf{J} \times \mathbf{B}) \quad \text{for angular momentum density.} \end{aligned} \tag{1.33}$$

Note that because the magnetic field is perpendicular to the velocity it does no work and does not contribute to the rate of change of energy density. The most well-known application of these ideas is the Poynting theorem, which states that the rate of energy transfer from charged matter in motion to the electromagnetic fields, $\mathrm{d}W_{field}/\mathrm{d}t$, is governed by

$$\frac{\mathrm{d}W_{field}}{\mathrm{d}t} = -\int_V \mathrm{d}V \mathbf{J} \cdot \mathbf{E} = \int_V \mathrm{d}V \left[\frac{1}{\mu_0} \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) + \frac{1}{2\mu_0} \frac{\partial \mathbf{B} \cdot \mathbf{B}}{\partial t} + \frac{\epsilon_0}{2} \frac{\partial \mathbf{E} \cdot \mathbf{E}}{\partial t} \right], \tag{1.34}$$

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which follows from replacing **J**, the mechanical variable, by its connection with the Ampère–Maxwell equation, and simplifying with integration by parts and Faraday's law. If the volume integration region is arbitrary this implies

$$-\frac{\partial}{\partial t}\left[\frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}+\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E}\right]=\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})+\mathbf{J}\cdot\mathbf{E}, \tag{1.35}$$

which can be viewed as an equation of motion relating the rate of electromagnetic energy density decrease of the electromagnetic field, given by $\partial w_{field}/\partial t$, to the flux density $\nabla\cdot\mathbf{S}$ at which w_{field} flows into other regions of space, and the rate of conversion of electromagnetic energy to mechanical energy density $\partial w_{mech}/\partial t$ of the charge density. These three terms are obtained from Equation (1.35) giving

$$\begin{aligned} w_{field} &= \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}+\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E},\\ \mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B},\\ \frac{\partial}{\partial t}w_{mech} &= \mathbf{J}\cdot\mathbf{E}, \end{aligned} \tag{1.36}$$

all evaluated at the $(\mathbf{r}, t)$ of interest. **S** is called the Poynting vector. So the equation of motion for w_{field} clearly has the form of a continuity equation with an added mechanical loss term. When a similar derivation is carried out for the electromagnetic momentum density, it is found to be $\mathbf{S}/c^2$. The electromagnetic momentum current density is known as the Maxwell stress tensor, $\boldsymbol{T}$, which is of utility for calculating electromagnetic forces on macroscopic objects.

We now shift our attention to the parallel discussion involving equations of motion and conservation laws for the continuum charged matter system.

1.12 Simple Continuum Mechanics of Charged Matter

Many expositions of electromagnetism give short shrift to the field theoretic equations that describe, on a macroscopic continuum level, the motion of charged matter exposed to electromagnetic and other forces. This is unfortunate for two reasons. First, the continuum description of matter in motion preceded electromagnetism and provided many of the concepts used in its development and description. These methods were well known to Maxwell and invaluable for anyone seeking a deep understanding and intuition for electromagnetism. Second, there are many practical problems where it is the motion of the charged matter that we are really interested in. Determining this motion requires knowing the electromagnetic forces. But knowing them does not solve the question of the motion which motivated the problem to start with. Thus, although the material presented below

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belongs to the formal study of continuum mechanics, we present it here for the curious reader who has not encountered it before.

As an example of continuum mechanics, we shall first focus on what is called an inviscid fluid, that is, a fluid without viscous forces. We shall additionally assume that all contact forces in the form of pressure gradients or viscosity-based frictional forces are absent. The remaining forces are called body forces. Historically, the most important example of a body force was an external gravitational field. In astrophysics, one is also interested in internal gravitational fields generated by the matter fields under observation. More relevant to us are electromagnetic body forces generated both internally and externally to the localized charged matter we are trying to describe.

Begin by considering a variable density charged fluid characterized by a charge density $\rho(\mathbf{r}, t)$, a mass density $\rho_M(\mathbf{r}, t)$, a velocity field $\mathbf{v}(\mathbf{r}, t)$, and a body force per unit volume $\mathbf{f}_B(\mathbf{r}, t)$, which for electromagnetic forces is $\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B}$. Mass and charge will be quantities that are strictly conserved. Other quantities, like momentum and kinetic energy of the fluid, are also conserved except when acted on by body forces. In the presence of body forces, the equations of motion may be cast in the form of conservation equations with a conservation breaking source term. Taking into account Newton's laws of motion we expect the nonconservation of momentum to be sourced by $\mathbf{f}_B$ and for kinetic energy by $\mathbf{v} \cdot \mathbf{f}_B$.

We begin by deriving mass conservation. Consider the motion of an infinitesimally localized volume ΔV containing a mass $\Delta M = \rho_M \Delta V$. During the motion, assume that the mass in the volume stays constant, which is what we mean by conservation of mass. But the density and volume element must both change along the motion.

$$\rho_M(\mathbf{r}_{\Delta V}(t), t)\Delta V(\mathbf{r}_{\Delta V}(t), t) = \Delta M = \text{const.} \tag{1.37}$$

where $\mathbf{r}_{\Delta V}(t)$ is no longer an independent variable. It describes the position of ΔV as it moves along with the fluid. For simplicity we shall call $\mathbf{r}_{\Delta V}(t)$ just $\mathbf{r}(t)$ in what follows.

A time derivative along the motion of a moving fluid element is called a substantive derivative and often indicated in fluid dynamics by the notation $\mathrm{D}/\mathrm{D}t$. In calculus and physics, this is called a total derivative and indicated by $\mathrm{d}/\mathrm{d}t$. The proper interpretation of $\mathrm{d}f/\mathrm{d}t$ is the rate of change of f of the fluid element as it moves about in space; it is not the rate of change of f of the fluid at a given point in space. The total time derivative for any function that is dragged along with the flow $\mathbf{r}(t)$ of material particles in the fluid is deduced from the chain rule as

$$\frac{\mathrm{d}}{\mathrm{d}t} f\,(\mathbf{r}(t), t) = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} \cdot \nabla f(\mathbf{r}(t), t) + \frac{\partial}{\partial t} f(\mathbf{r}, t). \tag{1.38}$$

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Thus, the change $\mathrm{d}f$ in f of a fluid element during the time $\mathrm{d}t$ consists of two parts: The first term corresponds to the difference in f at two points $\mathrm{d}r$ apart where $\mathrm{d}r$ is the distance moved by the fluid element in $\mathrm{d}t$; the second term corresponds to the change in f during $\mathrm{d}t$ at a fixed point in space. The reader must be cautioned that the symbols $\mathbf{r}(t)$ and $\mathbf{r}$ here mean very different things: $\mathbf{r}(t)$ is a chosen path in space, whereas $\mathbf{r}$ is the general spatial variable. Note that, for a particular $(\mathbf{r}, t)$ in the fluid the flow through that point actually defines $\mathbf{v}(\mathbf{r}, t)$ by

$$\mathbf{v}(\mathbf{r}, t) = \left.\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\right|_{\mathbf{r}}. \tag{1.39}$$

In terms of these concepts, the conservation of mass means

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta M = 0 = \frac{\mathrm{d}\rho_M \Delta V}{\mathrm{d}t} = \rho_M(\mathbf{r}(t), t)\frac{\mathrm{d}}{\mathrm{d}t}\Delta V(\mathbf{r}(t), t) + \Delta V(\mathbf{r}(t), t)\frac{\mathrm{d}}{\mathrm{d}t}\rho_M(\mathbf{r}(t), t). \tag{1.40}$$

Notice that from the conservation of mass and the divergence theorem applied to a very small volume ΔV, it follows that

$$\frac{\mathrm{d}\Delta V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Delta V} \mathrm{d}\mathbf{a}\cdot\mathbf{r}(t) = \int_{\partial\Delta V} \mathrm{d}\mathbf{a}\cdot\mathbf{v}(t) = \int_{\Delta V} \mathrm{d}V\nabla\cdot\mathbf{v}(t) = \Delta V\nabla\cdot\mathbf{v}, \tag{1.41}$$

where $\partial\Delta V$ is the boundary of the volume ΔV and $\mathrm{d}\mathbf{a}$ is the infinitesimal surface normal at a point on the surface of ΔV. When Equation (1.41) is used in Equation (1.40) gives

$$\frac{1}{\Delta V}\frac{\mathrm{d}\Delta V}{\mathrm{d}t} = \nabla\cdot\mathbf{v} = -\frac{1}{\rho_M}\frac{\mathrm{d}\rho_M}{\mathrm{d}t}. \tag{1.42}$$

This result provides an interesting interpretation of $\nabla\cdot\mathbf{v}$ in the context of motion along the flow. In cases where $\nabla\cdot\mathbf{v} = 0$, Equation (1.42) implies that the mass density does not change along the flow indicating that the fluid is incompressible.

Using the definition of the substantive derivative in Equation (1.38) to rewrite Equation (1.42) gives

$$\begin{aligned} 0 &= \rho_M\nabla\cdot\mathbf{v} + \frac{\mathrm{d}}{\mathrm{d}t}\rho_M, \\ 0 &= \rho_M\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\rho_M + \frac{\partial}{\partial t}\rho_M, \\ 0 &= \nabla\cdot(\rho_M\mathbf{v}) + \frac{\partial}{\partial t}\rho_M. \end{aligned} \tag{1.43}$$

This is the classic form of a differential equation connected with a local conservation law.

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Exactly the same steps can be used to express the local conservation of charge in terms of the charge and charge current densities via

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial}{\partial t}\rho(\mathbf{r}, t) = 0. \tag{1.44}$$

We now express Newton's law describing momentum changes in the fluid in terms of a local differential equation trying to get in as close a form as possible to a conservation law. The result follows from writing Newton's second law as

$$\begin{aligned} \frac{\mathrm{d}\Delta M\mathbf{v}}{\mathrm{d}t} = \Delta M\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho_M \Delta V \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \;\rightarrow\; \rho_M \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \mathbf{f}_B, \\ \rho_M \left((\mathbf{v}\cdot\nabla)\,\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \right) &= \mathbf{f}_B, \\ \nabla \cdot (\rho_M \mathrm{v}_i \mathbf{v}) + \frac{\partial}{\partial t}\left(\rho_M \mathrm{v}_i\right) &= f_{i,B}. \end{aligned} \tag{1.45}$$

The last equation follows from Equation (1.43) and the subscript i stands for any one of the Cartesian components of the vector quantities to which it is attached. Thus, Equation (1.45) shows that, unless there is a body force along the ith direction in the fluid, the momentum density along that direction $\rho_M \mathrm{v}_i$ is a conserved quantity.

At this point, you can probably write down the equation of motion for the kinetic energy in the fluid. It is

$$\nabla \cdot \left(\frac{1}{2}\rho_M (\mathbf{v}\cdot\mathbf{v})\mathbf{v} \right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_M \mathbf{v}\cdot\mathbf{v} \right) = \mathbf{v}\cdot\mathbf{f}_B. \tag{1.46}$$

Equations (1.43), (1.45), and (1.46) are called Euler's equations. In a multispecies system, there will be one set of conservation equations for each species. When the body force is electromagnetic, the electric and magnetic fields in each equation are total fields, but the charge and charge current densities are related to the mass and momentum density by the mass to charge ratio for the species. Thus, for the species, s, the fluid equations for mass and momentum are

$$\begin{aligned} \nabla \cdot (\rho_{sM}\,\mathbf{v}_s) + \frac{\partial}{\partial t}\rho_{sM} &= 0, \\ \nabla \cdot (\rho_{sM}\,\mathrm{v}_{si}\mathbf{v}_s) + \frac{\partial}{\partial t}\rho_{sM}\mathrm{v}_{si} &= \frac{q_s}{M_s}\rho_{sM}E_i + \frac{q_s}{M_s}\rho_{sM}\,(\mathbf{v}_s \times \mathbf{B})_i. \end{aligned} \tag{1.47}$$

In the equations describing the local conservation of mass and momentum, we have four unknown scalar functions ρ_{sM} and $\mathbf{v}_s$, as well as four coupled nonlinear

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partial differential equations. These are the equations of motion for the fluid. For charged matter, the body forces require a knowledge of ρ, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$, where ρ and $\mathbf{J}$ are related to ρ_M and $\mathbf{J}_M$, and Maxwell's equations are used to obtain $\mathbf{E}$ and $\mathbf{B}$. These equations of motion for the matter field are already much more complex than Maxwell's equations (which are at least linear) and we have not included contact pressure and viscous forces. When these other contact interactions and viscous forces are required or included in the equations of motion for an incompressible fluid, they are called the Navier–Stokes equations.

All fluid equations can be simplified and solved under special assumptions often used in solving Maxwell's equations. For example, static and steady-state equations can be deduced as well as those for incompressible fluids or fluids without angular momentum, etc. One interesting case is where a frictional damping force, proportional to velocity $\gamma \mathbf{v}$, is much stronger than the inertial and magnetic forces. The two fluid equations reduce to

$$\begin{aligned} \nabla \cdot (\rho_{sM}\, \mathbf{v}_s) + \frac{\partial}{\partial t}\rho_{sM} &= 0, \\ \rho_{sM}\gamma\, \mathbf{v}_s &= \frac{q_s}{M_s}\rho_{sM}\mathbf{E}. \end{aligned} \tag{1.48}$$

The recent implementation of numerical codes on very fast computers has also had a major impact on solving important fluids problems related to important issues like plasma fusion, aerodynamics, microfluidics, cosmology, and so on.

1.13 Problems

Problem 1: Assuming the sources are known, how many unknowns and how many equations are contained in Equation (1.1)? What does this suggest to you?

Problem 2: Assuming the sources ρ and $\mathbf{J}$ are known, how many unknowns are contained in the wave Equation (1.19)?

Problem 3: Provide the intrinsic dipole moments for an electron, proton, different isotopes of oxygen nuclei, water molecule, ammonia molecule, and carbon dioxide molecule.

Problem 4: Prove that the form of Maxwell's equations in Equation (1.26) follows from the fields defined in Equation (1.27).

Problem 5: Show that Equation (1.29) converts the form of Maxwell's equations in Equation (1.26) into the vacuum form of Maxwell's equations given in Equation (1.1). How does ρ as defined for Equation (1.26) transform to ρ appearing in Equation (1.1)?

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Problem 6: Consider a variable density, charged fluid characterized by a charge density $\rho(\mathbf{r}, t)$, a mass density ρ_M, and a velocity field $\mathbf{v}(\mathbf{r}, t)$. The fluid is described by the equation,

$$\rho_M(\mathbf{x}, t)\frac{\mathrm{d}\mathbf{v}(\mathbf{x}, t)}{\mathrm{d}t} = \mathbf{f}_{all\ else} + \rho(\mathbf{x}, t)\mathbf{E}(\mathbf{x}, t) + \rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t). \quad (1.49)$$

Body forces act per unit volume, and consist of electromagnetic forces, $\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B}$ and other forces such as the gravitational force. The term $\mathbf{f}_{all\ else}$ includes these additional body forces, as well as contact forces that infinitesimal fluid elements exert directly on the surface of any other neighboring element of fluid. Often the contact force on a small volume V manifests itself through a scalar pressure function $p(\mathbf{x}, t)$. The force per unit area is taken to be normal to the surfaces of the chosen volume element, giving for the contact force,

$$\mathbf{f}_{contact} = \iint p(\mathbf{x}, t)\mathrm{d}\mathbf{a}. \quad (1.50)$$

Part a: Show that including gravity the fluid equation (1.49) can be written as

$$\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}(\mathbf{x}, t) = \mathbf{g} - \frac{1}{\rho_M(\mathbf{x}, t)}\nabla p(\mathbf{x}, t) + \frac{\rho(\mathbf{x}, t)}{\rho_M(\mathbf{x}, t)}[\mathbf{E}(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t)], \quad (1.51)$$

and in the process explain where the derivative terms on the right-hand side of the equation come from.

Part b: This together with Maxwell's equations clearly does not contain enough information to solve for all the unknown fields. Find another field equation relating ρ_M to $\mathbf{v}$ and provide a physical motivation.

Part c: How much more information do you need to solve for all the fields? Make suggestions for where the additional field equations might come from.

Part d: What was Albert Einstein's PhD thesis about?

Part e: What was Niels Bohr's PhD thesis about?

Problem 7: (a) Prove that Maxwell's equations are consistent with and indeed require charge conservation. (b) Derive Maxwell's equations from Ampère–Maxwell, Faraday, and the charge conservation laws.

Problem 8: Consider an isolated conducting body of uniform conductivity. Imagine it is immersed in a steady electric field produced by bodies external to it. Can there be any static electric field inside the conducting body in the steady state? Imagine that at $t = 0$, through some mechanism, an excess charge density, ρ_0, is injected into a localized spot inside the conducting body. How long

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does it take to reach equilibrium and where does the charge end up after it does? Estimate this time for: a simple metal, and a good insulator.

Problem 9: Confirm or debug the claim that particles of the same charge-to-mass ratio take the same path when starting out with similar initial conditions in a static electric field.

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