Proc. Camb. Phil. Soc. (1965), **61**, 869
Printed in Great Britain

Vector bundles that fill n-space

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(*Received* 29 *December* 1964)

1. *Introduction.* The idea of *exact filling* bundle may be described roughly as follows. Suppose that ξ^k is a vector bundle with fibre R^k, total space $E(\xi^k)$ and base X. We say that ξ^k is a real k-plane bundle on X. Let ι^n be the trivial n-plane bundle on X so that $E(\iota^n) = X \times R^n$. A bundle monomorphism $j\colon \xi^k \to \iota^n$ defines a map $\tilde{j}\colon E(\xi^k) \to R^n$ obtained by composition of the embedding $E(\xi^k) \to E(\iota^n)$ and the product projection $E(\iota^n) \to R^n$. The map $\tilde{j}$ represents each fibre of ξ^k as a k-plane in R^n.

We are concerned with two properties of the image $\operatorname{Im}\tilde{j}$ of $\tilde{j}$ in R^n. Can j, n be chosen so that, firstly, $\operatorname{Im}\tilde{j} = R^n$ and, secondly, the image of no fibre under $\tilde{j}$ is covered completely by the images of the other fibres? Bundles ξ^k for which such a monomorphism j does exist are called *exact fillings*, and these are the main objects of our study. Precise definitions will be found in section 2.

In section 3 we collect a number of examples of exact fillings which are familiar from projective geometry. We also derive two methods for constructing new exact fillings from given exact fillings.

In section 4 we assume that X is a closed manifold and that ξ^k is an *interior filling* (for the definition see section 2; all known examples of exact fillings on manifolds are interior fillings). We deduce a condition on the characteristic classes of ξ^k which imposes severe restrictions on X and ξ^k: for example, ξ^k cannot be the tangent bundle of X (in the case when X is differentiable), nor can X be homeomorphic to a q-sphere unless $q = 1, 2, 4$, or 8.

An interesting class of exact fillings arises from differential geometry. The familiar concept of curves of constant width can be generalized to yield the concept of a *transnormal manifold*. It is shown in (11) that the normal planes of a transnormal manifold determine an interior filling in a natural way. The methods of this paper can then be used to obtain further results on transnormal manifolds.

The concepts of exact filling, interior filling and transnormal manifold were introduced by one of us (S. A. R.) in (8), (10) [see also (9), (11)]. The problem of classifying all exact fillings, or even all interior fillings on closed manifolds, remains largely unsolved. For example, we do not know whether there exist interior fillings on any closed surface X with negative Euler characteristic.

2. *Exact fillings.* The elementary results on exact fillings apply equally to real, complex and quaternionic vector bundles. We therefore let K denote any one of the real division algebras R (reals), C (complex numbers) and H (quaternions). Let K^m denote the (left) vector space of m dimensions over K.

Throughout this paper X will be a finite dimensional paracompact space and ξ^k

https://doi.org/10.1017/S0305004100039220 Published online by Cambridge University Press

a k-plane bundle on X with projection map $\pi: E(\xi^k) \to X$. These assumptions imply (see Milnor ((6)), Lemma 2·19) that there is a finite open covering $\{U_i; i = 1, \ldots, r\}$ of X such that $\xi^k|U_i$ is trivial for each i. Let $f_i: E(\xi^k|U_i) \to K^k$ be given by a trivialization and let $\{\phi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Define a function

$$E(\xi^k) \to X \times K^k \times \ldots \times K^k = X \times K^{rk}$$

by $e \to (\pi(e), \phi_1(\pi(e))f_1(e), \ldots, \phi_r(\pi(e))f_r(e))$. It is easy to check that this defines a monomorphism $j: \xi^k \to \iota^{rk}$. Hence (see Milnor ((6)), Theorem 2·20) we have

PROPOSITION 2·1. *For some $n \geqslant k$ there is a monomorphism $j: \xi^k \to \iota^n$.*

There is a commutative diagram of maps

$$\begin{array}{ccccc} & & E(\xi^k) & & \\ & \swarrow^{\pi} & \downarrow & \searrow^{\tilde{j}} & \\ X & \leftarrow & E(\iota^n) = X \times K^n & \to & K^n \end{array}$$

in which the horizontal arrows are product projections and the vertical arrow is induced by the monomorphism j.

DEFINITION 2·2. The monomorphism $j: \xi^k \to \iota^n$ is *exact* if, for all subspaces X_1, X_2 of X, $\tilde{j}(\pi^{-1}(X_1)) = \tilde{j}(\pi^{-1}(X_2))$ implies $X_1 = X_2$.

DEFINITION 2·3. The monomorphism $j: \xi^k \to \iota^n$ *fills* K^n if $\operatorname{Im}\tilde{j} = K^n$.

PROPOSITION 2·4. *If X is separable and metrizable then for some $n \geqslant k$ there exists an exact monomorphism $j: \xi^k \to \iota^n$.*

Proof. In this case each U_i is separable and metrizable. Hence there are, for some integer b depending on the dimension of X, embeddings $r_i: U_i \to K^b$ such that $r_i(U_i)$ is contained in the upper hemisphere of the unit sphere in K^b. Define a function

$$j_i: E(\xi^k/U_i) \to K^b \times K^{k-1} = K^{b+k-1}$$

by $j_i(e) = (y_1 r_i(\pi(e)), (y_2, \ldots, y_k))$ whenever $f_i(e) = (y_1, \ldots, y_k)$. Then the function

$$E(\xi^k) \to X \times K^{b+k-1} \times \ldots \times K^{b+k-1} = X \times K^{r(b+k-1)},$$

given by $e \to (\pi(e), \phi_1(\pi(e))j_1(e), \ldots, \phi_r(\pi(e))j_r(e))$, defines a monomorphism $j: \xi^k \to \iota^n$ where $n = r(b+k-1)$. It is straightforward to check that j is exact.

PROPOSITION 2·5. *For some $n \geqslant k$ there exists a monomorphism $j: \xi^k \to \iota^n$ that fills K^n.*

Proof. Consider a monomorphism $h: \xi^k \to \iota^m$ for some $m \geqslant k$. If $\operatorname{Im}\tilde{h} = K^m$ we choose $n = m$. Otherwise we assume, without loss of generality, that

$$(0, \ldots, 0, 1) \in K^m - \operatorname{Im}\tilde{h}.$$

If $r: \iota^m \to \iota^{m-1}$ is defined by $(x, (y_1, \ldots, y_m)) \to (x, (y_1, \ldots, y_{m-1}))$ then $rh: \xi^k \to \iota^{m-1}$ is a monomorphism. After repeating this argument not more than $m-k$ times we obtain a monomorphism $j: \xi^k \to \iota^n$ for which $\operatorname{Im}\tilde{j} = K^n$.

The above properties show that it is easy to construct monomorphisms $j: \xi^k \to \iota^n$ which are exact and to construct monomorphisms $j: \xi^k \to \iota^n$ that fill K^n. It is, however, a grave restriction on ξ^k to ask for a monomorphism having both properties.

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DEFINITION 2·6. An *exact filling*, or an *exact filling of* K^n, is a k-plane bundle ξ^k together with a monomorphism $j: \xi^k \to \iota^n$ for some $n \geqslant k$ that is both exact and fills K^n.

We conclude this section with some definitions and notations involving projective fibre bundles. Let P^{k-1} denote the $(k-1)$-dimensional projective space defined over K. There is associated to ξ^k a projective fibre bundle $P\xi^k$ with fibre P^{k-1}, bundle space $PE(\xi^k)$, and projection map $P\pi: PE(\xi^k) \to X$. A monomorphism $j: \xi^k \to \iota^n$ determines a commutative diagram of maps

$$\begin{array}{ccccc} & & PE(\xi^k) & & \\ & \swarrow^{P\pi} & \downarrow & \searrow^{P\tilde{\jmath}} & \\ X & \leftarrow & PE(\iota^n) = X \times P^{n-1} & \to & P^{n-1} \end{array}$$

in which the horizontal arrows are product projections and the vertical arrow is induced by the monomorphism j. We say that $b \in PE(\xi^k)$ is *simple* if $(P\tilde{\jmath})^{-1} P\tilde{\jmath}(b) = b$.

DEFINITION 2·7. An *interior filling* is an exact filling for which the set of simple points has a non-empty interior.

3. *Some examples.* In this section we first describe three examples of interior fillings. In the first two cases (3·1–3·2) the base X is a projective space; in the third (3·3) it is a certain open manifold. We then describe two methods of constructing new exact fillings from old by using Whitney sums (3·4) and projective fibre bundles (3·5). The latter construction yields examples of exact fillings on compact manifolds X which are not products of projective spaces. This answers a question in (8), section 1. Further examples of exact fillings, which arise in the study of transnormal manifolds, are given in (11).

3·1. *Hopf bundles.* Let ζ be the natural line bundle which associates to each point $b \in P^{n-1}$ the corresponding 1-plane in K^n. Then ζ is called the *Hopf bundle* on P^{n-1}. The definition implies that there is a monomorphism $j: \zeta \to \iota^n$. Clearly $\tilde{\jmath}$ fills K^n and is exact. Moreover, the map $P\tilde{\jmath}: PE(\zeta) \to P^{n-1}$ is a homeomorphism and hence every point of $PE(\zeta)$ is simple. Therefore ζ is an interior filling of K^n.

Remark. It is clear that every (complex) exact filling ξ^k of C^n determines a (real) exact filling η^{2k} of R^{2n}. Here η^{2k} is the $2k$-plane bundle obtained by ignoring the complex structure of ξ^k. Similarly, every quaternionic exact filling ξ^k of H^n determines a real exact filling η^{4k} of R^{4n} as well as a complex exact filling θ^{2k} of C^{2n}.

Using this remark we obtain

THEOREM 3·1. *Let $d = 1, 2$ or 4 according as $K = R, C$ or H. There exists a real d-plane bundle ξ^d on the $d(k-1)$-dimensional manifold P^{k-1} which is an interior filling of R^{dk}.*

In a similar way the Cayley numbers can be used to construct a real 8-plane bundle over an 8-sphere which is an interior filling of R^{16}.

3·2. *Secant bundles of rational normal curves.* For this example we assume $K = C$, so that P^r is homeomorphic to the r-fold symmetric product of P^1. Let $i: P^1 \to P^{n-1}$ be an embedding of P^1 as a rational normal curve of degree $n-1$. Every set of r points of $i(P^1)$ determines an r-secant $(r-1)$-plane P^{r-1} in P^{n-1} which is spanned by the set; for coincident points this is an appropriate osculating space.

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The set of r-secant $(r-1)$-planes may be represented by a projective fibre bundle $P\xi^r_{(n)}$ on P^r and a monomorphism $j\colon \xi^r_{(n)} \to \iota^n$, where $\xi^r_{(n)}$ is a complex r-plane bundle on P^r. In the special case $n = 2r$ there is exactly one r-secant $(r-1)$-plane through each point of $P^{n-1} - i(P^1)$. By the remark in 3·1 the real $2r$-plane bundle defined by $\xi^r_{(n)}$ is an interior filling of R^{4r}. The bundle dual to $\xi^r_{(n)}$ is called the *secant bundle* of the curve $i(P^1)$. Further details can be found in (12), section 1, and (13).

3·3. *An open base manifold.* For this example we assume $K = R$. Let Γ be a right circular cone in R^3 with vertex 0, whose generators form the set $\{g_s;\ s \in S^1\}$. Fix a base point $* \in S^1$, identify R^k with $R^3 \oplus R^{k-3}$ $(k \geqslant 3)$ and write Λ for the family of $(k-1)$-planes in R^k that contain $g_* \oplus R^{k-3}$. Then $\Lambda = \{\lambda_s\}$, where λ_s is the unique $(k-1)$-plane in Λ that contains g_s, if $s \neq *$, and where λ_* is tangent to Γ along g_*. Now let M denote the set of 2-planes B such that $\lambda_s \supset B \supset g_s$ for some $s \in S^1$, and

$$A = \{B;\ B \in M,\ B \subset g_* \oplus R^{k-3}\}.$$

Then M is a closed subspace of the Grassmannian of 2-planes in R^k and A is a closed subspace of M. Define $X = M - A$. Then $X = M$ for $k = 3$ and X is an open $(k-2)$-manifold for $k > 3$.

Let ξ^2 be the 2-plane bundle on X which associates to each point of X the corresponding plane $B \subset R^3 \oplus R^{k-3}$. There is a monomorphism $j\colon \xi^2 \to \iota^k$ which is exact and fills R^k. It is trivial to check that ξ^2 is an interior filling of R^k.

For $k = 3$, the bundle ξ^2 is the non-trivial 2-plane bundle on S^1. For $k = 4$ it is a non-trivial 2-plane bundle on a punctured Klein bottle.

3·4. *Whitney sums.* Let ξ^k be a k-plane bundle over X, and η^l an l-plane bundle over Y. The product projections $p\colon X \times Y \to X$, $q\colon X \times Y \to Y$ define bundles $p^*\xi^k$, $q^*\eta^l$ over $X \times Y$.

THEOREM 3·4. *Let ξ^k be an exact filling of K^n on X and η^l an exact filling of K^m on Y. Then the $(k+l)$-plane bundle $p^*\xi^k \oplus q^*\eta^l$ on $X \times Y$ is an exact filling of K^{n+m}.*

Proof. Let $j\colon \xi^k \to \iota^n$ and $i\colon \eta^l \to \iota^m$ be the monomorphisms defining the exact fillings. The monomorphism $p^*(j) \oplus q^*(i)\colon p^*\xi^k \oplus q^*\eta^l \to p^*\iota^n \oplus q^*\iota^m = \iota^{n+m}$ defines the required exact filling.

COROLLARY. *If ξ^k is an exact filling of K^n then $\xi^k \oplus \iota^m$ is an exact filling of K^{n+m}.*

Proof. Take Y to be a single point.

Remark. The proofs show that both Theorem 3·4 and its corollary remain true if 'exact' is replaced by 'interior' throughout.

The corollary implies that if ζ is the Hopf bundle on P^{n-1} then $\zeta \oplus \iota^m$ is an exact filling of K^{n+m}. The corresponding map $P\tilde{\jmath}\colon PE(\zeta \oplus \iota^m) \to P^{n+m-1}$ has the following interpretation in projective geometry. Let P^{m-1} and P^{n-1} be disjoint projective subspaces of P^{n+m-1}. Then $P\tilde{\jmath}$ maps the fibre of $PE(\zeta \oplus \iota^m)$ over $x \in P^{n-1}$ to the m-plane spanned by x and the $(m-1)$-plane P^{m-1}. Clearly the set of simple points is $(P\tilde{\jmath})^{-1}(P^{n+m-1} - P^{m-1})$ and therefore $\zeta \oplus \iota^m$ is an interior filling.

3·5. *Projective fibre bundles.* Let α be an r-plane bundle on X. By Proposition 2·1 there exists an integer s and an s-plane bundle β on X such that $\alpha \oplus \beta = \iota^{r+s}$ is the trivial $(r+s)$-plane bundle on X. The projective fibre bundle associated to α has fibre

https://doi.org/10.1017/S0305004100039220 Published online by Cambridge University Press

P^{r-1} and projection map $\varpi: PE(\alpha) \to X$. The vector bundle $\varpi^*\alpha$ on $PE(\alpha)$ has a subbundle ζ with fibre K which associates to each point $b \in PE(\alpha)$ the corresponding line in the fibre of α over $\varpi(b)$. There is a monomorphism $h: \zeta \to \varpi^*\alpha$ such that, for each $x \in X$, the restriction to $\varpi^{-1}(x) = P^{r-1}$ is the monomorphism defined in section 3·1.

THEOREM 3·5. *Let α be an r-plane bundle on X, and let s, β, $PE(\alpha)$, ϖ, ζ be defined as above. Let ξ be a vector bundle on X which is an exact filling of K^n. Then the vector bundle $\varpi^*\xi \oplus \zeta \oplus \varpi^*\beta$ over $PE(\alpha)$ is an exact filling of K^{n+r+s}.*

Proof. Let $j: \xi \to \iota^n$ be the monomorphism which defines the exact filling. The monomorphism $\varpi^*j: \varpi^*\xi \to \iota^n$ fills K^n but is not in general exact. Consider the monomorphisms

$$\varpi^*\xi \oplus \zeta \oplus \varpi^*\beta \xrightarrow{\omega^*j \oplus 1 \oplus 1} \iota^n \oplus \zeta \oplus \varpi^*\beta \xrightarrow{1 \oplus h \oplus 1} \iota^n \oplus \varpi^*\alpha \oplus \varpi^*\beta = \iota^{n+r+s}.$$

The composite monomorphism $\varpi^*\xi \oplus \zeta \oplus \varpi^*\beta \to \iota^{n+r+s}$ clearly fills K^{n+r+s}. It is also exact.

COROLLARY. *There is a real 3-plane bundle on the Klein bottle which is an exact filling of R^5.*

Proof. Take $K = R$ and α the non-trivial 2-plane bundle over S^1. Then $PE(\alpha)$ is a Klein bottle. We can choose ξ, β to be the non-trivial 1-plane bundle over S^1.

4. *Interior fillings on closed manifolds.* Throughout this paragraph we use the following notations which allow results to be interpreted for real, complex or quaternionic exact fillings. Let $G = Z_2$, Z or Z, and $d = 1, 2$ or 4 according as $K = R$, C or H. Let X be a compact connected dq-manifold without boundary, and write $H^*(X)$ for $H^*(X, G)$. We assume further that X is G-orientable (for $K = R$ this is no restriction) and that the k-plane bundle ξ^k on X is an interior filling of K^n.

The interior filling ξ^k is defined by a monomorphism $j: \xi^k \to \iota^n$. The simple points of ξ^k form a subset $S(j)$ of $PE(\xi^k)$ with non-empty interior $\operatorname{Int} S(j)$.

LEMMA 4·1. *There exists a $d(q+k-1)$-dimensional disc D in $S(j)$ such that $P\tilde{j}|D$ is a homeomorphism.*

Proof. Let $e \in \operatorname{Int} S(j)$ and set $x = P\pi(e)$, $b = P\tilde{j}(e) \in P^{n-1}$. Choose a $d(q+k-1)$-dimensional disc D in $\operatorname{Int} S(j)$ which contains e and put $M = P\pi(\operatorname{Int} D)$. For a suitably chosen (for instance, convex) D the set $\overline{M}$ is a dq-dimensional disc and $x \in M$. Giving P^{n-1} its standard Riemannian metric induced by the Euclidean metric of R^n, define $\phi: (X - M) \to R$ by setting

$$\phi(y) = \rho(b, P\tilde{j}(P\pi^{-1}(y))).$$

Since $b \notin P\tilde{j}(P\pi^{-1}(y))$ for any $y \in X - M$, and since $X - M$ is compact, it follows that $\phi(y) \geqslant \alpha > 0$ for some α and for all $y \in X - M$. Thus the open Riemannian disc

$$R_b = \{b'; b' \in P^{n-1}, \rho(b, b') < \alpha\}$$

is disjoint from $\tilde{j}(P\pi^{-1}(X - M))$. Further, since $P\tilde{j}$ is an embedding on each fibre of $PE(\xi^k)$, there is a disc neighbourhood V of b such that $V \subset R_b$ and V does not meet $P\tilde{j}(PE(\xi^k) - D)$. But $\operatorname{Im} P\tilde{j} = P^{n-1}$. Hence $V \subset P\tilde{j}(D)$. Without loss of generality we may assume that $V = P\tilde{j}(D)$. Thus $P\tilde{j}|D$ is a homeomorphism $D \to V$.

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THEOREM 4·2. *Under the above assumptions,* $q+k=n$.

Proof. By invariance of domain,

$$d(q+k-1) = \dim D = \dim V = \dim P^{n-1} = d(n-1).$$

Therefore $q+k=n$.

We now recall certain facts about the cohomology ring $H^*(PE(\xi^k))$. Let

$$c_i(\xi^k) \in H^{di}(X)$$

denote the ith Whitney, Chern or symplectic Pontrjagin class of ξ^k according as $K = R, C$ or H. We write

$$\sum_{i=0}^{q} c_i(\xi^k) = c(\xi^k) \quad \text{and} \quad \sum_{i=0}^{q} \bar{c}_i(\xi^k) = \bar{c}(\xi^k) = (c(\xi^k))^{-1}.$$

Let $P\pi\colon PE(\xi^k) \to X$ be the projection map and ζ the 1-plane bundle over $PE(\xi^k)$ defined in section 3·5. There is an exact sequence (see section 3·5)

$$0 \to \zeta \to P\pi^*\xi^k \to \gamma \to 0$$

in which, since γ is a $(k-1)$-plane bundle, $c_k(\gamma) = 0$. Let $h = -c_1(\zeta)$. Then

$$c(\gamma) = \bar{c}(\zeta)\, P\pi^* c(\xi^k) = \left(\sum_{i=0}^{q+k-1} h^i\right)\left(\sum_{j=0}^{k} P\pi^* c_j(\xi^k)\right)$$

and hence

$$c_k(\gamma) = h^k + h^{k-1} P\pi^* c_1(\xi^k) + \ldots + P\pi^* c_k(\xi^k) = 0. \tag{1}$$

In fact $P\pi^*\colon H^*(X) \to H^*(PE(\xi^k))$ is a monomorphism, and $H^*(PE(\xi^k))$ is a module over the image of $P\pi^*$ with a single generator h and a single relation (1) [for details see (1), (3)]. The definition of h implies that, if ξ' is a vector bundle over X' and $f\colon \xi^k \to \xi'$ is a bundle monomorphism, then $Pf^*h' = h$. Here $Pf\colon PE(\xi^k) \to PE(\xi')$ is the embedding determined by f and h' is the generator of $H^*(PE(\xi'))$ corresponding to h. In particular the restriction of h to any fibre $P\pi^{-1}(x) = P^{k-1}$ is the positive generator of $H^*(P^{k-1})$.

Since X is G-orientable, every element $b \in H^{dq}(X)$ can be evaluated on the fundamental cycle $[X]$ to give an element $b.[X] \in G$. Poincaré duality defines a homomorphism $P\pi_*\colon H^*(PE(\xi^k)) \to H^*(X)$ which preserves co-dimension. Let $[P\pi^{-1}(x)]$ be the homology cycle represented by a fibre. Then $h^{k-1}.[P\pi^{-1}(x)] = 1$ and therefore $P\pi_* h^{k-1}.[P\pi(P\pi^{-1}(x))] = 1$. It follows that

$$P\pi_* h^{k-1} = 1, \quad P\pi_* h^i = 0 \quad \text{for} \quad i < k-1. \tag{2}$$

LEMMA 4·3. $\bar{c}_j(\xi^k) = P\pi_* h^{j+k-1}$ *for* $0 \leqslant j \leqslant q$.

Proof. Relation (1) can be written in the form

$$\left(\sum_{i=0}^{q+k-1} h^i\right) P\pi^* c(\xi^k) = h^{k-1} + \sum_{l<k-1} h^l P\pi^* b_l,$$

where $b_l \in H^*(X)$. Since $P\pi_*(a.P\pi^* b) = P\pi_* a.b$ we deduce from (2) that

$$\left(\sum_{j=0}^{q} P\pi_* h^{j+k-1}\right) c(\xi^k) = P\pi_* h^{k-1} = 1.$$

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THEOREM 4·4. *Let ξ^k be an interior filling on a manifold X satisfying the above assumptions. Then $\bar{c}_q(\xi^k).[X] = \pm 1$.*

Proof. Choose a G-orientation on X, and the induced G-orientation on $PE(\xi^k)$. Since j fills K^n the degree, and therefore also the geometric degree (see (4)) of $P\tilde{j}\colon PE(\xi^k) \to P^{n-1}$ is non-zero. By Lemma 4·1 the geometric degree is 1. Therefore the absolute degree (see (4)) is 1. It follows that, if $h' \in H^d(P^{n-1})$ is the positive generator,

$$h^{n-1}.[PE(\xi^k)] = (P\tilde{j}^*h')^{n-1}.[PE(\xi^k)] = \pm (h')^{n-1}.P^{n-1} = \pm 1.$$

Hence $P\pi_* h^{n-1}.[X] = \bar{c}_q(\xi^k).[X] = \pm 1$. Note that the sign depends on the choice of orientation for X.

COROLLARY 4·5. *Let X be a differentiable q-manifold ($q > 0$). The tangent bundle τ of X is not an interior filling.*

Proof. By a theorem of Wu (see, for instance, (5)), the dual Stiefel–Whitney class $\bar{w}_q(\tau)$ of X is zero.

COROLLARY 4·6. *Let X be an m-sphere and ξ^k an interior filling on X. Then $m = 1, 2, 4$ or 8 if $K = R$; $m = 2$ or 4 if $K = C$; $m = 4$ if $K = H$.*

Proof. By Theorem 4·2, m is a multiple dq of d. Let η^l be an l-plane bundle on X such that $\xi^k \oplus \eta^l = \iota^{k+l}$. Then by Theorem 4·4, $c_q(\eta^l) = \pm 1$. If $K = R$ a theorem of Milnor (7) implies that $c_q(\eta^l) = 0$ unless $q = 1, 2, 4$ or 8. If $K = C$ then a theorem of Bott (2) implies that $c_q(\eta^l)$ is divisible by $(q-1)!$ and therefore $q = 1$ or 2. If $K = H$ then the same theorem implies that $c_q(\eta^l)$ is divisible by $(2q-1)!$ and therefore $q = 1$. The examples in section 3·1 and section 3·4 show that all the cases mentioned actually occur for any $k \geqslant 1$.

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