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QUANTIFIED MODAL LOGIC ON THE RATIONAL LINE

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Abstract. In the topological semantics for propositional modal logic, S4 is known to be complete for the class of all topological spaces, for the rational line, for Cantor space, and for the real line. In the topological semantics for quantified modal logic, QS4 is known to be complete for the class of all topological spaces, and for the family of subspaces of the irrational line. The main result of the current paper is that QS4 is complete, indeed strongly complete, for the rational line.

In the topological semantics for propositional modal logic (McKinsey, 1941; McKinsey & Tarski, 1944; Rasiowa & Sikorski, 1963), it is well known that

(S4*all*) S4 is complete for the class of all topological spaces,
(S4$\mathbb{Q}$) S4 is complete for the rational line,
(S4$\mathcal{C}$) S4 is complete for Cantor space, and
(S4$\mathbb{R}$) S4 is complete for the real line.[1]

Rasiowa & Sikorski (1963) extend the topological semantics to quantified modal logic. Let QS4 be classical first-order logic, without identity, enriched with a modal operator $\Box$ satisfying the axioms of S4. The four above results suggest four conjectures:

(QS4*all*) QS4 is complete for the class of all topological spaces,
(QS4$\mathbb{Q}$) QS4 is complete for the rational line,
(QS4$\mathcal{C}$) QS4 is complete for Cantor space, and
(QS4$\mathbb{R}$) QS4 is complete for the real line.

Rasiowa & Sikorski (1963) prove (QS4*all*) but leave (QS4$\mathbb{Q}$), (QS4$\mathcal{C}$), and (QS4$\mathbb{R}$) open.

(QS4$\mathbb{R}$) fails, since QS4 is not complete for any locally connected space (Theorem 3.4). The main result of the current paper is a strong version of (QS4$\mathbb{Q}$): QS4 is complete, indeed strongly complete, for $\mathbb{Q}$ with a constant countably infinite domain (Theorem 2.5). This result follows from a more general strong completeness theorem, Theorem 6.1.[2] (QS4$\mathcal{C}$) remains open.

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[1] See Rasiowa & Sikorski (1963), Theorem XI, 9.1, which is derived from McKinsey (1941) and McKinsey & Tarski (1944) . (S4$\mathbb{Q}$), (S4$\mathcal{C}$), and (S4$\mathbb{R}$) are special cases of Rasiowa & Sikorski (1963), Theorem XI, 9.1, (vii), which states that S4 is complete for any dense-in-itself metric space. Kremer (2013) strengthens this: S4 is *strongly* complete for any dense-in-itself metric space.

[2] The proof here grew out of my work on two-dimensional propositional modal logic: after reading that work, Valentin Shehtman drew my attention to its connection to topologically interpreted quantified modal logic. In particular, he alerted me to the result in Rasiowa & Sikorski (1963) that QS4 is sound and complete for all topological spaces with a constant domain, and claimed without proof that, in a language without function symbols, QS4 is complete for $\mathbb{Q}$ with a constant domain. The proof in the current paper is an application of the technique I had originally

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Analogous questions arise in the topological semantics for intuitionstic logic, as emphasized by an anonymous referee. In particular, it is well known (see McKinsey, 1941; McKinsey & Tarski, 1944; Rasiowa & Sikorski, 1963) that propositional H is complete for the class of all topological spaces, for the rational line, for Cantor space, and for the real line; regarding quantified intuitionistic logic, QH, it is natural to form conjectures (QH*all*), (QH$\mathbb{Q}$), (QH$\mathcal{C}$) and (QH$\mathbb{R}$) analogous to (QS4*all*), (QS4$\mathbb{Q}$), (QS4$\mathcal{C}$), and (QS4$\mathbb{R}$). The referee notes that if QS4 is complete for a space [with a constant domain D], then QH is complete for that space [with D]. A related observation is that if QS4 is strongly complete for a space [with a constant domain D], then QH is strongly complete for that space [with D]. So we have not only (QH*all*), already proved in Rasiowa & Sikorski (1963), but also a strong version of (QH$\mathbb{Q}$): QH is strongly complete for $\mathbb{Q}$ with a constant countably infinite domain. (QH$\mathbb{R}$) fails, since QH is not complete for any locally connected space (Remark 3.6, below). (QH$\mathcal{C}$) remains open, though a closely related conjecture is proved in Dragalin (1988) (see Remark 3.8, below). We will indicate in some footnotes and side remarks what happens in the intuitionistic case, though we will leave many of the details in the intuitionistic case to the reader.

§1. Preliminaries. Let $\mathcal{L}$ be a quantified modal language with a countably infinite set Var of variables; disjoint countable sets Pred_n of n-ary predicate symbols, for each $n \geq 1$; a set Names of names; disjoint countable sets Func_n of n-ary function symbols, for each $n \geq 1$; connective &, $\vee$, and $\neg$; a modal operator $\Box$; a quantifier $\forall$; and parentheses. We write $\Diamond A$ for $\neg\Box\neg A$, $(A \supset B)$ for $(\neg A \vee B)$, and $\exists x A$ for $\neg\forall x \neg A$. Let $\mathsf{Pred} = \bigcup_n \mathsf{Pred}_n$ and $\mathsf{Func} = \bigcup_n \mathsf{Func}_n$; we assume that Pred is nonempty. Note that $\mathcal{L}$ has no equals sign. If A is a formula, t is a term, and x is a variable, then $[\mathsf{t}/\mathsf{x}]A$ is the result of replacing every free occurrence of x in A with t. We say that t is *substitutible for* x *in* A iff no free occurrence of x in A is in the scope of any bound variable y, where y occurs in t. Given any set D, *D-terms*, *D-formulas*, and *D-sentences* are terms, formulas, and sentences in the language $\mathcal{L}(D)$, which is the result of expanding the language $\mathcal{L}$ so that every member of the set D is a name of $\mathcal{L}$. (Here we assume that $D \cap S = \emptyset$, if $S = \mathsf{Var}$, Pred, Names, or Func.) It will be useful to let $\mathsf{Term}(D)$ be the set of closed D-terms. Note that if D is infinite, then D and $\mathsf{Term}(D)$ have the same cardinality, since Names and Func are countable. Also note that, given any D-formula A, any variable x, and any $d \in D$, the D-sentence $[d/\mathsf{x}]A$ is the result of replacing every occurrence of x in A with d. We reserve the unprefixed expressions 'formula(s)' and 'sentence(s)' for formulas and sentences in the original language $\mathcal{L}$.

Let QS4 be the logic axiomatized in $\mathcal{L}$ as follows:

- Axioms and axiom schemes:
  - every instance, in $\mathcal{L}$, of a theorem of propositional S4;
  - $(\forall \mathsf{x} A \rightarrow A[\mathsf{t}/\mathsf{x}])$, where the term t is substitutible for x in A;
  - $\forall \mathsf{x}(A \rightarrow B) \rightarrow (\forall \mathsf{x} A \rightarrow \forall \mathsf{x} B)$; and
  - $A \rightarrow \forall \mathsf{x} A$, where x does not occur free in A.
- Rules: modus ponens, necessitation, and universal generalization.

developed for two-dimensional modal logic, extended to account for function symbols in the quantified language.

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It is well known that $(\Box\forall \mathsf{x}\mathsf{P}\mathsf{x} \supset \forall \mathsf{x}\Box\mathsf{P}\mathsf{x}) \in$ QS4, but $(\forall \mathsf{x}\Box\mathsf{P}\mathsf{x} \supset \Box\forall \mathsf{x}\mathsf{P}\mathsf{x}) \notin$ QS4. A nonempty finite set Γ of sentences of $\mathcal{L}$ is *consistent* iff the negation of their conjunction is not a theorem of QS4. A possibly infinite nonempty set Γ is *consistent* iff every nonempty finite subset is consistent.[3]

§2. Topological semantics. The topological semantics in this section is a terminological/notational variant of the topological semantics for quantified modal logic found in Rasiowa & Sikorski (1963) with terminology and notation adapted from Gabbay et al. (2009). We assume familiarity with the basics of point-set topology: Dugundji (1966) is a standard reference. A *predicate topological space* is an ordered triple $\mathbf{X} = \langle X, \tau, D\rangle$, where $\mathcal{X} = \langle X, \tau\rangle$ is a topological space and D is a nonempty domain. We say that $\mathbf{X}$ is *based on* $\mathcal{X}$. A *predicate topological model* is an ordered quartuple $\mathbf{M} = \langle X, \tau, D, V\rangle$, where $\mathbf{X} = \langle X, \tau, D\rangle$ is a predicate topological space, and

$$V : (X \times \mathsf{Pred}) \cup \mathsf{Names} \cup \mathsf{Func} \to \bigcup_{n\geq 1} \mathcal{P}(D^n) \cup D \cup (\bigcup_{n\geq 1} D^{D^n})$$

is such that

- $V(x, \mathsf{P}) \subseteq D^n$ for every $\mathsf{P} \in \mathsf{Pred}_n$,
- $V(\mathsf{c}) \in D$ for every $\mathsf{c} \in \mathsf{Names}$, and
- $V(\mathsf{f}) : D^n \to D$ for every $\mathsf{f} \in \mathsf{Func}_n$.

We say that $\mathbf{M}$ is *based on* $\mathbf{X}$.[4]

Suppose that $\mathbf{M} = \langle X, \tau, D, V\rangle$ is a predicate topological model. First, we define $Val(\mathsf{t}) \in D$ for every closed D-term t: $Val(d) = d$, if $d \in D$; if $\mathsf{c} \in \mathsf{Names}$ then $Val(\mathsf{c}) = V(\mathsf{c})$; and if $\mathsf{f} \in \mathsf{Func}_n$ and $\mathsf{t}_1, \ldots, \mathsf{t}_n$ are terms then $Val(\mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n) = V(\mathsf{f})(Val(\mathsf{t}_1), \ldots, Val(\mathsf{t}_n))$. Next, we define $\mathbf{M}, x \vDash A$, for each $x \in X$ and each D-sentence A as follows:

$\mathbf{M}, x \vDash \mathsf{P}\mathsf{t}_1 \ldots \mathsf{t}_n$	iff	$\langle Val(\mathsf{t}_1), \ldots, Val(\mathsf{t}_n)\rangle \in V(x, \mathsf{P})$, where $\mathsf{P} \in \mathsf{Pred}_n$
$\mathbf{M}, x \vDash \neg A$	iff	$\mathbf{M}, x \nvDash A$
$\mathbf{M}, x \vDash (A \,\&\, B)$	iff	$\mathbf{M}, x \vDash A$ and $\mathbf{M}, x \vDash B$
$\mathbf{M}, x \vDash (A \vee B)$	iff	$\mathbf{M}, x \vDash A$ or $\mathbf{M}, x \vDash B$
$\mathbf{M}, x \vDash \Box A$	iff	for some open set O, $x \in O$ and for every $y \in O$, $\mathbf{M}, y \vDash A$
$\mathbf{M}, x \vDash \forall \mathsf{x} A$	iff	for every $d \in D$, $\mathbf{M}, x \vDash [d/\mathsf{x}]A$ [5]

If Γ is a nonempty set of sentences, we say that $\mathbf{M}, x \vDash \Gamma$ iff $\mathbf{M}, x \vDash A$, for every $A \in \Gamma$. We say that $\mathbf{M} \vDash \Gamma$ iff $\mathbf{M}, x \vDash A$ for every $x \in X$. We say that $\mathbf{X} \vDash \Gamma$ iff $\mathbf{M} \vDash \Gamma$ for every

[3] In the intuitionistic case, the language has four primitive connectives, &, $\vee$, $\neg$, and $\supset$, two quantifiers $\forall$ and $\exists$, an no modal operators. Axioms for quantified intuitionistic logic, QH, are easily in the literature. The consistency of a nonempty set of sentences is defined analogously to the modal case, but in the intuitionistic case there is a more important notion of consistency. A *pair* $\langle \Gamma, \Delta\rangle$ of nonempty finite sets of sentences is *consistent* iff the formula $\bigwedge \Gamma \supset \bigvee \Delta$ is not a theorem of QH. A pair $\langle \Gamma, \Delta\rangle$ of possibly infinite nonempty sets of sentences is *consistent* iff every pair $\langle \Gamma', \Delta'\rangle$ is consistent, where Γ' is a finite subset of Γ and Δ' is a finite subset of Δ. Note that a set Γ is consistent iff the pair $\langle \Gamma, \{\bot\}\rangle$ is consistent where $\bot$ is any contradiction.

[4] In the intuitionistic case, we stipulate that the following set must be open in $\langle X, \tau\rangle$, whenever $\mathsf{P} \in \mathsf{Pred}_n$ and $d_1, \ldots, d_n \in D$: $\{x \in X : \langle d_1, \ldots, d_n\rangle \in V(x, \mathsf{P})\}$.

[5] In the intuitionistic case, we use the same clauses as in the modal case for atomic formulas, for & and for $\vee$, together with the following:

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M based on **X**. We say that $\mathcal{X} \vDash \Gamma$ iff $\mathbf{X} \vDash \Gamma$ for every **X** based on $\mathcal{X}$. If $\mathfrak{X}$ is a class of topological spaces [of predicate topological spaces], then we say that $\mathfrak{X} \vDash \Gamma$ iff $\mathcal{X} \vDash \Gamma$ [$\mathbf{X} \vDash \Gamma$] for every $\mathcal{X} \in \mathfrak{X}$ [$\mathbf{X} \in \mathfrak{X}$]. We write $\mathbf{M} \vDash A$ for $\mathbf{M} \vDash \{A\}$, and similarly with $\mathcal{X} \vDash A$, $\mathbf{X} \vDash A$, and $\mathfrak{X} \vDash A$.[6]

If $\mathbf{X} = \langle X, \tau, D\rangle$ is a predicate topological space and $x \in X$, then a nonempty set Γ of sentences is *satisfiable at x in* **X** iff $\mathbf{M}, x \vDash \Gamma$ for some predicate topological model **M** based on **X**. And Γ is *satisfiable in* **X** iff Γ is satisfiable at some x in **X**. If $\mathcal{X} = \langle X, \tau\rangle$ is a topological space, then Γ is *satisfiable in* $\mathcal{X}$ iff Γ is satisfiable in some predicate topological space based on $\mathcal{X}$. If $\mathfrak{X}$ is a class of topological spaces [predicate topological spaces], then Γ is *satisfiable in* $\mathfrak{X}$ iff Γ is satisfiable in some $\mathcal{X} \in \mathfrak{X}$ [some $\mathbf{X} \in \mathfrak{X}$].[7] QS4 is *sound for* a class $\mathfrak{X}$ of [predicate] topological spaces iff $\mathfrak{X} \vDash A$, for every sentence $A \in$ QS4; and QS4 is *complete for* a class $\mathfrak{X}$ of [predicate] topological spaces iff $A \in$ QS4 for every sentence A with $\mathfrak{X} \vDash A$.[8] Note that QS4 is complete for $\mathfrak{X}$ iff every nonempty finite consistent set of sentences is satisfiable in $\mathfrak{X}$.[9] QS4 is *strongly complete* for $\mathfrak{X}$ iff every nonempty consistent set of sentences is satisfiable in $\mathfrak{X}$.[10] If $\mathcal{X}$ [**X**] is a topological space [predicate topological space], we say that QS4 is sound, complete, or strongly complete for $\mathcal{X}$ [**X**] iff QS4 is sound, complete, or strongly complete for $\{\mathcal{X}\}$ [$\{\mathbf{X}\}$].[11]

In what follows, we assume that $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$, and that, for nonempty $X \subseteq \mathbb{R}$, $\tau_{\mathcal{X}}$ is the subspace topology induced by $\tau_{\mathbb{R}}$. It is easy to show that QS4 is sound for any class of predicate topological models. The main QS4-completeness result in Rasiowa & Sikorski (1963) is

THEOREM 2.1 (Rasiowa & Sikorski, 1963, XI, 10.2).

1. QS4 *is complete for the class of all predicate topological spaces; and*
2. *if D is countably infinite, then* QS4 *is complete for the following set of predicate topological spaces:* $\{\langle X, \tau_X, D\rangle : \emptyset \neq X \subseteq (\mathbb{R} - \mathbb{Q})\}$.

Rasiowa & Sikorski (1963) strengthen Theorem 2.1 by showing how to construct a family $\mathfrak{X}_0$ of subsets of the irrationals, $\mathbb{R} - \mathbb{Q}$, such that

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$\mathbf{M}, x \vDash \neg A$ iff for some open set O, $x \in O$ and for every $y \in O$, $\mathbf{M}, y \nvDash A$

$\mathbf{M}, x \vDash (A \supset B)$ iff for some open set O, $x \in O$ and for every $y \in O$ if $\mathbf{M}, y \vDash A$ then $\mathbf{M}, y \vDash B$

$\mathbf{M}, x \vDash \forall \mathsf{x} A$ iff for some open set O, $x \in O$ and for every $y \in O$ and every $d \in D$, $\mathbf{M}, y \vDash [d/\mathsf{x}]A$

$\mathbf{M}, x \vDash \exists \mathsf{x} A$ iff for some $d \in D$, $\mathbf{M}, x \vDash [d/\mathsf{x}]A$

[6] The definitions are the same in the intuitionistic case.

[7] In the intuitionistic case, we add the following definitions, where Γ and Δ are nonempty sets of sentences. The pair $\langle \Gamma, \Delta\rangle$ is *satisfiable at x in* **X** iff, for some predicate topological model **M** based on **X**, we have both $\mathbf{M}, x \vDash \Gamma$ and $\mathbf{M}, x \nvDash A$, for every $A \in \Delta$. The definitions of $\langle \Gamma, \Delta\rangle$ being satisfiable in a [class of] [predicate] topological space[s] then carry over from the modal case.

[8] Analogous definitions are given in the intuitionistic case.

[9] In the intuitionistic case, note that QH is complete for $\mathfrak{X}$ iff every consistent pair of nonempty finite sets is satisfiable in $\mathfrak{X}$.

[10] In the intuitionistic case, QH is *strongly complete* for $\mathfrak{X}$ iff every pair of nonempty consistent set of sentences is satisfiable in $\mathfrak{X}$.

[11] Similarly, in the intuitionistic case.

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THEOREM 2.2 (Rasiowa & Sikorski, 1963, XI, 11.2). *For every* $X \in \mathfrak{X}_0$ *and every countably infinite set* D, QS4 *is complete for* $\langle X, \tau_X, D\rangle$.[12]

REMARK 2.3. The construction in Rasiowa & Sikorski (1963) of the sets in $\mathfrak{X}_0$ is highly abstract, and reveals little about them. Rasiowa & Sikorski (1963) do point out that the set of all irrationals cannot be among them (p. 486). A careful investigation of the construction reveals that each $X \in \mathfrak{X}_0$ is of cardinality $2^{\aleph_0}$.

REMARK 2.4. Rasiowa & Sikorski (1963) prove exact intuitionistic analogs of Theorems 2.1 and 2.2, with QS4 replaced by QH: see Rasiowa & Sikorski (1963), X, 4.1 and 4.2. Rasiowa & Sikorski (1963) explicitly leave open the question of whether QH is complete for $\mathbb{R}$ with a countable domain and whether QH is complete for $(\mathbb{R} - \mathbb{Q})$ with a countable domain. The answers are no and yes: see Remarks 3.6 and 3.8, below, respectively. Indeed, as noted in Remark 3.6, QH is not complete for $\mathbb{R}$ regardless of the size of the domain.

The main result of the current paper strengthens Theorems 2.1 and 2.2:

THEOREM 2.5 (**Main Result**). *Suppose that* D *is countably infinite. Then* QS4 *is strongly complete for* $\langle \mathbb{Q}, \tau_{\mathbb{Q}}, D\rangle$, *and thus for* $\mathbb{Q}$.[13]

Given Lemma 6.2, Theorem 2.5 is a special case of Theorem 6.1.

REMARK 2.6. It follows from Theorem 2.5 that, if D is countably infinite, then QH is strongly complete for $\langle \mathbb{Q}, \tau_{\mathbb{Q}}, D\rangle$, and thus for $\mathbb{Q}$.

§3. Negative results and open questions. Rasiowa & Sikorski (1963) show that QS4 is *not* complete for any Baire space[14] with a countable domain. More precisely, if $\langle X, \tau\rangle$ is a Baire space, then QS4 is not complete for the following class of predicate topological spaces: $\{\langle X, \tau, D\rangle : D \text{ is nonempty and countable}\}$. In particular, let P be a unary predicate and let A be the following formula, from Rasiowa & Sikorski (1963, p. 487):

$$\neg\Box\exists\mathsf{x}(\mathsf{Px} \,\&\, \neg\Box\Diamond\mathsf{Px}).$$

Rasiowa & Sikorski (1963) show that A is not a theorem of QS4, but that $\langle X, \tau, D\rangle \vDash A$ whenever $\langle X, \tau\rangle$ is Baire and D is countable. Thus for example, QS4 is not complete for $\mathbb{R}$ with a countable domain, or for Cantor space with a countable domain. In the case of $\mathbb{R}$, we can strengthen this result: See Theorem 3.4.

REMARK 3.1. The assumption that the domain D is countable is required for this example to go through: note, for example, that $\langle \mathbb{R}, \tau_{\mathbb{R}}, \mathbb{R}\rangle \nvDash \neg\Box\exists\mathsf{x}(\mathsf{Px} \,\&\, \neg\Box\Diamond\mathsf{Px})$. To see this, let V be any valuation function such that $V(r, \mathsf{P}) = \{r\}$, for each $r \in \mathbb{R}$, and let $\mathbf{M} = \langle \mathbb{R}, \tau_{\mathbb{R}}, \mathbb{R}, V\rangle$. Then $\mathbf{M}, r \vDash \mathsf{P}r \,\&\, \neg\Box\Diamond\mathsf{P}r$, for each $r \in \mathbb{R}$. So $\mathbf{M}, r \vDash \exists\mathsf{x}(\mathsf{Px} \,\&\, \neg\Box\Diamond\mathsf{Px})$, for each $r \in \mathbb{R}$. So $\mathbf{M}, r \vDash \Box\exists\mathsf{x}(\mathsf{Px} \,\&\, \neg\Box\Diamond\mathsf{Px})$, for each $r \in \mathbb{R}$.

[12] The presentation in Rasiowa & Sikorski (1963) suggests that only one such subset X_0 of the irrationals is defined, but an inspection of the proof reveals that the construction does not determine a unique X_0.

[13] Clearly, the only feature of the domain of quantification relevant to strong completeness is its cardinality. So a proof of Theorem 2.5 for one countably infinite D suffices for every countably infinite D.

[14] A topological space $\mathcal{X}$ is a *Baire space* if the intersection of every countable family of open dense sets in $\mathcal{X}$ is dense. See, for example, Dugundji (1966, p. 249, Definition 10.1).

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So $\mathbf{M} \vDash \Box\exists\mathsf{x}(\mathsf{Px} \,\&\, \neg\Box\Diamond\mathsf{Px})$. So $\langle \mathbb{R}, \tau_{\mathbb{R}}, \mathbb{R}\rangle \nvDash \neg\Box\exists\mathsf{x}(\mathsf{Px} \,\&\, \neg\Box\Diamond\mathsf{Px})$. (This argument goes through for any space in which every point is in the boundary of an open set.)

REMARK 3.2. Rasiowa & Sikorski (1963) have no similar counterexample in the case of QH for Baire spaces with a countable domain. As it turns out, QH is not complete for $\mathbb{R}$: see Remark 3.6. But there is a Baire space for which QH is complete with a countable domain: see Remark 3.8.

DEFINITION 3.3. *A space $\mathcal{X} = \langle X, \tau\rangle$ is* connected *if it is not the union of two nonempty disjoint open sets. A subset $S \subseteq X$ is* connected *(in $\mathcal{X}$) if it is connected as subspace of $\mathcal{X}$. Note that an open subset of a space $\mathcal{X}$ is connected iff it is not the union of two nonempty disjoint open sets. A space is* locally connected *if it has a basis consisting of connected open sets. Note that $\mathbb{R}$ is locally connected, since it has as a basis the family of open intervals.*

THEOREM 3.4. QS4 *is not complete for any locally connected space. In particular,* QS4 *is not complete for* $\mathbb{R}$.

Proof. Let $\mathcal{X} = \langle X, \tau\rangle$ be a locally connected topological space with a basis $\mathcal{B}$ of connected open sets. Let P be a unary predicate, and let A be the formula,

$$\forall\mathsf{x}\Box\mathsf{Px} \,\&\, \Box\forall\mathsf{x}(\Box\mathsf{Px} \vee \Box\neg\mathsf{Px}) \supset \Box\forall\mathsf{x}\mathsf{Px}.^{15}$$

The formula A is not a theorem of QS4. We defer the proof of this until we have outlined the Kripke semantics for the language $\mathcal{L}$: see Remark 4.2 in Section 4.

To see that $\mathcal{X} \vDash A$, suppose not. Then there is a predicate topological model $\mathbf{M} = \langle X, \tau, D, V\rangle$ and a point $x \in X$ such that

$$\mathbf{M}, x \vDash \forall\mathsf{x}\Box\mathsf{Px}, \tag{1}$$

$$\mathbf{M}, x \vDash \Diamond\exists\mathsf{x}\neg\mathsf{Px}, \text{ and} \tag{2}$$

$$\mathbf{M}, x \vDash \Box\forall\mathsf{x}(\Box\mathsf{Px} \vee \Box\neg\mathsf{Px}). \tag{3}$$

By (3), there is a basis set $O \in \mathcal{B}$ such that $x \in O$ and,

$$(\forall y \in O)(\mathbf{M}, y \vDash \forall\mathsf{x}(\Box\mathsf{Px} \vee \Box\neg\mathsf{Px})). \tag{4}$$

By (2), there is a $z \in O$ and a $d \in D$, such that

$$\mathbf{M}, z \vDash \neg\mathsf{P}d \tag{5}$$

By (4),

$$\mathbf{M}, z \vDash (\Box\mathsf{P}d \vee \Box\neg\mathsf{P}d). \tag{6}$$

By (1),

$$\mathbf{M}, x \vDash \Box\mathsf{P}d. \tag{7}$$

By (5) and (6),

$$\mathbf{M}, z \vDash \Box\neg\mathsf{P}d. \tag{8}$$

By (4),

$$(\forall y \in O)(\mathbf{M}, y \vDash \Box\mathsf{P}d \text{ or } \mathbf{M}, y \vDash \Box\neg\mathsf{P}d). \tag{9}$$

[15] If the language has no unary predicates, then choose some nonunary predicate P' of arity, say n, and replace Px in the formula with $\mathsf{P}'\mathsf{x}\ldots\mathsf{x}$, with x occurring n times.

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Define $O^+, O^- \subseteq O$ as follows:

$$O^+ = \{y \in O : \mathbf{M}, y \vDash \Box \mathsf{P}d\}$$

$$O^- = \{y \in O : \mathbf{M}, y \vDash \Box\neg \mathsf{P}d\}$$

Note that O^+ and O^- are open. Given (7) and (8), O^+ and O^- are nonempty. And given (9), $O^+ \cup O^- = O$. But O is connected, since $O \in \mathcal{B}$: thus O cannot be the union of nonempty open sets, a contradiction. □

OPEN QUESTION 3.5. *Is* QS4 *[strongly] complete for Cantor space?*

REMARK 3.6. Like QS4, QH is not complete for any locally connected space, and hence not complete for $\mathbb{R}$. Suppose, in particular, that P is a unary predicate.[16] Then the argument in the proof of Theorem 3.4 can be amended to show that for the following formula A, which is not a theorem of QH, we have $\mathcal{X} \vDash A$ for every locally connected space $\mathcal{X}$:

$$\forall \mathsf{x}(\mathsf{Px} \vee \exists \mathsf{x}\neg \mathsf{Px}) \,\&\, \forall \mathsf{x}(\mathsf{Px} \vee \neg \mathsf{Px}) \supset (\forall \mathsf{xPx} \vee \exists \mathsf{x}\neg \mathsf{Px}).$$

OPEN QUESTION 3.7. *Is* QH *[strongly] complete for Cantor space [with a countable domain]?*

REMARK 3.8. Cantor space is well known to be homeomorphic to $\{0, 1\}^{\mathbb{N}}$, where $\{0, 1\}$ has the discrete topology and $\{0, 1\}^{\mathbb{N}}$ has the product topology. Consider the closely related space $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}$ has the discrete topology and $\mathbb{N}^{\mathbb{N}}$ has the product topology. Dragalin (1988) proves that QH is strongly complete for $\mathbb{N}^{\mathbb{N}}$ with a countable domain: see Dragalin (1988), Proposition 5.2, p. 110.[17] Since $\mathbb{N}^{\mathbb{N}}$ is a Baire space, QH is notably different from QS4: there is a Baire space for which QH is complete, indeed strongly complete, with a countable domain. Since $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the irrational line, QH is also strongly complete for the irrational line with a countable domain.

OPEN QUESTION 3.9. *Is* QS4 *[strongly] complete for* $\mathbb{N}^{\mathbb{N}}$*, or, equivalently, the irrational line?*

§4. Kripke semantics. A *Kripke frame* is an ordered pair $\mathcal{K} = \langle W, R\rangle$, where W is a nonempty set and $R \subseteq W \times W$. We say that $\mathcal{K}$ is *reflexive* [*transitive*, *symmetric*] iff R is, and that $r \in W$ is a *root* of $\mathcal{K}$ iff $\forall w \in W, rRw$. We say that $\mathcal{K}$ is *rooted* iff $\mathcal{K}$ has at least one root. Given $w \in W$, $R(w) =_{\text{df}} \{w' \in W : wRw'\}$.

Presently, we define *predicate frames*, the frame-theoretic analog to predicate topological spaces. Note that a predicate topological space comes equipped with a single domain over which the quantifiers range. By contrast, a predicate frame can come equipped with a different domain at each world. A [rooted] *predicate frame* is an ordered triple $\mathbf{K} = \langle W, R, D\rangle$, where $\mathcal{K} = \langle W, R\rangle$ is a [rooted] frame and D is a family, $(D_w)_{w\in W}$, of nonempty sets indexed by possible world in W, such that $wRw' \Rightarrow D_w \subseteq D_{w'}$. This last clause is a requirement that the domains be *expanding* along the accessibility relation. We say that $\mathbf{K}$ has a *constant* domain if $\forall w, w' \in W,\ D_w = D_{w'}$. We say that $\mathbf{K}$ is *countable* iff W is countable and each D_w is countable. We let $D_W =_{\text{df}} \bigcup_{w\in W} D_w$.

16 Follow footnote 15 if the language has no unary predicates.

17 I am grateful to an anonymous referee for this reference.

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A *predicate frame model* is an ordered quartuple $\mathbf{M} = \langle W, R, D, V\rangle$, where $\mathbf{X} = \langle W, R, D\rangle$ is a predicate topological space, and

$$V : (W \times \mathsf{Pred}) \cup \mathsf{Names} \cup \mathsf{Func} \rightarrow \bigcup_{n\geq 1} \mathcal{P}({D_W}^n) \cup D_W \cup (\bigcup_{n\geq 1} {D_W}^{{D_W}^n})$$

is such that

- $V(w, \mathsf{P}) \subseteq {D_w}^n$ for every $\mathsf{P} \in \mathsf{Pred}_n$,
- $V(\mathsf{c}) \in D_w$ for every $\mathsf{c} \in \mathsf{Names}$ and every $w \in W$, and
- $V(\mathsf{f}) : {D_W}^n \rightarrow D_W$ for every $\mathsf{f} \in \mathsf{Func}_n$, and $V(\mathsf{f})(d_1, \ldots, d_n) \in D_w$ for every $\mathsf{f} \in \mathsf{Func}_n$, every $w \in W$, and every $d_1, \ldots, d_n \in D_w$.

We say that $\mathbf{M}$ is *based on* $\mathbf{K}$.

Suppose that $\mathbf{M} = \langle W, R, D, V\rangle$ is a predicate frame model. We define $Val(\mathsf{t}) \in D_W$ for every closed D_W-term t exactly as in the topological semantics. Next, we define $\mathbf{M}, w \vDash A$, for each $w \in W$ and each D_w-sentence A as follows:

$$\begin{array}{rll}
\mathbf{M}, w \vDash \mathsf{P}\mathsf{t}_1 \ldots \mathsf{t}_n & \text{iff} & \langle Val(\mathsf{t}_1), \ldots, Val(\mathsf{t}_n)\rangle \in V(w, \mathsf{P}), \text{ where } \mathsf{P} \in \mathsf{Pred}_n \\
\mathbf{M}, w \vDash \neg A & \text{iff} & \mathbf{M}, w \nvDash A \\
\mathbf{M}, w \vDash (A \,\&\, B) & \text{iff} & \mathbf{M}, w \vDash A \text{ and } \mathbf{M}, w \vDash B \\
\mathbf{M}, w \vDash (A \vee B) & \text{iff} & \mathbf{M}, w \vDash A \text{ or } \mathbf{M}, w \vDash B \\
\mathbf{M}, w \vDash \Box A & \text{iff} & \text{for every } w' \in W, \text{ if } wRw' \text{ then } \mathbf{M}, w' \vDash A \\
\mathbf{M}, w \vDash \forall \mathsf{x} A & \text{iff} & \text{for every } d \in D_w, \mathbf{M}, w \vDash [d/\mathsf{x}]A
\end{array}$$

If Γ is a nonempty set of sentences, we say that $\mathbf{M}, w \vDash \Gamma$ iff $\mathbf{M}, w \vDash A$, for every $A \in \Gamma$. We say that $\mathbf{M} \vDash \Gamma$ iff $\mathbf{M}, w \vDash A$ for every $w \in W$. The definitions of $\mathbf{K} \vDash A$ and $\mathcal{K} \vDash A$ are standard. The definitions of satisfiability, soundness, completeness, and strong completeness are the obvious analogs to their counterparts in the topological semantics.

The following theorem is well known (see, e.g., Hughes & Cresswell, 1996 or Gabbay et al., 2009):

THEOREM 4.1. QS4 *is sound for the class of reflexive transitive predicate frames; and strongly complete for the class of countable rooted reflexive transitive predicate frames.*[18]

REMARK 4.2. Recall the formula A specified in the proof of Theorem 3.4: we promised a proof that A is not a theorem of QS4. Consider the following rooted reflexive transitive predicate frame model, $\mathbf{M} = \langle W, R, D, V\rangle$:

$$\begin{array}{rcl}
W & = & \{1, 2\} \\
R & = & \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\} \\
D_1 & = & \{3\} \\
D_2 & = & \{3, 4\} \\
V(1, \mathsf{P}) & = & \{3\} \\
V(2, \mathsf{P}) & = & \{3\}
\end{array}$$

It is easy to check that $\mathbf{M}, 1 \nvDash A$.

[18] Both Hughes & Cresswell (1996) and Gabbay et al. (2009) state this theorem for languages without function symbols, but the result extends to languages with function symbols.

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§5. p-morphisms. If $\mathcal{K} = \langle W, R\rangle$ is a reflexive transitive Kripke frame, then we say that a set $O \subseteq W$ is *open* iff $\forall w, w' \in W, w \in O \ \& \ wRw' \Rightarrow w' \in O$. Suppose that φ is a function from either a reflexive transitive Kripke frame or a topological space to a reflexive transitive Kripke frame or a topological space. We say that φ is *continuous* iff the preimage of every open set is open; that φ is *open* iff the image of every open set is open; and that φ is a *propositional p-morphism* iff φ is a continuous open surjection.

Gabbay *et al.* (2009) extend the notion of a p-morphism to the notion of a predicate p-morphism from one predicate Kripke frame to another. We slightly alter the definition in Gabbay *et al.* (2009) and define predicate p-morphisms from predicate topological spaces to predicate Kripke frames.

DEFINITION 5.1. *Suppose that* $\mathbf{K} = \langle W, R, D\rangle$ *is a reflexive transitive predicate frame and that* $\mathbf{X} = \langle X, \tau, D^*\rangle$ *is a predicate topological space, and that* $\mathbf{M} = \langle W, R, D, V\rangle$ *and* $\mathbf{M}^* = \langle X, \tau, D^*, V^*\rangle$ *are models based on* $\mathbf{K}$ *and* $\mathbf{X}$*, respectively.*

(i) A predicate p-morphism from $\mathbf{X}$ to $\mathbf{K}$ *is an ordered pair* $\varphi = \langle \varphi_0, \varphi_1\rangle$*, such that*

1. φ_0 *is a propositional p-morphism from* $\langle X, \tau\rangle$ *to* $\langle W, R\rangle$*;*
2. $\varphi_1 = (\varphi_{1x})_{x\in X}$ *is a family of functions indexed by the members of* X*;*
3. *every* $\varphi_{1x} : D^* \to D_{\varphi_0(x)}$ *is a surjective map; and*
4. *for every* $d \in D^*$ *and every* $x \in X$*, there is an open set* $O \subseteq X$*, such that both* $x \in O$ *and for every* $y \in O$*,* $\varphi_{1y}(d) = \varphi_{1x}(d)$*.*

(ii) A predicate p-morphism from $\mathbf{M}^*$ to $\mathbf{M}$ *is a predicate p-morphism from* $\mathbf{X}$ *to* $\mathbf{K}$ *such that, for every* $x \in X$*, for every* $\mathsf{P} \in \mathsf{Pred}_n$ *(*$n \geq 1$*), for every* $\mathsf{c} \in \mathsf{Names}$*, for every* $f \in \mathsf{Func}_n$ *(*$n \geq 1$*), and for every* $d_1, \ldots, d_n \in D^*$*,*

5. $\langle d_1 \ldots d_n\rangle \in V^*(x, \mathsf{P})$ *iff* $\langle \varphi_{1x}(d_1) \ldots \varphi_{1x}(d_n)\rangle \in V(\varphi_0(x), \mathsf{P})$*;*
6. $\varphi_{1x}(V^*(\mathsf{c})) = V(\mathsf{c})$*; and*
7. $\varphi_{1x}(V^*(\mathsf{f})(d_1, \ldots, d_n)) = V(\mathsf{f})(\varphi_{1x}(d_1), \ldots, \varphi_{1x}(d_n))$.

LEMMA 5.2. *If* $\varphi = \langle \varphi_0, \varphi_1\rangle$ *is a predicate p-morphism from the predicate topological model* $\mathbf{M}^* = \langle X, \tau, D^*, V^*\rangle$ *to the reflexive transitive predicate frame model* $\mathbf{M} = \langle W, R, D, V\rangle$*, then for every* D^**-term* t*,*

$$\textit{for every } x \in X,\ \varphi_{1x}(\mathit{Val}(\mathsf{t})) = \mathit{Val}(\varphi_{1x} \cdot \mathsf{t}),$$

where $\varphi_{1x} \cdot \mathsf{t}$ *is the* $D_{\varphi_0(x)}$*-term obtained from the* D^**-term* t *by replacing every occurrence in* t *of every* $d \in D^*$ *with* $\varphi_{1x}(d)$*.*

LEMMA 5.3. *If* $\varphi = \langle \varphi_0, \varphi_1\rangle$ *is a predicate p-morphism from the predicate topological model* $\mathbf{M}^* = \langle X, \tau, D^*, V^*\rangle$ *to the reflexive transitive predicate frame model* $\mathbf{M} = \langle W, R, D, V\rangle$*, then for every* D^**-sentence* B*,*

$$\textit{for every } x \in X,\ \mathbf{M}^*, x \vDash B \textit{ iff } \mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B,$$

where $\varphi_{1x} \cdot B$ *is the* $D_{\varphi_0(x)}$*-sentence obtained from the* D^**-sentence* B *by replacing every free occurrence in* B *of every* $d \in D^*$ *with* $\varphi_{1x}(d)$*.*

Proof. We prove this by strong induction on the complexity of B, that is, the number of quantifier- or connective-occurrences in B. As an inductive hypothesis (IH), suppose that for every D^*-sentence B' of complexity strictly less than the complexity of B, and every

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$x \in X$, we have $\mathbf{M}^*, x \vDash B'$ iff $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B'$. We will verify three cases: (1) B is atomic, (2) $B = \Box C$, and (3) $B = \forall \mathsf{x} C$.

Case (1). B is of the form $\mathsf{Pt}_1 \ldots \mathsf{t}_n$, where $\mathsf{P} \in \mathsf{Pred}_n$ and $\mathsf{t}_1 \ldots \mathsf{t}_n$ are D^*-terms. Note: $M^*, x \vDash \mathsf{Pt}_1 \ldots \mathsf{t}_n$ iff $\langle Val(\mathsf{t}_1), \ldots, Val(\mathsf{t}_n)\rangle \in V^*(x, \mathsf{P})$ iff $\langle \varphi_{1x}(Val(\mathsf{t}_1)), \ldots, \varphi_{1x}(Val(\mathsf{t}_n))\rangle \in V(\varphi_0(x), \mathsf{P})$ (by the definition of predicate p-morphism) iff $\langle Val(\varphi_{1x} \cdot \mathsf{t}_1), \ldots, Val(\varphi_{1x} \cdot \mathsf{t}_n)\rangle \in V(\varphi_0(x), \mathsf{P})$ (by Lemma 5.2) iff $M, \varphi_0(x) \vDash \mathsf{P}(\varphi_{1x} \cdot \mathsf{t}_1) \ldots (\varphi_{1x} \cdot \mathsf{t}_n)$ iff $M, \varphi_0(x) \vDash \varphi_{1x} \cdot \mathsf{Pt}_1 \ldots \mathsf{t}_n$.

Case (2). $B = \Box C$. Choose any $x \in X$. **Claim.** $\mathbf{M}^*, x \vDash B$ iff $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B$. *Proof.* ($\Rightarrow$) Suppose that $\mathbf{M}^*, x \vDash B$, that is, that $\mathbf{M}^*, x \vDash \Box C$. Then there's an open set $O \subseteq X$ such that $x \in O$ and for every $y \in O$, $\mathbf{M}, y \vDash C$. Let $d_1, \ldots, d_n$ be the members of D^* that occur as names in B. If this list is empty, let $O^* = O$. Otherwise, since φ is a predicate p-morphism, for each $i = 1, \ldots, n$, there is an open set $O_i \subseteq X$, such that $x \in O_i$ and for each $y \in O_i$, $\varphi_{1y}(d_i) = \varphi_{1x}(d_i)$. Let $O^* = O \cap \bigcap_{i=1}^{n} O_i$. So, for each $i = 1, \ldots, n$, and for each $y \in O^*$, $\varphi_{1y}(d_i) = \varphi_{1x}(d_i)$. So, for each $y \in O^*$, $\varphi_{1y} \cdot C = \varphi_{1x} \cdot C$.

We want to show that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B$, that is, that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot \Box C$. Since $\varphi_{1x} \cdot \Box C = \Box(\varphi_{1x} \cdot C)$, it suffices to show that $\mathbf{M}, \varphi_0(x) \vDash \Box(\varphi_{1x} \cdot C)$. So suppose that $w \in W$ and $\varphi_0(x) R w$. Note that $\varphi_0[O^*]$ (the image of O^* under φ_0) is open in W, since φ_0 is a propositional p-morphism. So, since $\varphi_0(x) \in \varphi_0[O^*]$ and $\varphi_0(x) R w$, we have $w \in \varphi_0[O^*]$. So $w = \varphi(y)$ for some $y \in O^*$. Note that $\mathbf{M}, y \vDash C$. So by the IH, $\mathbf{M}, w \vDash \varphi_{1y} \cdot C$. So $\mathbf{M}, w \vDash \varphi_{1x} \cdot C$, since $\varphi_{1y} \cdot C = \varphi_{1x} \cdot C$. Thus, for every $w \in W$, if $\varphi_0(x) R w$ then $\mathbf{M}, w \vDash \varphi_{1x} \cdot C$. So $\mathbf{M}, \varphi_0(x) \vDash \Box(\varphi_{1x} \cdot C)$, as desired.

($\Leftarrow$) Suppose that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B$, that is, that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot \Box C$. We want to show that $\mathbf{M}^*, x \vDash \Box C$. Note that $R(\varphi_0(x))$ is an open subset of W. So $O = {\varphi_0}^{-1}[R(\varphi_0(x))]$ (the preimage of $R(\varphi_0(x))$ under φ_0) is an open subset of X. Let $d_1, \ldots, d_n$ be the members of D^* that occur as names in B. If this list is empty, let $O^* = O$. Otherwise, since φ is a predicate p-morphism, for each $i = 1, \ldots, n$, there is an open set $O_i \subseteq X$, such that $x \in O_i$ and for each $y \in O_i$, $\varphi_{1y}(d_i) = \varphi_{1x}(d_i)$. Let $O^* = O \cap \bigcap_{i=1}^{n} O_i$. So, for each $i = 1, \ldots, n$, and for each $y \in O^*$, $\varphi_{1y}(d_i) = \varphi_{1x}(d_i)$. So, for each $y \in O^*$, $\varphi_{1y} \cdot C = \varphi_{1x} \cdot C$.

It will suffice to show that $\mathbf{M}^*, y \vDash C$, for every $y \in O^*$. So choose $y \in O^*$. So $y \in {\varphi_0}^{-1}[R(\varphi_0(x))]$. So $\varphi_0(y) \in R(\varphi_0(x))$. So $\varphi_0(x) R \varphi_0(y)$. Now $\mathbf{M}, \varphi_0(x) \vDash \Box(\varphi_{1x} \cdot C)$, since $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot \Box C$. So $\mathbf{M}, \varphi_0(y) \vDash \varphi_{1x} \cdot C$. So $\mathbf{M}, \varphi_0(y) \vDash \varphi_{1y} \cdot C$, since $\varphi_{1y} \cdot C = \varphi_{1x} \cdot C$. So, by the IH, $\mathbf{M}^*, y \vDash C$, as desired.

Case (3). $B = \forall \mathsf{x} C$. Choose any $x \in X$. **Claim.** $\mathbf{M}^*, x \vDash B$ iff $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B$. *Proof.* ($\Rightarrow$) Suppose that $\mathbf{M}^*, x \vDash B$, that is, that $\mathbf{M}^*, x \vDash \forall \mathsf{x} C$. Then, for every $d \in D^*$, we have $\mathbf{M}^*, x \vDash [d/\mathsf{x}]C$. We want to show that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B$, that is, that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot \forall \mathsf{x} C$, where $\varphi_{1x} \cdot \forall \mathsf{x} C = \forall \mathsf{x}(\varphi_{1x} \cdot C)$. So we want to show that, for every $d \in D_{\varphi_0(x)}$, we have $\mathbf{M}, \varphi_0(x) \vDash [d/\mathsf{x}](\varphi_{1x} \cdot C)$. Choose any $d \in D_{\varphi_0(x)}$. Since $\varphi_{1x} : D^* \to D_{\varphi_0(x)}$ is surjective, there is a $d^* \in D^*$ such that $\varphi_{1x}(d^*) = d$. So $\mathbf{M}^*, x \vDash [d^*/\mathsf{x}]C$. So by the IH, $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot [d^*/\mathsf{x}]C$. So $\mathbf{M}, \varphi_0(x) \vDash [d/\mathsf{x}](\varphi_{1x} \cdot C)$, as desired.

($\Leftarrow$) Suppose that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot B$, that is, that $\mathbf{M}, \varphi_0(x) \vDash \varphi_{1x} \cdot \forall \mathsf{x} C$. We want to show that $\mathbf{M}^*, x \vDash \forall \mathsf{x} C$. So choose $d \in D^*$. We want to show that $\mathbf{M}^*, x \vDash [d/\mathsf{x}]C$. Since $\varphi_{1x} : D^* \to D_{\varphi_0(x)}$, we have $\mathbf{M}, \varphi_0(x) \vDash [\varphi_{1x}(d)/\mathsf{x}](\varphi_{1x} \cdot C)$. So $\mathbf{M}, \varphi_0(x) \vDash (\varphi_{1x} \cdot [d/\mathsf{x}]C)$. So by the IH, $\mathbf{M}^*, x \vDash [d/\mathsf{x}]C$, as desired. □

COROLLARY 5.4. *Suppose that $\varphi = \langle \varphi_0, \varphi_1\rangle$ is a predicate p-morphism from the predicate topological space $\mathbf{X} = \langle X, \tau, D^*\rangle$ to the reflexive transitive predicate frame $\mathbf{K} = \langle W, R, D\rangle$. Then for every nonempty set Γ of formulas, if Γ is satisfiable in $\mathbf{K}$ then Γ is satisfiable in the predicate topological space $\mathbf{X}^* = \langle X, \tau, \mathsf{Term}(D^*)\rangle$.*

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REMARK 5.5. In most contexts, a p-morphism from $\mathbf{X}$ to $\mathbf{K}$ would transfer the satisfiability of Γ from $\mathbf{K}$ back to $\mathbf{X}$. But here, such a p-morphism transfers satisfiability from $\mathbf{K}$ not back to $\mathbf{X}$, but rather to a closely related predicate topological space, $\mathbf{X}^*$.

Proof of Corollary 5.4. Suppose that φ, $\mathbf{X}$, and $\mathbf{K}$ are as given, and that Γ is satisfiable in $\mathbf{K}$. Then there is a valuation V and a $w \in W$ such that $\mathbf{M}, w \vDash \Gamma$, where $\mathbf{M} = \langle W, R, D, V \rangle$.

We begin by specifying a new predicate p-morphism $\varphi^* = \langle {\varphi^*}_0, {\varphi^*}_1 \rangle$ from $\mathbf{X}^*$ to $\mathbf{K}$. Let ${\varphi^*}_0 = \varphi_0$. For each $x \in X$, we define the function ${\varphi^*}_{1x} : \mathsf{Term}(D^*) \to D_W$ recursively on $\mathsf{Term}(D^*)$ as follows:

- If $d \in D^*$, then ${\varphi^*}_{1x}(d) = \varphi_{1x}(d)$;
- if $\mathsf{c} \in \mathsf{Names}$, then ${\varphi^*}_{1x}(\mathsf{c}) = V(\mathsf{c})$; and
- if $\mathsf{f} \in \mathsf{Func}_n$ and $\mathsf{t}_1, \ldots, \mathsf{t}_n \in \mathsf{Term}(D)$, then ${\varphi^*}_{1x}(\mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n) = V(\mathsf{f})({\varphi^*}_{1x}(\mathsf{t}_1), \ldots, {\varphi^*}_{1x}(\mathsf{t}_n))$.

Claim. φ^* is indeed a predicate p-morphism from $\mathbf{X}^*$ to $\mathbf{K}$. *Proof.* We have to check that φ^* satisfies Clauses (1)–(4) in Definition 5.1, (i). φ^* satisfies Clause (1) since ${\varphi^*}_0 = \varphi_0$. φ^* satisfies Clause (2) by the definition of each ${\varphi^*}_{1x}$.

To check Clause (3), we need to check that (i) ${\varphi^*}_{1x}(\mathsf{t}) \in D_{{\varphi^*}_0(x)}$, for every $\mathsf{t} \in \mathsf{Term}(D^*)$; and (ii) ${\varphi^*}_{1x} : \mathsf{Term}(D^*) \to D_{{\varphi^*}_0(x)}$ is surjective. For (i), it suffices to note that (a) ${\varphi^*}_{1x}(d) = \varphi_{1x}(d) \in D_{{\varphi^*}_0(x)}$, for each $d \in D^*$; (b) ${\varphi^*}_{1x}(\mathsf{c}) = V(\mathsf{c}) \in D_w$, for every $w \in W$; and (c) D_w is closed under the function $V(\mathsf{f})$, for every $\mathsf{f} \in \mathsf{Func}$ and every $w \in W$. To see (ii), it suffices to note that $\varphi_{1x} : D^* \to D_{\varphi_0(x)}$ is surjective and that ${\varphi^*}_{1x}$ extends φ_{1x}.

For Clause (4), we want to show that for every $\mathsf{t} \in \mathsf{Term}(D^*)$ and every $x \in X$, there is an open set $O \subseteq X$, such that both $x \in O$ and for every $y \in O$, ${\varphi^*}_{1y}(\mathsf{t}) = {\varphi^*}_{1x}(\mathsf{t})$. We show this by induction on $\mathsf{Term}(D)$. For the base case, either $\mathsf{t} \in D$ or $\mathsf{t} \in \mathsf{Names}$. If $\mathsf{t} \in D$, then the result follows from the definition of ${\varphi^*}_{1x}(\mathsf{t})$ as $\varphi_{1x}(\mathsf{t})$ and the fact that φ satisfies Clause (4). If $\mathsf{t} \in \mathsf{Names}$, then the result follows from the fact that ${\varphi^*}_{1x}(\mathsf{t}) = V(\mathsf{t})$ for every $x \in X$. For the inductive step, suppose that $\mathsf{t} = \mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n$ and that, for each $i = 1, \ldots, n$ for every $x \in X$, there is an open set $O_i \subseteq X$, such that

$$\text{both } x \in O_i \text{ and for every } y \in O_i, {\varphi^*}_{1y}(\mathsf{t}_i) = {\varphi^*}_{1x}(\mathsf{t}_i). \qquad (*)$$

Fix $x \in X$, and choose open sets $O_i \subseteq X$ satisfying $(*)$. Let $O = \bigcap O_i$. Note that O is open and $x \in O$. Suppose that $y \in O$. Then every $y \in O_i$, for each i. So ${\varphi^*}_{1y}(\mathsf{t}_i) = {\varphi^*}_{1x}(\mathsf{t}_i)$, for each i. So ${\varphi^*}_{1y}(\mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n) = V(\mathsf{f})({\varphi^*}_{1y}(\mathsf{t}_1), \ldots, {\varphi^*}_{1y}(\mathsf{t}_n)) = V(\mathsf{f})({\varphi^*}_{1x}(\mathsf{t}_1), \ldots, {\varphi^*}_{1x}(\mathsf{t}_n)) = {\varphi^*}_{1x}(\mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n)$.

We now specify a valuation V^* for the predicate topological space $\mathbf{X}^*$.

- If $\mathsf{P} \in \mathsf{Pred}_n$ and $x \in X$, then $V^*(x, \mathsf{P}) = \{\langle \mathsf{t}_1 \ldots \mathsf{t}_n \rangle \in \mathsf{Term}(D^*)^n : \langle {\varphi^*}_{1x}(\mathsf{t}_1) \ldots {\varphi^*}_{1x}(\mathsf{t}_n) \rangle \in V({\varphi^*}_0(x), \mathsf{P})\}$;
- if $\mathsf{c} \in \mathsf{Names}$, then $V^*(\mathsf{c}) = \mathsf{c}$, and
- if $\mathsf{f} \in \mathsf{Func}_n$, then $V^*(\mathsf{f})(\mathsf{t}_1, \ldots, \mathsf{t}_n) = \mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n$.

Claim. φ^* is a predicate p-morphism from the predicate topological model $\mathbf{M}^* = \langle X, \tau, \mathsf{Term}(D^*), V^* \rangle$ to the predicate Kripke model $\mathbf{M} = \langle W, R, D, V \rangle$. *Proof.* We have to check that φ^* satisfies Clauses (5)–(7) in Definition 5.1 (ii), for every $x \in X$, every $\mathsf{P} \in \mathsf{Pred}_n$ $(n \geq 1)$, every $\mathsf{c} \in \mathsf{Names}$, every $f \in \mathsf{Func}_n$ $(n \geq 1)$, and every

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$\mathsf{t}_1, \ldots, \mathsf{t}_n \in \mathsf{Term}(D^*)$. Clause (5) is immediate by the definition of $V^*(x, P)$. As for Clause (6), note that $\varphi^*{}_{1x}(V^*(\mathsf{c})) = \varphi^*{}_{1x}(\mathsf{c}) = V(\mathsf{c})$. As for Clause (7), note that $\varphi^*{}_{1x}(V^*(\mathsf{f})(\mathsf{t}_1, \ldots, \mathsf{t}_n)) = \varphi^*{}_{1x}(\mathsf{ft}_1 \ldots \mathsf{t}_n) = V(\mathsf{f})(\varphi^*{}_{1x}(\mathsf{t}_1), \ldots, \varphi^*{}_{1x}(\mathsf{t}_n))$. □

COROLLARY 5.6. *Suppose that* $\varphi = \langle \varphi_0, \varphi_1 \rangle$ *is a predicate p-morphism from the predicate topological space* $\mathbf{X} = \langle X, \tau, D^* \rangle$ *to the reflexive transitive predicate frame* $\mathbf{K} = \langle W, R, D \rangle$*, where* D^* *is infinite. Then for every nonempty set* Γ *of formulas, if* Γ *is satisfiable in* $\mathbf{K}$ *then* Γ *is satisfiable in* $\mathbf{X}$.

Proof. By Corollary 5.4, Γ is satisfiable in the predicate topological space $\mathbf{X}^* = \langle X, \tau, \mathsf{Term}(D^*) \rangle$. Since D^* is infinite, it has the same cardinality as $\mathsf{Term}(D^*)$. And since the cardinality of the domain of quantification is the only feature of it relevant to satisfiability of any set of sentences, Γ is also satisfiable in the predicate topological space $\mathbf{X}^* = \langle X, \tau, D^* \rangle$. □

§6. A general strong completeness theorem. A subset of a topological space is *clopen* iff it is both closed and open. A topological space is *zero-dimensional* iff it has a basis of clopen sets.[19] A topological space is *frame-simulating* iff every countable rooted reflexive transitive Kripke frame is an image of it under a propositional p-morphism.[20] Our general strong completeness theorem is

THEOREM 6.1. *Suppose that* $\mathcal{X} = \langle X, \tau \rangle$ *is a frame-simulating zero-dimensional topological space and that* D *is of the same cardinality as* X*. Then* QS4 *is strongly complete for the predicate topological space* $\langle X, \tau, D \rangle$.

Before proving Theorem 6.1, we note that our main result, Theorem 2.5, is a corollary to Theorem 6.1 and the following lemma:

LEMMA 6.2. $\mathbb{Q}$ *is* (i) *zero-dimensional and* (ii) *frame-simulating.*

Proof. (i) is well-known. For a basis of clopen sets, consider the sets of the form $\{x \in \mathbb{Q} : a < x < b\}$, where a and b are irrational.

To show (ii), let $2^{<\omega}$ be the set of finite binary sequences, ordered as follows: $b \leq b'$ iff b is an initial segment of b'. The *infinite binary tree* is the Kripke frame $\langle 2^{<\omega}, \leq \rangle$. We write $\langle\rangle$ for the empty sequence, which is the root of this Kripke frame. If $b \in 2^{<\omega}$, we write $b0$ [$b1$] for b concatenated with 0 [1]. And we define $\leq(b) =_{\mathrm{df}} \{b' \in 2^{<\omega} : b \leq b'\}$. Claim (ii) follows from (ii.a) and (ii.b):

(ii.a) There is a propositional p-morphism from $\mathbb{Q}$ to $2^{<\omega}$: this is proved in the course of the proof of Theorem 2.4 in van Benthem *et al.* (2006). (ii.b) Every countable rooted reflexive transitive Kripke frame is the image of $2^{<\omega}$ under some propositional p-morphism. Claim (ii.b) is Lemma 3.3 in Kremer (2013): we reproduce a proof of (ii.b) here, since the proof in Kremer (2013) is incorrect.[21]

[19] This is the standard definition—see, for example, Steen & Seebach (1970). Other sources have variants: for example, Engelking (1989) adds that the space must be T_1.

[20] This is not a standard notion: we have not seen it defined anywhere in the literature.

[21] Claim (ii.b) strengthens a well-known result, due originally to Dov Gabbay, independently discovered by Johan van Benthem, and proved in Goldblatt (1980): any finite rooted reflexive transitive Kripke frame is the image of $2^{<\omega}$ under some propositional p-morphism.

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Proof of (ii.b). Suppose that $\langle W, R\rangle$ is a countable Kripke frame with root r. We will, in effect, unravel $\langle W, R\rangle$ into $2^{<\omega}$. For each $w \in W$, let $succ_0(w), succ_1(w), succ_2(w), \ldots$ be an enumeration of $R(w)$ in which every member of $R(w)$ occurs infinitely often. We also need a function $zero : 2^{<\omega} \to \mathbb{N}$, defined as follows: $zero(\langle\rangle) = 0$; $zero(b0) = zero(b) + 1$; and $zero(b1) = 0$. Note that $zero(b)$ is simply the number of uninterrupted occurrences of 0 at the end of b: e.g., $zero(001101000) = 3$, $zero(100001) = 0$, and $zero(000100) = 2$. Now we define the p-morphism $\varphi : 2^{<\omega} \to W$ recursively as follows: $\varphi(\langle\rangle) = r$; $\varphi(b0) = \varphi(b)$; and $\varphi(b1) = succ_{zero(b)}(\varphi(b))$.

We have to check that φ actually is a p-morphism. To see that φ is surjective, suppose that $w \in W$. Then $w = succ_n(r)$ for some $n \in \mathbb{N}$. Let 0^n be the sequence of n 0's. And note that $\varphi(0^n 1) = succ_{zero(0^n)}(r) = succ_n(r) = w$. We must also show that φ is continuous and open. For continuity, it suffices to show that the preimage of $R(w)$ is open for every $w \in W$: note that the preimage of $R(w)$ is $\bigcup_{\varphi(b)=w} \leq(b)$. For openness, it suffices to show that the image of $\leq(b)$ is open for every $b \in 2^{<\omega}$: note that the image of $\leq(b)$ is $R(\varphi(b))$. □

Proof of Theorem 6.1. Suppose that $\mathcal{X} = \langle X, \tau\rangle$ is a frame-simulating zero-dimensional topological space and that D is of the same cardinality as X. Also suppose that Γ is a consistent set of sentences. First note that X is infinite, since $\mathcal{X}$ is frame-simulating. Given Theorem 4.1, Γ is satisfiable in some countable rooted reflexive transitive predicate frame $\mathbf{K} = \langle W, R, D'\rangle$. Below, we will specify a nonempty set D^* of the same infinite cardinality as X, and a predicate p-morphism φ from $\langle X, \tau, D^*\rangle$ to $\mathbf{K}$. Given Corollary 5.6, Γ will be satisfiable in the predicate topological space $\langle X, \tau, D^*\rangle$. Thus, since D^* and D have the same cardinality, Γ will be satisfiable in the predicate topological space $\langle X, \tau, D\rangle$.

It remains to specify D^* and the predicate p-morphism φ. Since $\mathcal{X}$ is frame-simulating, there is a propositional p-morphism φ_0 from $\mathcal{X}$ to $\langle W, R\rangle$. Let $D^* = \{\langle x, d\rangle : x \in X \,\&\, d \in D'_{\varphi_0(x)}\}$. Note that D^* has the same cardinality as X. Below, we will define functions $(\varphi_{1x})_{x\in X}$ so that

1. every $\varphi_{1x} : D^* \to D'_{\varphi_0(x)}$ is a surjective map; and
2. for every $\langle y, d\rangle \in D^*$ and every $x \in X$, there is an open set $O \subseteq X$, such that both $x \in O$ and for every $z \in O$, $\varphi_{1z}(\langle y, d\rangle) = \varphi_{1x}(\langle y, d\rangle)$.

Thus we will have a predicate p-morphism $\varphi = \langle\varphi_0, \varphi_1\rangle$ from the predicate topological space $\mathbf{X} = \langle X, \tau, D^*\rangle$ to the predicate frame $\mathbf{K} = \langle W, R, D'\rangle$. So it remains to specify the functions $(\varphi_{1x})_{x\in X}$ satisfying (1) and (2).

Note that, for any $x \in X$, the set $R(\varphi_0(x))$ is open in the frame $\langle W, R\rangle$. So, since φ_0 is continuous, the set ${\varphi_0}^{-1}[R(\varphi_0(x))]$ is open in $\mathcal{X}$. So, since $\mathcal{X}$ is zero-dimensional, for every $x \in X$, there is a clopen set O_x with $x \in O_x \subseteq {\varphi_0}^{-1}[R(\varphi_0(x))]$.

Choose any root world $r \in W$, and choose any $d_r \in D'_r$. Note that, since $r \in W$ is a root, we have $d_r \in D'_w$ for every $w \in W$. For each $x \in X$, define $\varphi_{1x} : D^* \to D'_W$ as follows:

$$\varphi_{1x}(\langle y, d\rangle) \quad = \quad \begin{cases} d \text{ if } x \in O_y \\ d_r \text{ if } x \notin O_y \end{cases}$$

Claim. $\varphi_{1x} : D^* \to D'_{\varphi_0(x)}$. *Proof.* Suppose $\langle y, d\rangle \in D^*$. If $x \notin O_y$, then $\varphi_{1x}(\langle y, d\rangle) = d_r \in D'_{\varphi_0(x)}$. On the other hand, suppose that $x \in O_y$. Then $\varphi_0(x) \in \varphi_0[O_y]$. So $\varphi_0(x) \in R(\varphi_0(y))$. So $\varphi_0(y) R \varphi_0(x)$. Also, $d \in D'_{\varphi_0(y)}$ since $\langle y, d\rangle \in D^*$. So $\varphi_{1x}(\langle y, d\rangle) = d \in D'_{\varphi_0(x)}$.

Claim. $\varphi_{1x} : D^* \to D'_{\varphi_0(x)}$ is surjective. *Proof.* Suppose that $d \in D'_{\varphi_0(x)}$. Note that $\langle x, d\rangle \in D^*$ and that $x \in O_x$. So $\varphi_{1x}(\langle x, d\rangle) = d$.

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Claim. For every $\langle y, d\rangle \in D^*$ and every $x \in X$, there is an open set $O \subseteq X$, such that $x \in O$ for every $z \in O$, $\varphi_{1z}(\langle y, d\rangle) = \varphi_{1x}(\langle y, d\rangle)$. *Proof.* Choose a $\langle y, d\rangle \in D^*$ and an $x \in X$. If $x \in O_y$, let $O = O_y$. Then note that $x \in O$, and for every $z \in O$, we have $\varphi_{1z}(\langle y, d\rangle) = d = \varphi_{1x}(\langle y, d\rangle)$. On the other hand, if $x \notin O_y$, then let $O = X - O_y$. Note that O is open, since O_y is clopen. Also note that $x \in O$, and for every $z \in O$, we have $\varphi_{1z}(\langle y, d\rangle) = d_r = \varphi_{1x}(\langle y, d\rangle)$. □

§7. Concluding remarks.

REMARK 7.1. Our proof of the strong completeness of QS4 for $\mathbb{Q}$, with a countable domain, relies on two features of $\mathbb{Q}$: it is zero-dimensional and frame-simulating. $\mathbb{R}$, for which QS4 is not complete, is neither zero-dimensional nor frame-simulating (for the latter point, see Lemma 4.6 of Kremer (2013), which follows from the Baire Category Theorem). Cantor space, for which the question of [strong] completeness remains open, is zero-dimensional but not frame-simulating (again, see Lemma 4.6 of Kremer, 2013).

Zero-dimensionality is clearly insufficient for strong completeness: QS4 is not complete for any topological space with only one point. Being frame-simulating is also insufficient for strong completeness. Consider the infinite binary tree, $\langle 2^{<\omega}, \leq\rangle$, defined in the proof of Lemma 6.2. Let $\tau_{2^{<\omega}}$ consist of all sets closed under $\leq$: it is easy to see that $\tau_{2^{<\omega}}$ is a topology on $2^{<\omega}$. By Lemma 3.3 in Kremer (2013) (Claim (ii.b) in the proof of Lemma 6.2), the topological space $\langle 2^{<\omega}, \tau_{2^{<\omega}}\rangle$ is frame-simulating. But, we claim, QS4 is not complete for this space. To see this, choose any nonempty domain D, and note that the predicate topological space $\langle 2^{<\omega}, \tau_{2^{<\omega}}, D\rangle$ validates exactly the same sentences as the constant domain predicate Kripke frame $\langle 2^{<\omega}, \leq, D'\rangle$, where $D'_w = D$ for each $w \in 2^{<\omega}$. So $\langle 2^{<\omega}, \tau_{2^{<\omega}}\rangle \models (\forall \mathsf{x}\Box \mathsf{P}\mathsf{x} \supset \Box\forall \mathsf{x}\mathsf{P}\mathsf{x})$, even though $(\forall \mathsf{x}\Box \mathsf{P}\mathsf{x} \supset \Box\forall \mathsf{x}\mathsf{P}\mathsf{x}) \notin$ QS4.

REMARK 7.2. The proof of our main result depends on the countability of the language $\mathcal{L}$. But even if $\mathcal{L}$ were not countable, QS4 would be complete, if not strongly so, for $\mathbb{Q}$ with a countable domain. For, to show that any finite consistent set Γ of sentences is satisfiable in $\mathbb{Q}$ with a countable domain, it suffices to consider only the predicates, names, and function symbols occurring in Γ.

REMARK 7.3. If there are sufficiently many predicates in the language, then QS4 is not strongly complete for $\mathbb{Q}$ with domains of even arbitrary cardinality. To see this, let W be some set of cardinality strictly greater than the cardinality of $\mathcal{P}(\mathbb{Q})$, that is, strictly greater than $2^{\aleph_0}$. And suppose that there's a unique predicate P_w for each $w \in W$. Consider the following set of sentences: $\Gamma = \{\Diamond(\exists \mathsf{x}\mathsf{P}_w\mathsf{x} \,\&\, \neg\exists \mathsf{x}\mathsf{P}_{w'}\mathsf{x}) : w, w' \in W \,\&\, w \neq w'\}$. To see that Γ is consistent, it suffices (by soundness) for every nonempty finite subset of Γ to be satisfiable. So suppose that Γ' is some nonempty finite subset of Γ, and let

$$\begin{aligned} S \;=\; & \{w \in W : \Diamond(\exists \mathsf{x}\mathsf{P}_w\mathsf{x} \,\&\, \neg\exists \mathsf{x}\mathsf{P}_{w'}\mathsf{x}) \in \Gamma', \text{for some } w' \in W\} \\ & \cup \{w \in W : \Diamond(\exists \mathsf{x}\mathsf{P}_{w'}\mathsf{x} \,\&\, \neg\exists \mathsf{x}\mathsf{P}_w\mathsf{x}) \in \Gamma', \text{for some } w' \in W\}. \end{aligned}$$

Note that S is finite. Choose some $w_0 \in W - S$, and let $X = S \cup \{w_0\}$. Impose the following topology τ on X: $O \subseteq X$ is open iff $O = X$ or $O \subseteq S$. Let $D = \{1\}$. And let $\mathbf{M} = \langle X, \tau, D, V\rangle$ be any predicate topological model where, for $w, w' \in S$,

$$\begin{aligned} V(w_0, \mathsf{P}_w) &= \emptyset, \text{ and} \\ V(w', \mathsf{P}_w) &= \begin{cases} \{1\}, \text{ if } w' = w, \text{ and} \\ \emptyset, \text{ if } w' \neq w. \end{cases} \end{aligned}$$

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Note: For distinct $w, w' \in S$, we have $\mathbf{M}, w \vDash (\exists \mathsf{x} \mathsf{P}_w \mathsf{x} \,\&\, \neg \exists \mathsf{x} P_{w'} \mathsf{x})$. Thus, $\mathbf{M}, w_0 \vDash \Gamma'$.

To see that Γ is not satisfiable in $\mathbb{Q}$, suppose that it is: so there is some predicate topological model $\mathbf{M} = \langle \mathbb{Q}, \tau_{\mathbb{Q}}, D, V \rangle$ and some $q_0 \in \mathbb{Q}$ such that $\mathbf{M}, q_0 \vDash \Gamma$. For each $w \in W$, let $S_w = \{q \in \mathbb{Q} : \mathbf{M}, q \vDash \exists \mathsf{x} \mathsf{P}_w \mathsf{x}\}$. Note that, if $w, w' \in W$ and $w \neq w'$, then $\mathbf{M}, q_0 \vDash \Diamond(\exists \mathsf{x} \mathsf{P}_w \mathsf{x} \,\&\, \neg \exists \mathsf{x} \mathsf{P}_{w'} \mathsf{x})$: thus $q_0 \in Cl(S_w - S_{w'})$. Thus the sets $S_w \subseteq \mathbb{Q}$ are all distinct for $w \in W$. But this cannot be, since the cardinality of W is greater than the cardinality of $\mathcal{P}(\mathbb{Q})$.

We suspect that a similar argument could be constructed for any uncountable W—even if the cardinality of W is not greater than $2^{\aleph_0}$—but we haven't seen how to do it.

REMARK 7.4. Let full second order S4, FOS4, be quantified S4 with identity in a language with names, predicate symbols, and function symbols. Neither reflexive transitive predicate frame models nor predicate topological models provide an adequate semantics for FOS4, since the following nontheorem is validated by every predicate frame and by every predicate topological space: $\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} \neq \mathsf{y} \supset \Box \mathsf{x} \neq \mathsf{y})$. As Kohei Kishida noted in private email correspondence, this is why Shehtman and others invented the notion of a *Kripke sheaf*, a generalization of predicate frames with expanding domains. Kripke sheaves allow an object in one world to be 'identified' with distinct objects in another world, allowing us to invalidate $\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} \neq \mathsf{y} \supset \Box \mathsf{x} \neq \mathsf{y})$. Kishida notes that FOS4 is complete in the Kripke sheaf semantics, if we interpret names and function symbols as certain monotone maps. Awodey & Kishida (2008) generalize the Kripke sheaf semantics to a semantics using topological sheaves. (See also Kishida, 2006.) As far as we know, this is the first topological sheaf semantics for FOS4 that explicitly provides interpretations for names and function symbols: Awodey & Kishida (2008) and Kishida (2006) interpret them as certain continuous functions. It follows trivially from Theorem 2.5 that QS4 (without identity) is complete for sheaves over $\mathbb{Q}$. We conclude with three open questions.

OPEN QUESTION 7.5. *Is* FOS4 *complete for sheaves over* $\mathbb{Q}$*? Is* QS4 *without identity complete for sheaves over* $\mathbb{R}$*? Is* FOS4 *compete for sheaves over* $\mathbb{R}$*?*

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