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C^1-stably shadowable chain components

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Abstract. Let p be a hyperbolic periodic saddle of a diffeomorphism f on a closed C^∞ manifold M, and let $C_f(p)$ be the chain component of f containing p. In this paper, we show that if $C_f(p)$ is C^1-stably shadowable, then (i) $C_f(p)$ is the homoclinic class of p and admits a dominated splitting $E \oplus F$ (where the dimension of E is equal to that of the stable eigenspace of p); (ii) the $C_f(p)$-germ of f is expansive if and only if $C_f(p)$ is hyperbolic; and (iii) when M is a surface, $C_f(p)$ is locally maximal if and only if $C_f(p)$ is hyperbolic.

1. *Introduction*

1.1. *Context.*

The notion of pseudo-orbits very often appears in several branches of the modern theory of dynamical systems and, in particular, the pseudo-orbit shadowing property, as well as expansivity, usually plays an important role in the investigation of stability and ergodic properties.

Let M be a closed C^∞ manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced by a Riemannian metric on the tangent bundle. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b} \subset M$ $(-\infty \le a < b \le \infty)$ is called a *δ-pseudo-orbit* of $f \in \mathrm{Diff}(M)$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$.

Denote by $f_{|A}$ the restriction of f to a subset A of M. Let $\Lambda \subset M$ be a closed f-invariant set. We say that $f_{|\Lambda}$ has the *shadowing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of f (where $-\infty \le a < b \le \infty$), there is a $y \in M$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$ (see [**18, 26**]). Notice that only δ-pseudo-orbits of f contained in Λ can be ϵ-shadowed, but a shadowing point $y \in M$ is not necessarily contained in Λ.

We say that f has the *shadowing property* if $M = \Lambda$ in the above definition. This property does not depend on the metric used and is preserved under topological conjugacy. It is easy to see that $f_{|\Lambda}$ has the shadowing property if and only if $f_{|\Lambda}^n$ has the shadowing property for every $n \in \mathbb{Z}$.

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For given points $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo-orbit $\{x_i\}_{i=0}^{b_\delta}$ $(b_\delta > 0)$ of f such that $x_0 = x$ and $x_{b_\delta} = y$. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set of points $\{x \in M : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. It is easy to see that the set is closed and $f(\mathcal{R}(f)) = \mathcal{R}(f)$. Clearly, if $P(f)$ denotes the set of periodic points and $\Omega(f)$ is the non-wandering set of f, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. It is known that if f has the shadowing property, then $\Omega(f) = \mathcal{R}(f)$. For more information on the relationship between chain recurrence and the shadowing property, see **[4, 35]**.

It is well-known that if f satisfies both Axiom A and the strong transversality condition, then f has the shadowing property (see **[26, 30]**). Since such a system is structurally stable, there exists a C^1-neighborhood $\mathcal{U}(f)$ of f such that every element $g \in \mathcal{U}(f)$ has the shadowing property, because g is conjugate to f. The author proved in **[31]** that if there is a C^1-neighborhood $\mathcal{U}(f)$ of f whose elements all have the shadowing property, then f satisfies both Axiom A and the strong transversality condition. Thus, the C^1-interior of the set of diffeomorphisms possessing the shadowing property can be characterized as the set of diffeomorphisms satisfying both Axiom A and the strong transversality condition.

Let $M = M^2$ be a surface, and let $f \in \mathrm{Diff}(M^2)$ satisfy Axiom A. The so-called C^0-transversality condition for f is introduced in **[32]**. Roughly speaking, we say that the stable manifold $W^s(z)$ and the unstable manifold $W^u(z)$ of $z \in M^2$ are C^0-transverse at z if these sets cross at z in the same way that the graph of $y = x^3$ crosses the x-axis at the origin in the xy-plane. We say that f satisfies the C^0-transversality condition if all the stable and unstable manifolds are C^0-transverse at any point of their intersection. It has been proved that f satisfies the C^0-transversality condition if and only if f has the shadowing property (see **[28, 32]**); thus the set of dynamical systems possessing the shadowing property makes up a larger class than the structurally stable systems.

It is proved in **[23,** Proposition 1] that every element in the C^1-interior of the set of diffeomorphisms f such that $f_{|\Omega(f)}$ has the shadowing property satisfies both Axiom A and the no-cycle condition. On the other hand, it is known that if f satisfies both Axiom A and the no-cycle condition, then $\Omega(f) = \mathcal{R}(f)$, so that $f_{|\mathcal{R}(f)}$ has the shadowing property. Thus, when we restrict our attention to the chain recurrent set, the C^1-interior of the set of diffeomorphisms f such that $f_{|\mathcal{R}(f)}$ has the shadowing property is characterized as the set of diffeomorphisms satisfying both Axiom A and the no-cycle condition.

The relation $\leftrightsquigarrow$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain components* (or *chain recurrence classes*) of f (see **[2, 4, 5, 11, 35]**). For dynamical systems satisfying Axiom A, hyperbolic basic sets are basic subsystems that possess many important dynamical properties, and have been investigated extensively in the context of both stability theory and ergodic theory. In view of shadowing theory for dynamical systems, chain components are among the basic objects of investigation, and are natural candidates to replace hyperbolic basic sets. In fact, it is known by **[5]** that, in the C^1-generic context, every chain component with a periodic point is a homoclinic class (the definition of which is given below). Some of the properties of chain components (Lyapunov functions, filtrations, etc; see **[11]**) are interesting subjects in the present context.

Let $\{g_t\}_{t \in \mathbb{R}}$ be the bifurcating one-parameter family of diffeomorphisms on a 2-disk $D^2 \subset \mathbb{R}^2$ derived from a hyperbolic horseshoe that possesses a homoclinic tangency point

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q_0 of g_0 associated with the hyperbolic saddle fixed point p; see Figure 1(b) in §2 and [**25,** Chapter 5]. Owing to the existence of a homoclinic tangency, the chain component $C_{g_0}(p)$ of g_0 containing p does not coincide with its homoclinic class, and g_0 does not have the shadowing property as g_0 fails to satisfy the C^0-transversality condition. More precisely, it will be shown in §2.2 that $g_{0|C_{g_0}(p)}$ does not have the shadowing property. Furthermore, we can choose $t > 0$ arbitrarily small such that $g_{t|C_{g_t}(p)}$ does not have the shadowing property, where $C_{g_t}(p)$ is the chain component of g_t containing p. Hence, g_0 is in the closure (with respect to the C^1-topology) of the set of semi-local systems which do not have the shadowing property. The critical saddle-node horseshoe considered in [**12, 24**] is another example exhibiting the same property (see §2.3). Thus it is natural to study semi-local systems which satisfy the shadowing property with some stability.

The purpose of this paper is to characterize the chain component containing a hyperbolic periodic point by making use of the shadowing property under the C^1-open condition.

This paper has the same philosophy as that of some previous papers that study 'hyperbolic-like properties' implied by some stable topological or dynamical property—for instance, stable ergodicity in the series of works of Pugh, Shub and their collaborators (see [**10**]), stable (robust) transitivity in [**14**], and stable expansivity in [**24, 34**], among others. In this paper, we use (and improve) some of the techniques introduced by Liao and Mañé for proving the C^1-stability conjecture and which have recently been developed further by some of the authors cited above.

Finally, we close this subsection by pointing out one of the key principles of this paper. The two main obstructions to shadowing are the existence of homoclinic tangencies and of heterodimensional cycles. Such dynamical phenomena easily give pseudo-orbits which cannot be shadowable (see §§2, 4 and 5). There are examples in [**3, 7, 38**], for dimensions greater than or equal to 3, such that non-shadowability follows from the existence of heterodimensional cycles, and in [**26, 28, 32**] these are related to homoclinic tangencies. Of course, semi-local systems exhibiting these dynamical phenomena are not hyperbolic. In this paper, the hyperbolicity of a chain component containing a hyperbolic saddle periodic point under the C^1-open condition on the shadowing property is proved by ruling out homoclinic tangencies and heterodimensional cycles.

1.2. *Notation and definitions.* Throughout this paper our space M is a closed C^∞ manifold, and we let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Let $d(\cdot, \cdot)$ be the distance on M induced by a Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

For $f \in \mathrm{Diff}(M)$, we denote a C^1-neighborhood of f by $\mathcal{U}(f)$. Our results will be obtained through several perturbations of a map with respect to the C^1-topology. Let f be a map under consideration, and let $\mathcal{U}(f)$ be given. Each perturbation is *always* arbitrarily small with respect to the C^1-topology, so that the perturbed map g near f is still contained in $\mathcal{U}(f)$.

Let us list some standard symbols that will be used in this paper:

$B_\epsilon(x)$	the closed ball centered at $x \in M$ with radius $\epsilon > 0$;
$\mathrm{int}\, A$	the interior of a set $A \subset M$;

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$\mathcal{O}_f(x)$	the f-orbit of $x \in M$, i.e. $\{f^n(x)\}_{n\in\mathbb{Z}}$;
$\pi(p)$	the (minimum) period of $p \in P(f)$;
E^s_p (E^u_p)	the stable (unstable) eigenspace of a hyperbolic periodic point $p \in P(f)$;
index(p)	the index of a hyperbolic periodic point $p \in P(f)$, i.e. the dimension of E^s_p;
$E_x(\epsilon)$	a closed ball (linear piece) of the linear subspace $E_x \subset T_xM$ centered at the origin O_x with radius $\epsilon > 0$;
$\|\cdot\|$	the usual metric on $\mathbb{R}^{\dim M}$, where dim M denotes the dimension of M.

Besides the above, we introduce a so-called continuation for a hyperbolic periodic point. Let $p \in P(f)$ be hyperbolic; then there exist $\epsilon = \epsilon(p) > 0$ and $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, there is a unique hyperbolic periodic point $p_g \in B_\epsilon(p)$ of g with the same period as p and which has index$(p_g) =$ index(p). Such a point p_g is called the *continuation* of p.

Finally, for notational simplicity, the symbols showing dependence on the dynamics will be omitted in some cases if there is no risk of misunderstanding.

1.3. *Statement of the results.* Let $p \in P(f)$ be hyperbolic, and let $C_f(p)$ be the chain component of f containing p. If p is a sink or a source, then $C_f(p)$ is the periodic orbit of p itself. Hence we consider the case where p is a saddle; that is, of the eigenvalues of $D_p f^{\pi(p)}$, none of them has modulus equal to one, at least one of them is greater than one, and at least one of them is smaller than one.

Let $q \in P(f)$ be a hyperbolic saddle. We say that p and q are *homoclinically related*, and write $p \sim q$, if there are non-empty transverse intersections between $W^s(p)$ and $W^u(q)$, and between $W^u(p)$ and $W^s(q)$. It is clear that if $p \sim q$, then index$(p) =$ index(q).

Let $H_f(p)$ be the *homoclinic class* of p; that is, the closure of the set of all $q \in P(f)$ such that $p \sim q$. Note that, by Smale's transverse homoclinic point theorem in [**36**], $H_f(p)$ coincides with the closure of the set of transverse homoclinic points $x \in W^s(p) \cap W^u(p)$. When a homoclinic class is not hyperbolic, it may contain periodic points having different indices. Actually, there are examples of diffeomorphisms with homoclinic classes that contain hyperbolic periodic points with different indices in a robust way; for instance, see [**6, 9, 13–15, 20**]. Set $H_f(\mathcal{O}_f(p)) = H_f(p) \cup \cdots \cup H_f(f^{\pi(p)-1}(p))$; then $H_f(\mathcal{O}_f(p)) \subset C_f(p)$ but these sets do not coincide in general (see §§2.2 and 2.3). Obviously, $H_f(\mathcal{O}_f(p))$ is a closed f-invariant set, and it is known that $f_{|H_f(\mathcal{O}_f(p))}$ is transitive.

Let Λ be an f-invariant set of M. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$\|D_x f^n_{|E_x}\| \cdot \|D_{f^n(x)} f^{-n}_{|F_{f^n(x)}}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. Let Λ admit a dominated splitting $E \oplus F$. We now list the main properties of a dominated splitting that will be used in this paper; for more information, see [**9,** Appendix B.1].

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- *Continuity*. The dominated splitting $E \oplus F$ is continuous, meaning that the fibers E_x and F_x depend continuously on the point $x \in \Lambda$.
- *Extension to the closure*. The dominated splitting $E \oplus F$ can be extended to a dominated splitting defined over the closure of Λ.
- *Constant dimension in transitive components*. The dimensions of sub-bundles E and F are constant in transitive components of Λ.
- *Uniqueness*. Assume that there is a dominated splitting $G \oplus H$ over Λ such that $\dim E \leq \dim G$. Then we have $E \subset G$.

We say that $C_f(p)$ is C^1-*stably shadowable* if there exists $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, $g_{|C_g(p_g)}$ has the shadowing property (examples and counterexamples of C^1-stably shadowable chain components are given in the next section). In this paper, the following results are obtained.

THEOREM A. *If $C_f(p)$ is C^1-stably shadowable, then there is a $\mathcal{U}(f)$ such that for each $g \in \mathcal{U}(f)$, $C_g(p_g)$ coincides with $H_g(\mathcal{O}_g(p_g))$ and admits a dominated splitting $T_{C_g(p_g)}M = E(g) \oplus F(g)$ satisfying* $\dim E(g) = \mathrm{index}(p_g)$.

The dominated splitting $E(g) \oplus F(g)$ in this theorem satisfies

$$E_x(g) = T_x W^s(x, g) \quad \text{and} \quad F_x(g) = T_x W^u(x, g)$$

for any transverse homoclinic point x of saddles homoclinically related to p_g by construction. Since the set of such points is dense in $C_g(p_g)$, the dimension of the sub-bundle $E(g)$ is constant (see §6).

Generically in $\mathrm{Diff}(M)$, a part of the result of Theorem A holds without the shadowing assumption. Indeed, it is proved in [**5**] that, generically, if a chain component contains a periodic point, then the chain component and the homoclinic class of the periodic point coincide (see also [**9,** Chapter 10]).

We can also prove a certain kind of uniformity of the hyperbolicity for hyperbolic periodic points of the chain components of C^1-stably shadowable diffeomorphisms.

THEOREM B. *Let $C_f(p)$ be C^1-stably shadowable. Then there exist $\mathcal{U}(f)$ and constants $m > 0$, $C > 0$ and $0 < \lambda < 1$ such that for each $g \in \mathcal{U}(f)$, if $q \in C_g(p_g) \cap P(g)$ $(\pi(q) \geq m)$ is hyperbolic, then*

$$\prod_{i=0}^{k-1} \left\| D_{g^{im}(q)} g^m_{|E^s_{g^{im}(q)}} \right\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \left\| D_{g^{-im}(q)} g^{-m}_{|E^u_{g^{-im}(q)}} \right\| < C\lambda^k,$$

where $k = [\pi(q)/m]$ (and $[\cdot]$ represents the integer part).

At this point we note that an analogous theorem was shown in [**34,** Theorem 2] under the condition that $p \sim q$ for C^1-persistently expansive homoclinic classes (see the definition below). We remove this condition for C^1-stable shadowing diffeomorphisms.

Let Λ be a closed f-invariant set of M. We say that $f_{|\Lambda}$ is *expansive* if there is a constant $c > 0$ such that for any $x, y \in \Lambda$, $d(f^n(x), f^n(y)) \leq c$ $(n \in \mathbb{Z})$ implies $x = y$. This property does not depend on the metric used, and it is easy to see that $f_{|\Lambda}$ is expansive if and only if $f^n_{|\Lambda}$ is expansive for $n \in \mathbb{Z}$. Such a constant c is said to be an *expansivity*

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constant of $f_{|\Lambda}$, and we say that $f_{|\Lambda}$ is *c-expansive*. We say that f is *expansive* if $M = \Lambda$ in the above definition.

We say that *the Λ-germ of f is expansive* if there is a constant $c > 0$ such that for any $x \in \Lambda$ and $y \in M$, $d(f^n(x),\ f^n(y)) \leq c$ $(n \in \mathbb{Z})$ implies $x = y$.

It is well known that every expansive system $f_{|\Lambda}$ has a hyperbolic-like property on the local stable and unstable sets; that is, for any $y \in W^{s\ (u)}_{\mathrm{loc}}(x)$ $(x,\ y \in \Lambda)$,

$$d(f^n(x),\ f^n(y)) \to 0 \quad \text{as } n \to \infty\ (-\infty). \tag{$*$}$$

If the Λ-germ of f is expansive, then property $(*)$ holds in a neighborhood of Λ. We need this property to prove Theorem C below (see also the next paragraph of the theorem).

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n_{|E^s_x}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}_{|E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. It is well-known that if Λ is a hyperbolic set of f, then the Λ-germ of f is expansive. Indeed, if Λ is hyperbolic, then there is a compact neighborhood U of Λ such that any pseudo-orbit of f in U is shadowed by a unique true f-orbit (see [**18,** Theorem 18.1.3]). The germ-expansivity immediately follows from this fact.

We have the following characterization of the chain component $C_f(p)$ for a C^1-stably shadowable diffeomorphism.

THEOREM C. *Let $C_f(p)$ be C^1-stably shadowable. Then the $C_f(p)$-germ of f is expansive if and only if $C_f(p)$ coincides with $H_f(\mathcal{O}_f(p))$ and is hyperbolic.*

The proof of the 'if' part of Theorem C is clear. One of the problems we are faced with in the proof of the 'only if' part is the following. Let $q \in C_f(p)$ be a periodic point (we do not know whether q is hyperbolic or not at this moment). If it were *always* true for q that $W^s_{\mathrm{loc}}(q) \cap W^u(p) \neq \emptyset$ and $W^u_{\mathrm{loc}}(q) \cap W^s(p) \neq \emptyset$ with respect to perturbations, then we can show that every periodic point in $C_f(p)$ is hyperbolic under the condition that $C_f(p)$ is C^1-stably shadowable. Here $W^s_{\mathrm{loc}}(q)$ and $W^u_{\mathrm{loc}}(q)$ are just the local stable and unstable sets satisfying property $(*)$; we do not know whether these sets are manifolds or not. But if we do not have such a property, then upon perturbing f to g it may happen that the intersection points of the sets 'escape' from the chain component of p_g. To avoid this problem, we assume germ-expansivity for $f_{|C_f(p)}$ (see Proposition 8.4). Of course, it is natural to try to remove this assumption.

We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. When we focus on surfaces, we can replace the germ-expansivity condition on $C_f(p)$ by the local maximality of it.

THEOREM D. *Let $M = M^2$ be a surface, and let $C_f(p)$ be C^1-stably shadowable. Then $C_f(p)$ is locally maximal if and only if $C_f(p)$ coincides with $H_f(\mathcal{O}_f(p))$ and is hyperbolic.*

The 'if' part of this theorem is proved in [**2,** Proposition 2.9] by making use of the notion of so-called filtering neighborhoods of chain recurrent sets (see [**2,** Proposition 2.4]). This result is independent of the dimension of M. We give here a rather basic proof of the 'if'

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part based on a conventional approach (see Proposition 9.1). The 'only if' part will be proved by showing that if $C_f(p)$ is C^1-stably shadowable and locally maximal, then the $C_f(p)$-germ of f is expansive.

There are some results closely related to Theorem D, with the same flavor. First, there is the C^1-dichotomy for surface diffeomorphisms in [**29**], 'hyperbolicity versus approximation by homoclinic tangencies'. Roughly speaking, C^1-stably shadowable systems cannot be approximated by tangencies (see §3, Proposition 3.1) and thus, by [**29**], they must be hyperbolic. Of course, our result is semi-local and does not follow immediately from the above-mentioned paper. Note that there is a semi-local version of the above result in [**2,** Theorems 1 and 3]. Theorem D is not a direct consequence of these results (at least this is not obvious), but these results clearly are closely related.

1.4. *Further comments.* In contrast with our results, there are several results which show that C^1-robustly non-hyperbolic dynamics on M prevents stable shadowing (see [**3, 7, 38**]). Recently, it was proved in [**3**] that (in the C^1-generic setting) locally maximal homoclinic classes which are not hyperbolic do not have the shadowing property (and more general results were obtained there). This fact, together with our results, naturally raise the question of whether there exist some examples of C^1-stably shadowable chain components which are not hyperbolic. The author has no results relevant to this problem at present.

In [**1**], two C^1-open generic sets were introduced: *tame* diffeomorphisms, which have a finite number of homoclinic classes and whose non-wandering sets admit partitions into a finite number of disjoint transitive sets; and *wild* diffeomorphisms, which have an infinite number of (disjoint different) homoclinic classes (for a survey of this subject, see [**9,** Chapter 10]). It was also proved in [**3**] that there is a residual set $\mathcal{R} \subset \mathrm{Diff}(M)$ such that if $f \in \mathcal{R}$ is tame, then the following two conditions are equivalent: (a) f satisfies both Axiom A and the no-cycle condition; (b) f has the shadowing property. Of course, it seems worthwhile to study C^1-stably shadowable chain components in view of the above two C^1-open generic sets. The author will consider this topic in the near future.

As mentioned before, this paper has the same philosophy as that of some previous works that study hyperbolic-like properties implied by some stable topological property. In fact, the results obtained in [**24**] and [**34**] are closely related to ours. The authors of those papers investigated diffeomorphisms with C^1-robust and C^1-persistent expansivity on the homoclinic classes, and characterized the sets under such C^1-open conditions (see also [**19**]). In particular, the assertions of Theorems A–C in the present paper were proved in [**34**] for C^1-persistently expansive homoclinic classes. Let me now try to outline the results of [**24**] and [**34**] briefly; please see the original papers for the details.

Let $p \in P(f)$ be a hyperbolic saddle (we assume that $f(p) = p$ for simplicity). We say that f is of C^1-*robustly expansive homoclinic class* $H_f(p)$ if there exist $\mathcal{U}(f)$ and $c > 0$ such that $g_{|H_g(p_g)}$ is c-expansive for all $g \in \mathcal{U}(f)$. Let M^3 be a three-dimensional manifold, and suppose that $f \in \mathrm{Diff}(M^3)$ is of C^1-robustly expansive homoclinic class $H_f(p)$. Then it was proved in [**24**] that there is a $\mathcal{U}(f)$ such that for any g in an open dense subset of $\mathcal{U}(f)$, $H_g(p_g)$ is hyperbolic.

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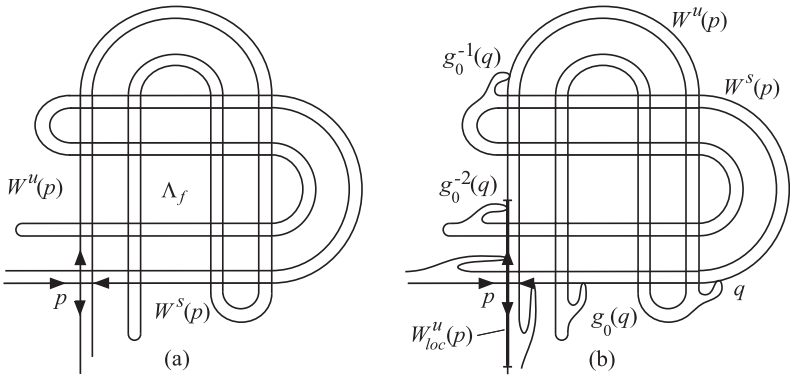


FIGURE 1. Horseshoes.

In [**34**] a weaker notion is introduced as follows: we say that $f \in \mathrm{Diff}(M)$ is of C^1*-persistently expansive homoclinic class* $H_f(p)$ if there is a $\mathcal{U}(f)$ such that $g_{|H_g(p_g)}$ is expansive for all $g \in \mathcal{U}(f)$ (i.e. the expansivity constant is not uniform). It is proved that if f is of C^1-persistently expansive homoclinic class $H_f(p)$ and the $H_f(p)$-germ of f is expansive, then it is hyperbolic (there is no restriction on the dimension of M). We follow the spirit of [**34**] and apply some of the ideas therein to make the aforementioned characterization of the chain components of C^1-stable shadowing diffeomorphisms.

2. *Examples and counterexamples*

In this section, we give examples and counterexamples of C^1-stably shadowable chain components.

2.1. *Hyperbolic horseshoe: a C^1-stably shadowable set.* Let $D^2 \subset \mathbb{R}^2$ be a 2-disk, and let f be Smale's horseshoe map on D^2 with a hyperbolic saddle fixed point p; see Figure 1(a). Denote by Λ_f the hyperbolic horseshoe containing p. Since this is Ω-stable, $\Lambda_f = H_f(p) = C_f(p)$ is a typical example of a C^1-stably shadowable chain component.

2.2. *Horseshoe with a homoclinic tangency: not a C^1-stably shadowable set.* Let $\{g_t\}_{t\in\mathbb{R}}$ be the bifurcating one-parameter family of diffeomorphisms on D^2 derived from a horseshoe Λ_f that exhibits a homoclinic tangency q_0 of g_0 associated with the fixed point p (see [**25,** Chapter 5]). Then $H_{g_0}(p) = \Lambda_f$ and $C_{g_0}(p) = \Lambda_f \cup \mathcal{O}_{g_0}(q_0)$; see Figure 1(b). We show here that (i) $g_{0|C_{g_0}(p)}$ does not have the shadowing property (hence $C_{g_0}(p)$ is not C^1-stably shadowable), and (ii) we can choose $t > 0$ arbitrarily small such that $g_{t|C_{g_t}(p)}$ does not have the shadowing property. Thus g_0 is in the closure of the set of non-shadowable semi-local systems.

For simplicity, denote g_0 and q_0 by g and q, respectively. Let $W^u_{\mathrm{loc}}(p)$ be the local unstable manifold of p, and fix $k > 0$ such that $g^{-k}(q) \in \mathrm{int}\, g(W^u_{\mathrm{loc}}(p)) \setminus W^u_{\mathrm{loc}}(p)$. Suppose $k = 1$ for simplicity; see Figure 1(b). For $\epsilon > 0$, denote by $C^u_\epsilon(q)$ the connected component of $g^2(W^u_{\mathrm{loc}}(p)) \cap B_\epsilon(q)$ containing q. It is easy to see that there is $\epsilon_1 > 0$ such

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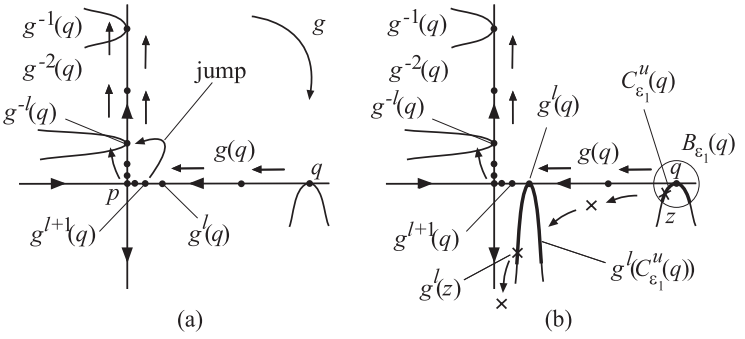


FIGURE 2. Pseudo-orbit which cannot be shadowable.

that $C^u_{\epsilon_1}(q) = g^2(W^u_{\mathrm{loc}}(p)) \cap B_{\epsilon_1}(q)$. We may suppose further that if $|g^{-n}(q) - g^{-n}(y)| < \epsilon_1$ for all $n \geq 0$, then $g^{-2}(y) \in W^u_{\mathrm{loc}}(p)$.

Assume, for contradiction, that $g_{|C_g(p)}$ has the shadowing property, and let $\delta = \delta(\epsilon_1) > 0$ be the number in the definition. Take $l > 0$ such that $|g^l(q) - g^{-l}(q)| < \delta$; then

$$\{\ldots, g^{-2}(q), g^{-1}(q), q, g(q), \ldots, g^{l-1}(q), g^{-l}(q), g^{-l+1}(q), \ldots\} \subset C_g(p)$$

is a δ-pseudo-orbit of g composed of two segments of true g-orbits; see Figure 2(a). Since $g_{|C_g(p)}$ has the shadowing property, there is a z (near q) ϵ_1-shadowing the above pseudo-orbit. Thus

$$\max\{|g^{-n}(q) - g^{-n}(z)|, |g^{-l+n}(q) - g^{l+n}(z)|\} < \epsilon_1 \quad \text{for all } n \geq 0. \qquad (**)$$

From this, $g^{-2}(z) \in W^u_{\mathrm{loc}}(p)$ so that, by the choice of ϵ_1, we have $z \in C^u_{\epsilon_1}(q)$. If $z = q$, then the forward orbit of $g^l(z)$ cannot ϵ_1-shadow the forward orbit of $g^{-l}(q)$ because the ω-limit set of $\mathcal{O}_g(g^l(z)) = \mathcal{O}_g(g^l(q))$ is p. If $z \neq q$, then the forward orbits of $g^l(z)$ and $g^{-l}(q)$ are going in opposite directions since p is hyperbolic; see Figure 2(b). This contradicts $(**)$, and hence (i) is proved.

To prove (ii), observe that we can choose $t > 0$ arbitrarily small such that there is a homoclinic tangency q_t (near q_0) of g_t associated with p. Thus, by the same reasoning stated above, it can easily be checked that $g_{t\,|C_{g_t}(p)}$ does not have the shadowing property.

2.3. *Critical saddle-node horseshoe: not a C^1-stably shadowable set.* Let $\varphi : D^2 \to D^2$ be the saddle-node horseshoe map constructed in [**24,** §2.2] possessing a saddle-node fixed point $\tilde{p}$; see Figure 3(a). It is stated in [**24**] that the saddle-node horseshoe Λ_φ is conjugate to Smale's horseshoe Λ_f. Thus $\varphi_{|\Lambda_\varphi}$ has the shadowing property. Observe that there is a hyperbolic saddle fixed point $\tilde{q}$ in Λ_φ that has negative stable and unstable eigenvalues. Of course, we consider here the chain recurrent set $C_\varphi(\tilde{q})$ containing $\tilde{q}$.

It is easy to see that $\Lambda_\varphi\ (= H_\varphi(\tilde{q})) \subsetneq C_\varphi(\tilde{q})$. Indeed, if we take $r \in W^u(\tilde{p})$ as in Figure 3(a), then $r \in C_\varphi(\tilde{q}) \setminus \Lambda_\varphi$ since $\varphi^{\pm n}(r) \to \tilde{p}$ as $n \to \infty$.

In this subsection, we show that: (i) owing to the existence of a saddle-node fixed point $\tilde{p}$, $\varphi_{|C_\varphi(\tilde{q})}$ does not have the shadowing property, and thus $C_\varphi(\tilde{q})$ is not C^1-stably shadowable; (ii) if we construct a diffeomorphism $\hat{\varphi}$ arbitrarily near φ which exhibits

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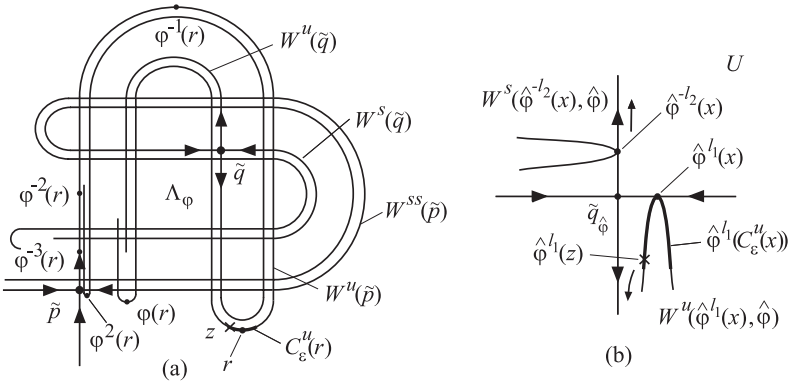


FIGURE 3. Critical saddle-node horseshoe and its perturbation.

a homoclinic tangency associated with $\tilde{q}$, then $\hat{\varphi}_{|C_{\hat{\varphi}}(\tilde{q}_{\hat{\varphi}})}$ does not have the shadowing property. Therefore φ is in the closure of the set of non-shadowable semi-local systems.

To show (i), let $r \in W^u(\tilde{p})$ be as in Figure 3(a). For contradiction, suppose that $\varphi_{|C_\varphi(\tilde{q})}$ has the shadowing property. Fix $\epsilon > 0$ small enough and denote by $C^u_\epsilon(r)$ the connected component of $W^u(\tilde{p})$ containing r, defined similarly as in the example of §2.2 (see Figure 3(a)). Let $\delta = \delta(\epsilon) > 0$ be the number in the definition of the shadowing property for $\varphi_{|C_\varphi(\tilde{q})}$, and take $l > 0$ such that $|\varphi^{-l}(r) - \varphi^l(r)| < \delta$. Then the union of two segments of true φ-orbits

$$\{\ldots, \varphi^{-2}(r), \varphi^{-1}(r), r, \varphi(r), \ldots, \varphi^{l-1}(r), \varphi^{-l}(r), \varphi^{-l+1}(r), \ldots\} \subset C_\varphi(\tilde{q})$$

is a δ-pseudo-orbit of φ. Thus the same reasoning used in §2.2 shows that there is $z \in C^u_\epsilon(r)$ (near r) ϵ-shadowing the above pseudo-orbit. Observe that $\varphi^n(z) \to \tilde{p}$ as $n \to \infty$ since $\tilde{p}$ is a saddle-node point. Thus the forward φ-orbit of z cannot ϵ-shadow the forward φ-orbit of $\varphi^{-l}(r)$, and this is a contradiction.

To prove (ii), we perturb φ to $\hat{\varphi}$ (arbitrarily near φ) such that the saddle-node fixed point $\tilde{p}$ disappears and there exists a homoclinic tangency point x associated to $\tilde{q}_{\hat{\varphi}}$. Clearly, $x \in C_{\hat{\varphi}}(\tilde{q}_{\hat{\varphi}})$. Observe that this tangency is not (necessarily) isolated as the ones in previous cases.

Suppose, for contradiction, that $\hat{\varphi}_{|C_{\hat{\varphi}}(\tilde{q}_{\hat{\varphi}})}$ has the shadowing property. Fix $\epsilon > 0$ small enough, and let $C^u_\epsilon(x)$ be the connected component of $W^u(\tilde{q}_{\hat{\varphi}}, \hat{\varphi})$ containing x, defined similarly as in §2.2. Let $\delta = \delta(\epsilon) > 0$ be the number in the definition of the shadowing property for $\hat{\varphi}_{|C_{\hat{\varphi}}(\tilde{q}_{\hat{\varphi}})}$. Take $l_1, l_2 > 0$ such that both $\hat{\varphi}^{l_1}(x)$ and $\hat{\varphi}^{-l_2}(x)$ are in a small neighborhood U of $\tilde{q}_{\hat{\varphi}}$ and $|\hat{\varphi}^{l_1}(x) - \hat{\varphi}^{-l_2}(x)| < \delta$; see Figure 3(b). Then the union of two segments of true $\hat{\varphi}$-orbits

$$\{\ldots, \hat{\varphi}^{-2}(x), \hat{\varphi}^{-1}(x), x, \hat{\varphi}(x), \ldots, \hat{\varphi}^{l_1-1}(x), \hat{\varphi}^{-l_2}(x), \hat{\varphi}^{-l_2+1}(x), \ldots\} \subset C_{\hat{\varphi}}(\tilde{q}_{\hat{\varphi}})$$

is a δ-pseudo-orbit of $\hat{\varphi}$. Thus there is $z \in C^u_\epsilon(x)$ (near x) ϵ_1-shadowing the above pseudo-orbit.

Recall that the unstable eigenvalue of $D_{\tilde{q}_{\hat{\varphi}}}\hat{\varphi}$ is negative. If l_1 and l_2 have been chosen such that the connected component of $W^u(\hat{\varphi}^{l_1}(x), \hat{\varphi})$ in U containing $\hat{\varphi}^{l_1}(x)$ and the

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connected component of $W^s(\hat{\varphi}^{-l_2}(x), \hat{\varphi})$ in U containing $\hat{\varphi}^{-l_2}(x)$ are on different sides from each other across the local stable manifold of $\tilde{q}_{\hat{\varphi}}$ (see Figure 3(b)), then the forward orbits of $\hat{\varphi}^{l_1}(z)$ and $\hat{\varphi}^{-l_2}(x)$ are going in opposite directions since $\tilde{q}_{\hat{\varphi}}$ is hyperbolic. Thus the $\hat{\varphi}$-orbit of z cannot ϵ-shadow the pseudo-orbit, giving a contradiction.

3. *Preliminaries*

The purpose of this section is to state our main propositions (Propositions 3.1 and 3.2) and to prove some consequences of them. For the sake of simplicity, we suppose throughout this section that $f(p) = p$ is a hyperbolic saddle.

Denote by $homo_f(p)$ the set of homoclinic points associated with p, i.e.

$$homo_f(p) = \{x \in M : x \in W^s(p) \cap W^u(p)\},$$

and let

$$\tilde{H}_f(p) = \overline{homo_f(p)}.$$

This set is f-invariant, closed, and we have $H_f(p) \subset \tilde{H}_f(p) \subset C_f(p)$. We say that $\tilde{H}_f(p)$ is the *generalized homoclinic class* of p. Observe that if $q \in \tilde{H}_f(p) \cap P(f)$ is hyperbolic, then q is neither a sink nor a source.

In general, $H_f(p)$ does not coincide with $\tilde{H}_f(p)$ (hence, in general, $H_f(p) \subsetneqq C_f(p)$). Indeed, let $\{g_t\}_{t\in\mathbb{R}}$ be the bifurcating one-parameter family treated in §2.2, with a homoclinic tangency q_0 of g_0 associated with p. Then $H_{g_0}(p) = \Lambda_f$ and $\tilde{H}_{g_0}(p) = \Lambda_f \cup \mathcal{O}_{g_0}(q_0)$. Here, the existence of the homoclinic tangency is the obstruction for the generalized homoclinic class being the homoclinic class. Also, $\tilde{H}_f(p)$ does not coincide with $C_f(p)$ in general. In fact, if $\varphi : D^2 \to D^2$ is the map in §2.3, then $H_\varphi(\tilde{q}) = \tilde{H}_\varphi(\tilde{q}) = \Lambda_\varphi \subsetneqq C_\varphi(\tilde{q})$. The existence of the saddle-node point (non-hyperbolic periodic point) $\tilde{p}$ is the obstruction for the chain recurrent set being the generalized homoclinic class. However, these phenomena cannot occur if $C_f(p)$ is C^1-stably shadowable (see Lemma 3.6 below).

We say that $\tilde{H}_f(p)$ is *C^1-stably shadowable* if there exists $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, $g_{|\tilde{H}_g(p_g)}$ has the shadowing property. The C^1-stable shadowability prevents the existence of non-transverse intersections between the invariant manifolds of periodic points, i.e. homoclinic tangencies and heterodimensional cycles.

PROPOSITION 3.1. (Non-existence of non-transverse homoclinic points) *Let $\tilde{H}_f(p)$ be C^1-stably shadowable. Then there exists $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, if $x \in homo_g(p_g)$ then*

$$T_x M = T_x W^s(p_g, g) + T_x W^u(p_g, g) = T_x W^s(p_g, g) \oplus T_x W^u(p_g, g).$$

Thus, $H_g(p_g) = \tilde{H}_g(p_g)$ for any $g \in \mathcal{U}(f)$.

Recall that a non-hyperbolic homoclinic class $H_f(p)$ may contain saddle periodic points with different indices (see [**6, 9, 13–15, 20**]). Thus the chain component $C_f(p)$ may contain saddle periodic points with different indices, because $H_f(p) \subset C_f(p)$. However, if $C_g(p_g)$ admits a dominated splitting and $g_{|C_g(p_g)}$ has the shadowing property for any g near f, then such a case cannot happen. More precisely, we have the following.

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PROPOSITION 3.2. *Suppose that there exists $\mathcal{U}(f)$ such that for each $g \in \mathcal{U}(f)$, $C_g(p_g)$ admits a dominated splitting $T_{C_g(p_g)}M = E(g) \oplus F(g)$ with $\dim E(g) = \mathrm{index}(p_g)$, and $g_{|C_g(p_g)}$ has the shadowing property. Then, for any $g \in \mathcal{U}(f)$, if $q \in C_g(p_g) \cap P(g)$ is hyperbolic, then $\mathrm{index}(q) = \mathrm{index}(p_g)$.*

The following corollary, which asserts the non-existence of heterodimensional cycles, is a consequence of Propositions 3.1 and 3.2 (but not immediate). As stated previously, Theorem B has been obtained in [**34**] for homoclinic classes in the context of C^1-persistently expansive diffeomorphisms under the condition that $q \sim p$. We can remove this condition for C^1-stable shadowing diffeomorphisms by the following.

COROLLARY 3.3. (Non-existence of heterodimensional cycles) *Suppose that there exists $\mathcal{U}(f)$ such that for each $g \in \mathcal{U}(f)$, $C_g(p_g)$ admits a dominated splitting $T_{C_g(p_g)}M = E(g) \oplus F(g)$ with $\dim E(g) = \mathrm{index}(p_g)$, and $g_{|C_g(p_g)}$ has the shadowing property. Then there exists $\mathcal{V}(f) \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{V}(f)$, if $q \in C_g(p_g) \cap P(g)$ is hyperbolic, then $q \sim p_g$.*

Remark 3.4. Under the hypothesis of Corollary 3.3, if $q \in C_g(p_g) \cap P(g)$ ($g \in \mathcal{V}(f)$) is hyperbolic, then $E^s_q = E_q(g)$ and $E^u_q = F_q(g)$ by the uniqueness property of the dominated splitting. Let me emphasize that this property does not imply the hyperbolicity of the dominated splitting.

Propositions 3.1 and 3.2 are proved in §§4 and 5, respectively.

3.1. *Consequences of Proposition 3.1.* In this subsection, we prove two consequences of Proposition 3.1, which will mainly be used in the proof of Theorem A. To begin with, we prepare one lemma.

Let Λ be a closed f-invariant set. For $\epsilon > 0$, let

$$W^s_\epsilon(x) = \{y \in M : d(f^n(x),\ f^n(y)) \le \epsilon \ \text{ for all } n \ge 0\},$$
$$W^u_\epsilon(x) = \{y \in M : d(f^{-n}(x),\ f^{-n}(y)) \le \epsilon \ \text{ for all } n \ge 0\}$$

be the *local stable set* and the *local unstable set* of $x \in \Lambda$, respectively. If Λ is hyperbolic, then there is $\epsilon_0 > 0$ such that for any $0 < \epsilon \le \epsilon_0$, the above sets are C^1-embedded disks. Since $f(p) = p$ is a hyperbolic saddle, we have the following.

LEMMA 3.5. *There exists $\epsilon_0 > 0$ such that for any g near f, $d(g^i(x), g^i(p_g)) \le \epsilon_0$ for all $i \ge 0$ implies $x \in W^s_{\epsilon_0}(p_g, g)$, and $d(g^{-i}(x), g^{-i}(p_g)) \le \epsilon_0$ for all $i \ge 0$ implies $x \in W^u_{\epsilon_0}(p_g, g)$.*

The first result proves a part of Theorem A.

LEMMA 3.6. *Let $C_f(p)$ be C^1-stably shadowable, and let $\mathcal{U}(f)$ be given by Proposition 3.1. Then for any $g \in \mathcal{U}(f)$, we have $H_g(p_g) = \tilde{H}_g(p_g) = C_g(p_g)$.*

Proof. Let $\mathcal{U}(f)$ be given by Proposition 3.1, and fix any $g \in \mathcal{U}(f)$. We show that $H_g(p_g) = C_g(p_g)$. Recall that $\tilde{H}_g(p_g)$ is the closure of the set of homoclinic points of g associated with p_g. By Proposition 3.1, to obtain the conclusion, it is enough to

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show that every $x \in C_g(p_g)$ is an accumulation of points in $\tilde{H}_g(p_g)$; that is, for any $x \in C_g(p_g) \setminus \{p_g\}$ and any $\epsilon > 0$, there exists $y \in B_\epsilon(x) \cap homo_g(p_g)$.

If we construct a pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}}$ (with $x_0 = x$) such that x_i converges to p_g as $i \to \pm\infty$, then intuitively one can see that the shadowing point y (near x) is in $homo_g(p_g)$. Recall the notation $\leftrightsquigarrow$. Since $x \leftrightsquigarrow p_g$, for any $\delta > 0$ there exists a periodic δ-pseudo-orbit $\{x_i\}_{i=-n_1}^{n_2}$ of g such that $x_{-n_1} = p_g$, $x_0 = x$ and $x_{n_2} = p_g$ for some $n_1 = n_1(\delta)$ and $n_2 = n_2(\delta) > 0$. We don't know whether or not this periodic δ-pseudo-orbit is contained in $C_g(p_g)$ in general, but it was proved in [**35,** Proposition 1.6] that $\{x_i\}_{i=-n_1}^{n_2} \subset C_g(p_g)$. We extend the pseudo-orbit by setting $x_i = g^{i+n_1}(p_g)$ for $i \le -n_1$ and $x_i = g^{i-n_2}(p_g)$ for $i \ge n_2$, and thus $x_i \to p_g$ as $i \to \pm\infty$.

For every $0 < \epsilon < \min\{\epsilon_0, d(x, p_g)/2\}$, let $0 < \delta = \delta(\epsilon) \le \epsilon$ be the number in the definition of the shadowing property for $g_{|C_g(p_g)}$. Then there is $y \in B_\epsilon(x)$ such that $d(g^i(y), x_i) < \epsilon$ and $d(g^{-i}(y), x_{-i}) < \epsilon$ for all $i \ge 0$. Since $d(g^{n_2+i}(y), g^i(p_g)) < \epsilon$ for all $i \ge 0$, $g^{n_2}(y) \in W^s_{\epsilon_0}(p_g)$ by Lemma 3.5, and from this we have $y \in W^s(p_g)$. The same reasoning shows that $y \in W^u(p_g)$. Thus $y \in W^s(p_g) \cap W^u(p_g) \cap B_\epsilon(x) \subset \tilde{H}_g(p_g)$. □

Remark 3.7. Let $C_f(p)$ be C^1-stably shadowable, and let $\mathcal{U}(f)$ be given by Proposition 3.1. Then for any $g \in \mathcal{U}(f)$, by our definition of the homoclinic class $H_g(p_g)$, hyperbolic periodic points q of g satisfying $q \sim p_g$ are dense in $C_g(p_g)$. This fact will be used in the proofs of Theorems C and D.

To state the second result, let us define the angle between two subspaces of $\mathbb{R}^{\dim M}$. Let E and F be two subspaces of $\mathbb{R}^{\dim M}$ such that $E \oplus F = \mathbb{R}^{\dim M}$. Hence $\dim F = \dim E^\perp$ and F is the graph of the linear map $L : E^\perp \to E$ defined as follows: given $v \in E^\perp$ there exists a unique pair of vectors $u \in E$, $w \in F$ such that $v + u = w$; define $L(v) = u$ so that L is linear and $graph(L) = F$. We define the angle $\angle(E, F)$ between E and F as $\|L\|^{-1}$. In particular, $\angle(E, E^\perp) = +\infty$ (see [**22, 34**]).

For any g near f and for any $x \in homo_g(p_g)$, define $E_x(g)$ and $F_x(g)$ by the subspaces of T_xM tangent to $W^s(p_g, g)$ and $W^u(p_g, g)$ at the point x, respectively; that is,

$$E_x(g) = T_x W^s(p_g, g) \quad \text{and} \quad F_x(g) = T_x W^u(p_g, g).$$

Observe that $\dim E_x(g) = \text{index}(p_g)$. It will be shown later in §6 that these subspaces construct a dominated splitting on the chain recurrent set containing p_g, under the condition that $C_f(p)$ is C^1-stably shadowable.

By Proposition 3.1, we can show that the angle between the above two subspaces at x is bounded from below.

LEMMA 3.8. (cf. [**34,** Proposition 1]) *Let $C_f(p)$ be C^1-stably shadowable, and let $\mathcal{U}(f)$ be given by Proposition 3.1. Then there exists $\gamma > 0$ such that for any $g \in \mathcal{U}(f)$, $\angle(E_x(g), F_x(g)) > \gamma$ for any $x \in homo_g(p_g)$.*

Proof. Let $\mathcal{U}(f)$ be given by Proposition 3.1. If the assertion of this lemma is false, then for any $n > 0$, there exist $g_n \in \mathcal{U}(f)$ and $x_n \in homo_{g_n}(p_{g_n})$ such that $\angle(E_{x_n}(g_n), F_{x_n}(g_n)) \le 1/n$. Thus, for sufficiently large $n > 0$, by slightly perturbing g_n at $g_n^{-1}(x_n)$ we can construct $\tilde{g} \in \mathcal{U}(f)$ near g_n such that $x_n \in homo_{\tilde{g}}(p_{\tilde{g}})$ and $E_{x_n}(\tilde{g}) \cap F_{x_n}(\tilde{g}) \neq \{0\}$. This contradicts Proposition 3.1. □

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3.2. *Proof of Corollary 3.3.* Before proving the corollary, we prepare one simple lemma which is proved in a similar manner to Lemma 3.6. This lemma will also be used in the proof of Proposition 3.2 later on.

LEMMA 3.9. *Suppose that $f_{|C_f(p)}$ has the shadowing property. If $q \in C_f(p) \cap P(f)$ is a hyperbolic saddle, then $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$.*

Proof. Suppose that $f_{|C_f(p)}$ has the shadowing property, and let $q \in C_f(p) \cap P(f)$ be a hyperbolic saddle. Choose $\epsilon_0 > 0$ such that both $W^\sigma_{\epsilon_0}(p)$ and $W^\sigma_{\epsilon_0}(q)$ (where $\sigma = s$ or u) are C^1-embedded disks (see Lemma 3.5). Let $0 < \delta = \delta(\epsilon_0) < \epsilon_0$ be the number in the definition of the shadowing property for $f_{|C_f(p)}$. Since $q \in C_f(p)$, there exists a δ-pseudo-orbit $\{x_i\}_{i=0}^{n_\delta}$ of f such that $x_0 = p$ and $x_{n_\delta} = q$. We can assume that $\{x_i\}_{i=0}^{n_\delta} \subset C_f(p)$ by [**35,** Proposition 1.6].

Now, we extend the δ-pseudo-orbit by putting $x_{-i} = f^{-i}(p)$ and $x_{n_\delta+i} = f^i(q)$ for all $i \geq 0$. Since $f_{|C_f(p)}$ has the shadowing property, there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon_0$ for all $i \in \mathbb{Z}$. Thus $y \in W^u_{\epsilon_0}(p)$ and $f^{n_\delta}(y) \in W^s_{\epsilon_0}(q)$ by Lemma 3.5, so that $y \in W^u(p) \cap f^{-n_\delta}(W^s(q))$. Since $f(p) = p$, we have $f^{n_\delta}(y) \in W^u(p) \cap W^s(q)$. In an analogous way we can show that $W^s(p) \cap W^u(q) \neq \emptyset$. □

End of the proof of Corollary 3.3. Suppose that f satisfies the assumption of the corollary, and let $\mathcal{U}(f)$ be as in the assumption. Observing that $C_f(p)$ is C^1-stably shadowable, denote here by $\mathcal{V}(f)$ ($\subset \mathcal{U}(f)$) the neighborhood given by Proposition 3.1. Thus, for any $g \in \mathcal{V}(f)$, $T_xM = T_xW^s(p_g, g) + T_xW^u(p_g, g)$ for all $x \in homo_g(p_g)$ by the proposition. Fix any $g \in \mathcal{V}(f)$ and any hyperbolic periodic saddle $q \in C_g(p_g) \cap P(g)$, and we have $q \sim p_g$. It follows from Proposition 3.2 that $\mathrm{index}(q) = \mathrm{index}(p_g)$. By Lemma 3.9, we can pick $x \in W^s(p_g, g) \cap W^u(q, g)$ and $y \in W^u(p_g, g) \cap W^s(q, g)$ since $g_{|C_g(p_g)}$ has the shadowing property. We show that these intersections are transverse. First, we prove that $T_xM = T_xW^s(p_g, g) + T_xW^u(q, g)$ by contradiction. Suppose that $T_xM \neq T_xW^s(p_g, g) + T_xW^u(q, g)$; then, since $\mathrm{index}(q) = \mathrm{index}(p_g)$, there is $\bar{g} \in \mathcal{V}(f)$ near g such that:

- $x \in W^s(p_{\bar{g}}, \bar{g}) \cap W^u(q_{\bar{g}}, \bar{g})$ and $y \in W^u(p_{\bar{g}}, \bar{g}) \cap W^s(q_{\bar{g}}, \bar{g})$;
- $T_xM \neq T_xW^s(p_{\bar{g}}, \bar{g}) + T_xW^u(q_{\bar{g}}, \bar{g})$, but $T_yM = T_yW^u(p_{\bar{g}}, \bar{g}) + T_yW^s(q_{\bar{g}}, \bar{g})$.

Since $T_yM = T_yW^u(p_{\bar{g}}, \bar{g}) + T_yW^s(q_{\bar{g}}, \bar{g})$, by the λ-lemma we can construct $\tilde{g} \in \mathcal{V}(f)$ near $\bar{g}$ such that there is $\tilde{x} \in W^s(p_{\tilde{g}}, \tilde{g}) \cap W^u(p_{\tilde{g}}, \tilde{g})$ near x (so $\tilde{x} \in homo_{\tilde{g}}(p_{\tilde{g}})$) satisfying $T_{\tilde{x}}M \neq T_{\tilde{x}}W^s(p_{\tilde{g}}, \tilde{g}) + T_{\tilde{x}}W^u(p_{\tilde{g}}, \tilde{g})$. This contradicts Proposition 3.1 since $\tilde{x} \in homo_{\tilde{g}}(p_{\tilde{g}})$ and $\tilde{g} \in \mathcal{V}(f)$. Therefore $T_xM = T_xW^s(p_g, g) + T_xW^u(q, g)$. The proof of $T_yM = T_yW^u(p_g, g) + T_yW^s(q, g)$ is analogous. □

4. *Proof of Proposition 3.1*

We divide the proof of into two cases: the two-dimensional case and the general case. The proof of the two-dimensional case is simple and will be a good reference for the proof of the general case. Throughout this section, suppose that $f(p) = p$ is a hyperbolic saddle and that

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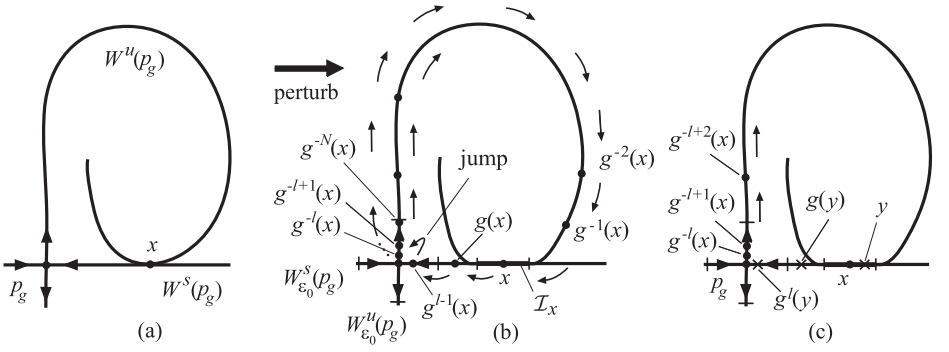


FIGURE 4. Two-dimensional case.

- the generalized homoclinic class $\tilde{H}_f(p)$ is C^1-stably shadowable; that is, there is $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, $g_{|\tilde{H}_g(p_g)}$ has the shadowing property.

The proof is done by contradiction. Assume that there are $g \in \mathcal{U}(f)$ and $x \in W^s(p_g, g) \cap W^u(p_g, g) \setminus \{p_g\}$ such that

$$T_x M \neq T_x W^s(p_g, g) + T_x W^u(p_g, g). \tag{1}$$

The strategy of the proof is: by assumption (1), we construct a new diffeomorphism near g (by perturbation) and a special pseudo-orbit of the perturbed map in the generalized homoclinic class. Since the perturbation is small, the perturbed map has the shadowing property so that the above pseudo-orbit must be shadowed by some g-orbit. But we show that the shadowing g-orbit cannot in fact shadow the above pseudo-orbit. This gives a contradiction.

4.1. *Proof of Proposition 3.1: the two-dimensional case.* In this case, $\dim W^\sigma(p_g) = 1$ for $\sigma = s, u$; see Figure 4(a).

4.1.1. *Construction of a stable–unstable set.* Pick $r > 0$ with $B_r(x) \cap B_r(g^k(x)) = \emptyset$ for all $k \neq 0$. Since $T_x W^u(p_g, g) = T_x W^s(p_g, g)$ (see (1)), by perturbing g to $\tilde{g}$ at $g^{-1}(x)$ we can find a small arc $\mathcal{I}_x$ such that $x \in \operatorname{int} \mathcal{I}_x$ and

$$\mathcal{I}_x \subset W^s(p_{\tilde{g}}, \tilde{g}) \cap W^u(p_{\tilde{g}}, \tilde{g}) \cap B_r(x).$$

For simplicity, we denote $\tilde{g}$ by g and $W^\sigma(p_{\tilde{g}}, \tilde{g})$ by $W^\sigma(p)$ for $\sigma = s, u$ (see Figure 4(b)). Clearly, $\mathcal{I}_x \subset \tilde{H}_g(p)$. We call $\mathcal{I}_x$ the *stable–unstable set of* x because the set is contained in both the stable and the unstable manifolds of p_g.

4.1.2. *Construction of a unstable connected component.* Fix $\epsilon_0 > 0$ such that both $W^s_{\epsilon_0}(p)$ and $W^u_{\epsilon_0}(p)$ are C^1-embedded disks and disjoint (see Lemma 3.5). Let $N > 0$ be the smallest integer such that $g^{-N}(x) \in W^u_{\epsilon_0}(p)$. Thus $x \in g^N(W^u_{\epsilon_0}(p))$. Clearly, $g^{-N}(x) \notin W^s_{\epsilon_0}(p)$, and so we let

$$\epsilon_1 = \min\{\epsilon_0,\ d(g^{-N}(x),\ W^s_{\epsilon_0}(p))\}/2 > 0. \tag{2}$$

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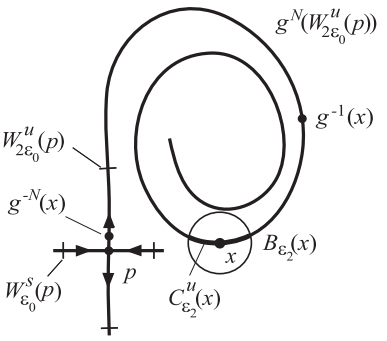


FIGURE 5. The unstable connected component of x.

For any $\epsilon > 0$, define

$$C^u_\epsilon(x) \text{ to be the connected component of } B_\epsilon(x) \cap g^N(W^u_{2\epsilon_0}(p)) \text{ containing } x.$$

We call $C^u_\epsilon(x)$ the *unstable connected component of* x. Since the distance from x to p with respect to the inner metric on $W^u(p)$ is finite, we have the following result which is independent of the dimension of the unstable manifold.

LEMMA 4.1. [**27, 31**] *There is* $\epsilon_2 \in (0, \epsilon_1]$ *such that for any* $0 < \epsilon \le \epsilon_2$, *if* $d(g^{-n}(x), g^{-n}(y)) < \epsilon$ *for all* $n \ge 0$, *then* $y \in C^u_\epsilon(x)$.

Proof. Recall that $x \in g^N(W^u_{\epsilon_0}(p)) \subset g^N(W^u_{2\epsilon_0}(p))$. In general, there are finitely many connected components in $B_\epsilon(x) \cap g^N(W^u_{2\epsilon_0}(p))$, and the above $C^u_\epsilon(x)$ is the one containing x. However, if we take $\epsilon_2 \in (0, \epsilon_1]$ small enough, then since $g^N(W^u_{2\epsilon_0}(p))$ is a compact subset of $W^u(p)$, for any $\epsilon_2 \in (0, \epsilon_1]$ the number of connected components of $B_\epsilon(x) \cap g^N(W^u_{2\epsilon_0}(p))$ is just one and must coincide with $C^u_\epsilon(x)$ (see Figure 5).

If $0 < \epsilon \le \epsilon_2$ and $d(g^{-n}(x), g^{-n}(y)) < \epsilon$ for all $n \ge 0$, then by Lemma 3.5 $g^{-N}(y) \in W^u_{2\epsilon_0}(p)$ so that $y \in B_\epsilon(x) \cap g^N(W^u_{2\epsilon_0}(p)) = C^u_\epsilon(x)$. Thus $y \in C^u_\epsilon(x)$. □

Since $x \in \operatorname{int} \mathcal{I}_x$, we may assume that $B_{\epsilon_2}(x) \cap \mathcal{I}_x \subset \operatorname{int} \mathcal{I}_x$, reducing ϵ_2 if necessary. Thus

$$C^u_{\epsilon_2}(x) \subset \mathcal{I}_x. \tag{3}$$

Now, let us construct a special pseudo-orbit of g. Before constructing the pseudo-orbit, let us import the notion of elementary pseudo-orbits from [**38**]. We say that a δ-pseudo-orbit $\{x_i\}_{i=a}^b$ is *elementary* if there is $a < i_0 < b$ such that $g(x_i) = x_{i+1}$ for all i except for $i = i_0$.

Recalling that $g_{|\tilde{H}_g(p_g)}$ has the shadowing property, let $0 < \delta = \delta(\epsilon_2) \le \epsilon_2$ be the number associated with the property. Take $l > N$ such that $d(g^l(x), g^{-l}(x)) < \delta$ and $g^{l-1}(x) \in W^s_{\epsilon_0/2}(p)$. Then the union of two segments of true g-orbits

$$\mathcal{PO}_\delta = \{\ldots, g^{-2}(x), g^{-1}(x), x, g(x), \ldots, g^{l-1}(x), g^{-l}(x), g^{-l+1}(x), \ldots, g^{-N}(x), \ldots\}$$

is an elementary δ-pseudo-orbit of g; see Figure 4(b). Clearly, $\mathcal{PO}_\delta \subset \tilde{H}_g(p)$.

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4.1.3. *End of the proof of Proposition 3.1: the two-dimensional case.* Since $g_{|\tilde{H}_g(p)}$ has the shadowing property, $\mathcal{PO}_\delta$ must be ϵ_2-shadowed by some g-orbit; that is, we can pick $y \in B_{\epsilon_2}(x)$ such that:

- $d(g^{-n}(x), g^{-n}(y)) < \epsilon_2$ for all $n \geq 0$;
- $d(g^n(x), g^n(y)) < \epsilon_2$ for $0 \leq n \leq l-1$; and
- $d(g^{-l+n}(x), g^{l+n}(y)) < \epsilon_2$ for $n \geq 0$.

Thus $d(g^{-n}(x), g^{-n}(y)) < \epsilon_2$ for all $n \geq 0$ implies that $y \in \mathcal{I}_x$ (since $y \in C^u_{\epsilon_2}(x)$ by Lemma 4.1 and (3)), and hence $y \in W^s(p)$. Therefore $g^{2l-N}(y) \in W^s_{\epsilon_0}(p)$ by Lemma 3.5 since $\epsilon_2 \leq \epsilon_0/2$ (see Figure 4(c)). However, we see that $d(g^{-N}(x), g^{2l-N}(y)) < \epsilon_2$ so that $d(g^{-N}(x), W^s_{\epsilon_0}(p)) < \epsilon_2 \leq \epsilon_1$. This contradicts (2), and the proof is complete. □

Remark 4.2. By the perturbation for g at $g^{-1}(x)$, $W^s(p, g)$ is slightly deformed in a small neighborhood of $g^{-n}(x) = \tilde{g}^{-n}(x)$ $(n \geq 1)$. Observe that $g^n(x) = \tilde{g}^n(x)$ for $n \geq 0$, and the connected component of the deformed stable manifold $W^s(p_{\tilde{g}}, \tilde{g})$ containing x coincides with $W^s(p, g)$ in a neighborhood of x. Thus such a deformation does not have an effect on the argument.

4.2. *Proof of Proposition 3.1: the general case.* Recall condition (1), and let $T_{p_g}M = E^s_{p_g} \oplus E^u_{p_g}$ be the hyperbolic splitting of p_g. We treat the situation where either $\dim E^s_{p_g} \geq 2$ or $\dim E^u_{p_g} \geq 2$. Suppose that

- $\dim E^s_{p_g} \geq 2$ (the other case is similar).

We perturb g several times. The following well-known lemma plays an important role in these perturbations of the map.

LEMMA 4.3. *Let $f \in \mathrm{Diff}(M)$ and let $\mathcal{U}(f)$ be a C^1-neighborhood of f. Then there are $\delta > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ such that for any given $g \in \mathcal{U}_0(f)$, finite set $\{x_1, x_2, \ldots, x_N\}$, neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \leq \delta$ for all $1 \leq i \leq N$, there exists $\hat{g} \in \mathcal{U}(f)$ such that $\hat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \ldots, x_N\} \cup (M \setminus U)$, and $D_{x_i}\hat{g} = L_i$ for all $1 \leq i \leq N$.*

Proof. See [**16,** Lemma 1.1] or [**22,** Lemma II.2]. □

4.2.1. *Local linearization.* Take $\epsilon_0 > 0$ such that $W^\sigma_{\epsilon_0}(p_g, g)$ $(\sigma = s, u)$ are C^1-disks. First of all, by Lemma 4.3 we linearize g at p_g with respect to the exponential coordinates $\exp_{p_g}$, without changing its stable and unstable manifolds near the non-transverse intersection point x. More precisely, we construct $g_1 \in \mathcal{U}(f)$ near g satisfying the following assumptions (a.1)–(a.3).

(a.1) There exists a sufficiently small $0 < \alpha < \epsilon_0$ such that

$$g_1(y) = \begin{cases} \exp_{p_g} \circ D_{p_g}g \circ \exp^{-1}_{p_g}(y) & \text{if } y \in B_{\alpha/4}(p_g), \\ g(y) & \text{if } y \notin B_\alpha(p_g), \end{cases}$$

and thus $g_1(p_g) = p_g$ and $D_{p_g}g_1 = D_{p_g}g$.

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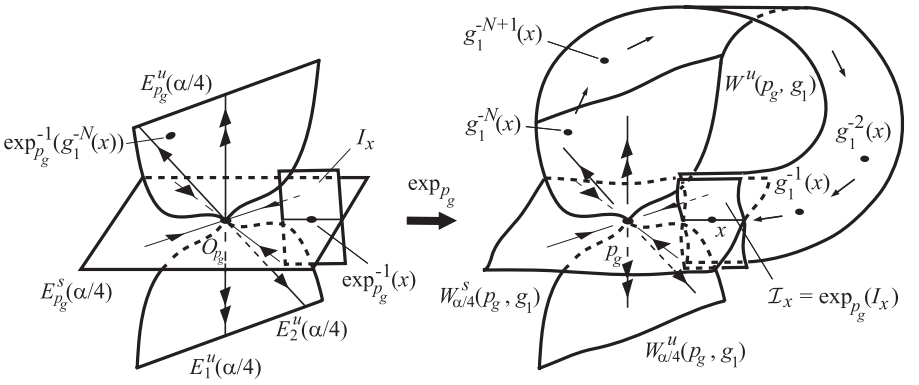


FIGURE 6. Local linearization and an affine unstable piece.

(a.2) The non-transverse intersection point x is still the non-transverse intersection point for g_1, i.e. $x \in W^s(p_g, g_1) \cap W^u(p_g, g_1)$,

$$T_x W^s(p_g, g) = T_x W^s(p_g, g_1) \quad \text{and} \quad T_x W^u(p_g, g) = T_x W^u(p_g, g_1),$$

so that $T_x M \neq T_x W^s(p_g, g_1) + T_x W^u(p_g, g_1)$ by (1).

Observe that both $W^s_{\alpha/4}(p_g, g_1)$ and $W^u_{\alpha/4}(p_g, g_1)$ are 'linear pieces' with respect to the exponential coordinates, i.e.

$$W^\sigma_{\alpha/4}(p_g, g_1) = \exp_{p_g}(E^\sigma_{p_g}(\alpha/4)) \quad \text{for } \sigma = u, s$$

(see [**31**]). By slightly modifying $D_{p_g} g$, we may suppose that

(a.3) $D_{p_g} g = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$ with respect to the hyperbolic splitting (so $D_{p_g} g_{|E^s_{p_g}} = A$ and $D_{p_g} g_{|E^u_{p_g}} = B$), and that the multiplicity of any eigenvalue for B is one; that is,

$$B = \begin{pmatrix} B_1 & & O \\ & \ddots & \\ O & & B_{k'} \end{pmatrix},$$

with each B_i being a (at most 2×2) matrix corresponding to the eigenvalue λ_i, where $1 < |\lambda_1| < \cdots < |\lambda_{k'}|$ and $1 \leq k' \leq \dim E^u_{p_g}$.

We may suppose further, by iterating x by g_1 if necessary, that $x \in \operatorname{int} W^s_{\alpha/4}(p_g, g_1)$. Finally, take the smallest integer $N > 0$ such that

$$g_1^{-N}(x) \in \operatorname{int} W^u_{\alpha/4}(p_g, g_1). \tag{4}$$

Figure 6 shows the case where $\dim E^s_{p_g} = \dim E^u_{p_g} = 2$.

4.2.2. *Construction of an affine unstable piece.* This part corresponds to §4.1.1 in the previous subsection. Take $0 < r < \alpha/8$ small enough so that:

- $B_r(g_1^{-i}(x)) \cap B_r(g_1^{-j}(x)) = \emptyset$ whenever $0 \leq i \neq j \leq N-1$; and
- $B_r(g_1^{-i}(x)) \cap B_r(p_g) = \emptyset$ for $0 \leq i \leq N-1$.

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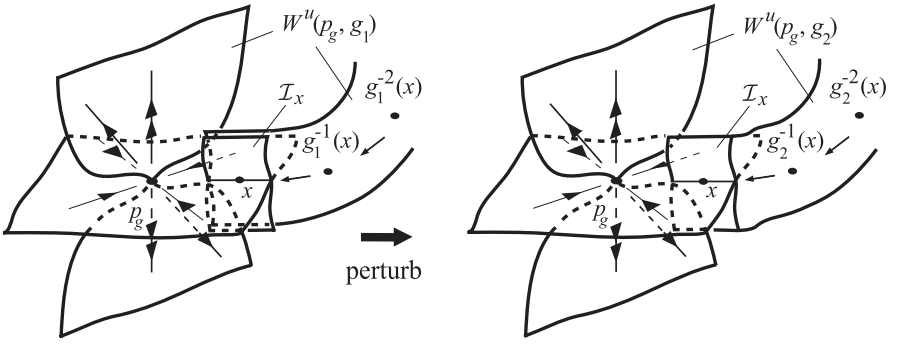


FIGURE 7. The perturbation of g_1.

Let $T_{p_g}M(\alpha/4) = \{v \in T_{p_g}M : \|v\| \le \alpha/4\}$ and choose a small 'affine piece' $I_x \subset T_{p_g}M(\alpha/4)$ centered at $\exp_{p_g}^{-1}(x)$ with dimension equal to $\dim E^u_{p_g}$, such that

$$T_x\mathcal{I}_x = T_x W^u(p_g, g_1).$$

Here $\mathcal{I}_x = \exp_{p_g}(I_x)$; see Figure 6. The next lemma is obtained by reducing the diameter of I_x (if necessary) and modifying the map g_1 in $B_r(g_1^{-1}(x))$; see Figure 7.

LEMMA 4.4. *There exists $g_2 \in \mathcal{U}(f)$ near g_1 such that $g_2(g_1^{-1}(x)) = x$, $\mathcal{I}_x \subset W^u(p_g, g_2)$ and $T_x\mathcal{I}_x = T_x W^u(p_g, g_2)$.*

It is easy to see that $g_2(p_g) = p_g$, $D_{p_g}g_2 = D_{p_g}g$, $T_x W^s(p_g, g_2) = T_x W^s(p_g, g_1)$ and $T_x W^u(p_g, g_2) = T_x W^u(p_g, g_1)$ by construction. Thus

$$T_x M \ne T_x W^s(p_g, g_2) + T_x W^u(p_g, g_2). \tag{5}$$

Observe that $\mathcal{I}_x$ is not necessarily a stable–unstable set of x (with respect to g_2 and p_{g_2}); this differs from the two-dimensional case described in §4.1.1.

4.2.3. *Local identification with the linear part.* By assumption (a.1), the local system $g_{2|B_{\alpha/4}(p_g)}$ is conjugate to the linear part $D_{p_g}g$ under the exponential map. Since our argument is almost local, to avoid complexity in the presentation caused by the exponential map, we shall hereafter identify $g_{2|B_{\alpha/4}(p_g)}$ with its derivative $D_{p_g}g$ by assuming that $B_{\alpha/4}(p_g) \subset \mathbb{R}^{\dim M}$ and p_g is the origin $O \in \mathbb{R}^{\dim M}$. Moreover, for notational simplicity, we denote g_2 by g and $W^\sigma_{\alpha/4}(p_g, g_2)$ by $W^\sigma_{\alpha/4}(p_g)$ $(\sigma = s, u)$.

Let us list some changes in notation and the properties preserved by the above identification:

(a.4) $W^\sigma_{\alpha/4}(p_g) = E^\sigma_{p_g}(\alpha/4)$ for $\sigma = s, u$;

(a.5) $I_x \subset W^u(p_g)$ and $\dim I_x = \dim E^u_{p_g}$ by Lemma 4.4.

Reducing the diameter of I_x if necessary, we may suppose further that:

(a.6) $g^{-N}(I_x) \subset W^u_{\alpha/4}(p_g)$ by condition (4).

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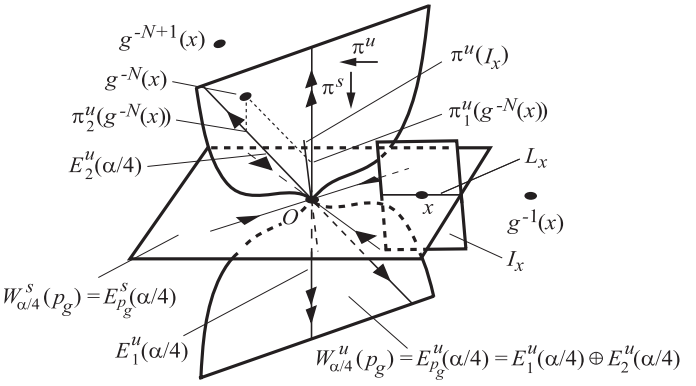


FIGURE 8. The dynamics of g after the identification.

4.2.4. *Projections.* Let $\pi^u : B_{\alpha/4}(p_g) \to E^u_{p_g}$ and $\pi^s : B_{\alpha/4}(p_g) \to E^s_{p_g}$ be the projections parallel to $E^s_{p_g}$ and $E^u_{p_g}$, respectively. Denote by $E_1^u (\subset E^u_{p_g})$ the eigenspace corresponding to the strongest eigenvalue $\lambda_{k'}$, and by $E_2^u (\subset E^u_{p_g})$ the eigenspace corresponding to the remaining eigenvalues. Thus $E^u_{p_g} = E_1^u \oplus E_2^u$. Finally, denote by π_1^u and π_2^u the projections to E_1^u and E_2^u parallel to $E_2^u \oplus E^s_{p_g}$ and $E_1^u \oplus E^s_{p_g}$, respectively; see Figure 8.

For any $(v^s, v_1^u, v_2^u) \in E^s_{p_g} \oplus E_1^u \oplus E_2^u = E^s_{p_g} \oplus E^u_{p_g}$, we define a metric on $B_{\alpha/4}(p_g)$ $(\subset \mathbb{R}^{\dim M})$ by

$$\|(v^s, v_1^u, v_2^u)\| = \max\{|v^s|, |v_1^u|, |v_2^u|\}.$$

We may suppose that this metric is equivalent to the original metric d on M.

Put $L_x = I_x \cap E^s_{p_g}(\alpha/4)$. Then:

(a.7) $\dim \pi^u(I_x) = \dim I_x - \dim L_x = \dim E^u_{p_g} - \dim L_x$.

Observe that $1 \le \dim L_x \le \dim E^s_{p_g}$ by (5) and that L_x is the stable–unstable set of x.

Now let us divide the proof of Proposition 3.1 into three cases. Recall, from (a.3), that $D_{p_g} g_{|E^u_{p_g}} = B$ has a diagonal form with elements B_i $(1 \le i \le k' \le \dim E^u_{p_g})$, where B_i is a (at most 2×2) matrix corresponding to the eigenvalue λ_i such that $1 < |\lambda_1| < \cdots < |\lambda_{k'}|$ and each eigenvalue has multiplicity one. There are the following three possibilities:

(1) $I_x = L_x$;

(2) $I_x \ne L_x$ with

(2.1) $k' > 1$; or

(2.2) $k' = 1$.

Case (1). Suppose that $I_x = L_x$. Then $\pi^u(I_x) = O$, and so I_x is just the stable–unstable set of x as in the two-dimensional case (see §4.1.1). The codimension one case (where $\dim E^u_{p_g} = 1$) is this, and the proof is the same as that for the two-dimensional case. Indeed, if we construct an elementary pseudo-orbit of g in $\tilde{H}_g(p_g)$ composed of two segments of the true g-orbit of x by just 'jumping' from $W^s_{\alpha/4}(p_g)$ to $W^u_{\alpha/4}(p_g)$ in a similar way, then it is easy to see that such a pseudo-orbit cannot be shadowed by any true g-orbit. This is a contradiction since $g \in \mathcal{U}(f)$, thus Proposition 3.1 is proved in Case (1). □

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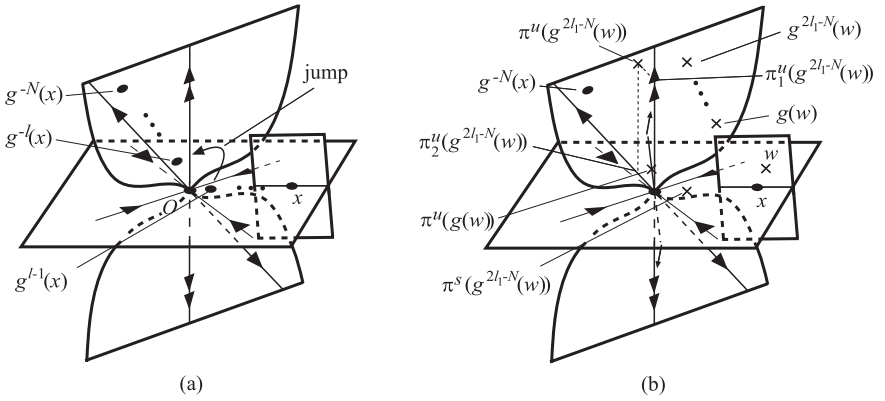


FIGURE 9. An elementary pseudo-orbit and the shadowing orbit; Case (2.1).

Case (2.1). Suppose that $I_x \neq L_x$ (hence $\pi^u(I_x) \neq O$) and $k' > 1$. Let me remark first that the following argument does *not* depend on whether $\lambda_{k'}$ is complex or not.

With a further modification of the map g at $g^{-N+1}(x)$ and $g^{-1}(x)$, we may suppose that:

(a.8) $\pi_2^u(g^{-N}(x)) \neq 0$, i.e. $g^{-N}(x) \notin E_1^u(\alpha/4)$; and

(a.9) $\pi_1^u(I_x) \neq \{0\}$, i.e. $\pi^u(I_x) \neq E_2^u$.

See Figure 8.

To control the behavior of shadowing orbits, we need the following lemma. Denote by $\mathcal{C}(p_g)$ a vertical cone in $E_{p_g}^u$ containing E_1^u such that $\pi^u(I_x) \subset \mathcal{C}(p_g)$ (see (a.9)). We have $g(\mathcal{C}(p_g) \cap B_{\alpha/4}(p_g)) \subset \mathcal{C}(p_g)$ from (a.3). Thus the next lemma is easily checked by a standard argument about $\mathcal{C}(p_g)$.

LEMMA 4.5. *For any $\epsilon > 0$ there is an integer $l_0 = l_0(\epsilon) > 0$ such that if $l \geq l_0$ and $g^n(w) \in B_{\alpha/4}(p_g)$ $(w \in I_x \setminus \{x\})$ for $1 \leq n \leq l$, then*

$$\frac{|\pi_2^u(g^l(w))|}{|\pi_1^u(g^l(w))|} < \epsilon.$$

Since $\pi_2^u(g^{-N}(x)) \neq 0$ by (a.8), we let

$$\epsilon_1 = d(g^{-N}(x),\ E_1^u(\alpha/4))/2 > 0. \tag{6}$$

Let us define the unstable connected component of x as follows: for any $\epsilon > 0$, let

$$C_\epsilon^u(x) \text{ be the connected component of } B_\epsilon(x) \cap g^N(W_{\alpha/4}^u(p_g)) \text{ containing } x.$$

Then, by Lemma 4.1, there is $0 < \epsilon_2 \leq \min\{\epsilon_1, \alpha/8\}$ such that for any $0 < \epsilon \leq \epsilon_2$, if $d(g^{-n}(x),\ g^{-n}(w)) < \epsilon$ for all $n \geq 0$, then $w \in C_\epsilon^u(x)$. We may suppose that $C_{\epsilon_2}^u(x) \subset I_x$, reducing ϵ_2 if necessary.

Now, let us construct a special pseudo-orbit of g. Recall that $g_{|\tilde{H}_g(p_g)}$ has the shadowing property. Let $l_0 = l_0(\epsilon_2) > N$ be given by Lemma 4.5, and let $0 < \delta = \delta(\epsilon_2) \leq \epsilon_2$ be the number in the definition of the shadowing property for $g_{|\tilde{H}_g(p_g)}$. Take $l_1 = l_1(\delta) \geq l_0$ such that:

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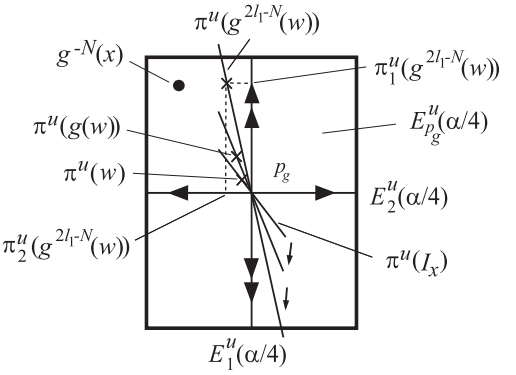


FIGURE 10. Behavior of the shadowing orbit; Case (2.1).

- if $l \geq l_1$, then $d(g^l(x), g^{-l}(x)) < \delta$; and
- if $l \geq l_1$ and $g^n(w) \in B_{\alpha/4}(p_g)$ $(w \in I_x)$ for $0 \leq n \leq l$, then $|\pi^s(g^l(w))| < \epsilon_2$.

It is easy to see that

$$\mathcal{PO}_\delta = \{\ldots, g^{-2}(x), g^{-1}(x), x, g(x), \ldots, g^{l_1-1}(x), g^{-l_1}(x), g^{-l_1+1}(x), \ldots, g^{-N}(x), \ldots\}$$

is an elementary δ-pseudo-orbit of g in $\tilde{H}_g(p_g)$; see Figure 9(a).

4.2.5. *End of the proof of Proposition 3.1 in Case (2.1).* Since $g_{|\tilde{H}_g(p_g)}$ has the shadowing property, $\mathcal{PO}_\delta$ must be ϵ_2-shadowed by some g-orbit; that is, there is $w \in B_{\epsilon_2}(x) \setminus \{x\}$ such that:

- $d(g^{-n}(x), g^{-n}(w)) < \epsilon_2$ for all $n \geq 0$;
- $d(g^n(x), g^n(w)) < \epsilon_2$ for $0 \leq n \leq l_1 - 1$; and
- $d(g^{-l_1+n}(x), g^{l_1+n}(w)) < \epsilon_2$ for $n \geq 0$.

By Lemma 4.1, $w \in C_{\epsilon_2}^u(x) \subset I_x \setminus \{x\}$; see Figure 9(b). Moreover, since $g^n(w) \in B_{\alpha/4}(p_g)$ for $0 \leq n \leq 2l_1 - N$, by Lemma 4.5 we have that

$$|\pi_2^u(g^{2l_1-N}(w))| \leq \epsilon_2 \cdot |\pi_1^u(g^{2l_1-N}(w))| \leq \epsilon_2 \cdot \frac{\alpha}{4} < \epsilon_2$$

(see Figure 10). Thus $d(g^{2l_1-N}(w), E_1^u(\alpha/4)) < \epsilon_2$ since $|\pi^s(g^{2l_1-N}(w))| < \epsilon_2$. On the other hand, because $d(g^{-N}(x), g^{2l_1-N}(w)) < \epsilon_2$, we have $d(g^{-N}(x), E_1^u(\alpha/4)) < 2\epsilon_2$. This contradicts (6) and so Proposition 3.1 is proved in the Case (2.1). □

Case (2.2). Suppose that $I_x \neq L_x$ (hence $\pi^u(I_x) \neq O$) and $k' = 1$. Thus $\dim E_{p_g}^u = 2$, and so $\dim \pi^u(I_x) = 1$. We remark that, referring to (a.3), the eigenvalue λ_1 corresponding to $B = B_1$ must be complex. By slightly modifying $D_{p_g} g_{|E_{p_g}^u} = B_1$ if necessary, we may assume that $g_{|\{0\} \times E_{p_g}^u(\alpha/4)}$ is a linear expanding irrational rotation map.

Let us construct a special pseudo-orbit of g (the construction is slightly complicated). Put $\epsilon = \frac{\alpha}{8|\lambda_1|} \sin 1 > 0$, and let $0 < \delta = \delta(\epsilon) \leq \epsilon$ be the number in the definition of the

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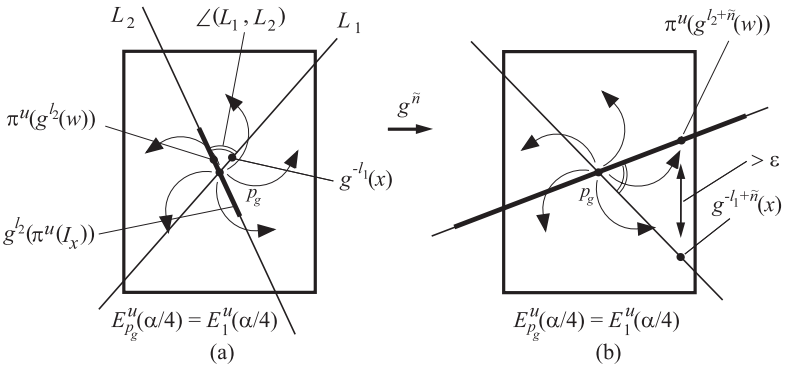


FIGURE 11. Behavior of the shadowing orbit; Case (2.2).

shadowing property for $g_{|\tilde{H}_g(p_g)}$. We can choose two integers $l_1 = l_1(\delta)$, $l_2 = l_2(\delta) > 0$ such that

$$d(g^{-l_1}(x), g^{l_2}(x)) < \delta \quad \text{and} \quad \angle(L_1, L_2) > 1.$$

Here $L_1 = \{t \cdot g^{-l_1}(x) : t \in \mathbb{R}\}$, and L_2 is a linear subspace containing $g^{l_2}(\pi^u(I_x))$ such that $\dim \pi^u(I_x) = \dim L_2 = 1$. Thus L_1, L_2 are one-dimensional linear subspaces of $E^u_{p_g}$; see Figure 11(a). Of course, $\angle(\cdot, \cdot)$ is the angle between linear subspaces of $E^u_{p_g}$ (see §3.1). We can pick the above l_1 and l_2 arbitrarily large since $g_{|\{0\}\times E^u_{p_g}(\alpha/4)}$ is a linear expanding irrational rotation map. Clearly,

$$\mathcal{PO}_\delta^{l_1,l_2} = \{\ldots, g^{-2}(x), g^{-1}(x), x, g(x), \ldots, g^{l_2-1}(x), g^{-l_1}(x), g^{-l_1+1}(x), \ldots\}$$

is an elementary δ-pseudo-orbit of g in $\tilde{H}_g(p_g)$.

4.2.6. *End of the proof of Proposition 3.1 in Case (2.2).* Since $g_{|\tilde{H}(p_g)}$ has the shadowing property, reasoning similar to that used in the previous case shows that there exists $w \in B_\epsilon(x) \cap I_x$ whose g-orbit ϵ-shadows the above $\mathcal{PO}_\delta^{l_1,l_2}$. We may assume that $|\pi^s(g^{l_2}(w))| < \epsilon$ since l_2 is sufficiently large, and so $|\pi^s(g^{l_2+i}(w))| < \epsilon$ for all $i \geq 0$ while $g^{l_2+i}(w) \in B_{\alpha/4}(p_g)$.

Since the above angle is preserved by the map $g\ (= D_{p_g}g)$ in $B_{\alpha/4}(p_g)$, we deduce that w cannot ϵ-shadow $\mathcal{PO}_\delta^{l_1,l_2}$ (see Figure 11(b)). Indeed, take $\tilde{n} = \tilde{n}(\epsilon) > 0$ such that $g^{-l_1+n}(x) \in B_{\alpha/8|\lambda_1|}(p_g)$ for $0 \leq n \leq \tilde{n} - 1$ and $g^{-l_1+\tilde{n}}(x) \notin B_{\alpha/8|\lambda_1|}(p_g)$. It is easy to see that $\pi^u(g^{l_2+\tilde{n}}(w)) \in D_{p_g}g^{\tilde{n}}(L_2) \cap B_{\alpha/4}(p_g)$, $g^{-l_1+\tilde{n}}(x) \in D_{p_g}g^{\tilde{n}}(L_1) \cap B_{\alpha/4}(p_g)$ and

$$\angle(D_{p_g}g^{\tilde{n}}(L_1), D_{p_g}g^{\tilde{n}}(L_2)) = \angle(L_1, L_2).$$

Thus

$$d(g^{l_2+\tilde{n}}(w), g^{-l_1+\tilde{n}}(x)) \geq d(\pi^u(g^{l_2+\tilde{n}}(w)), g^{-l_1+\tilde{n}}(x)) > \frac{\alpha}{8|\lambda_1|} \sin 1 = \epsilon.$$

This is a contradiction, since $d(g^{l_2+\tilde{n}}(w), g^{-l_1+\tilde{n}}(x)) < \epsilon$. The proof is therefore complete. □

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5. *Proof of Proposition 3.2*

Throughout this section, suppose that $f(p) = p$ is a hyperbolic saddle and that:

- there is $\mathcal{U}(f)$ such that for each $g \in \mathcal{U}(f)$, the chain component $C_g(p_g)$ admits a dominated splitting $T_{C_g(p_g)}M = E(g) \oplus F(g)$ with $\dim E(g) = \mathrm{index}(p_g) = i$ $(0 < i < \dim M)$, and $g_{|C_g(p_g)}$ has the shadowing property.

We show that for any $g \in \mathcal{U}(f)$, if $q \in C_g(p_g) \cap P(g)$ is hyperbolic, then $\mathrm{index}(q) = \mathrm{index}(p_g)$. Assume, for contradiction, that there exist $g \in \mathcal{U}(f)$ and a hyperbolic periodic point $q \in C_g(p_g) \cap P(g)$ such that

$$\mathrm{index}(q) = j < i = \mathrm{index}(p_g) \tag{7}$$

(since $g^{-1}_{|C_g(p_g)}$ also has the shadowing property, the case $\mathrm{index}(q) > \mathrm{index}(p_g)$ is analogous). For simplicity, we suppose $g(q) = q$; the general case is treated similarly.

5.1. *Outline of the proof.* The strategy of the proof is as follows: by assuming (7), we construct a new diffeomorphism near g (by perturbation) and a special elementary pseudo-orbit of the perturbed map in the chain component. Since the perturbed map has the shadowing property in the chain component, this pseudo-orbit must be shadowed by some g-orbit. However, by making use of the dominated splitting, we shall prove that this is impossible, leading to a contradiction.

First, by applying Lemma 3.9 to $g_{|C_g(p_g)}$, we construct a heterodimensional cycle between p_g and q in $C_g(p_g)$, i.e.

$$x \in W^s(p_g) \cap W^u(q) \neq \emptyset \quad \text{and} \quad y \in W^s(q) \cap W^u(p_g) \neq \emptyset \quad (x,\, y \in C_g(p_g)).$$

Note that, by (7), y is a non-transverse intersection point between $W^s(q)$ and $W^u(p_g)$.

Next, we linearize g at q (denoting the perturbed map by g again), and identify the map with its derivative $D_q g$ in a small neighborhood, $B_{\alpha/4}(q)$ say, by assuming that $B_{\alpha/4}(q) \subset \mathbb{R}^{\dim M}$ and q is the origin O. We may suppose that both y and x are in $\mathrm{int}\big(B_{\alpha/4}(q)\big)$ by iterating them with g and g^{-1}, respectively, if necessary. By perturbing the map again at $g^{-1}(y)$ we construct a small 'unstable affine piece' $I_y \subset B_{\alpha/4}(q)$, which is a neighborhood of y in the unstable manifold $W^u(p_g)$ of p_g.

Since $g_{|B_{\alpha/4}(q)}$ is identified with its linear part $D_q g$, we may suppose that the neighborhood $B_{\alpha/4}(q)$ is of the form $E^s_q(\alpha/4) \times E^u_q(\alpha/4)$; see Figure 12. Of course, these are the local (linear) stable and the unstable manifolds of q, i.e.

$$W^s_{\alpha/4}(q) = E^s_q(\alpha/4) \quad \text{and} \quad W^u_{\alpha/4}(q) = E^u_q(\alpha/4).$$

It is assumed that $C_g(p_g)$ admits a dominated splitting $E \oplus F$. By the uniqueness of the dominated splitting, we have $T_y I_y = F_y$, $F_q \subset E^u_q$ and $E^s_q \subset E_q$. Put $E^u_1 = E_q \cap E^u_q$ and $E^u_2 = F_q$, then $E^u_q = E^u_1 \oplus E^u_2$. By modifying the map at $g(x)$, we may suppose that $x \notin E^u_2(\alpha/4)$. Take $\epsilon = d(x, E^u_2(\alpha/4))/2 > 0$. Recalling that $g_{|C_g(p_g)}$ has the shadowing property, let $0 < \delta = \delta(\epsilon) \leq \epsilon$ be the number in the definition of the property. Take $l = l(\epsilon) > 0$ such that $d(g^l(y),\, g^{-l}(x)) < \delta$. Then the union of two segments of true g-orbits

$$\mathcal{PO}_\delta = \{\ldots,\, g^{-2}(y),\, g^{-1}(y),\, y,\, g(y),\, \ldots,\, g^{l-1}(y),\, g^{-l}(x),\, g^{-l+1}(x),\, \ldots,\\ x,\, g(x),\, \ldots\}$$

is an elementary δ-pseudo-orbit of g in $C_g(p_g)$; see Figure 13.

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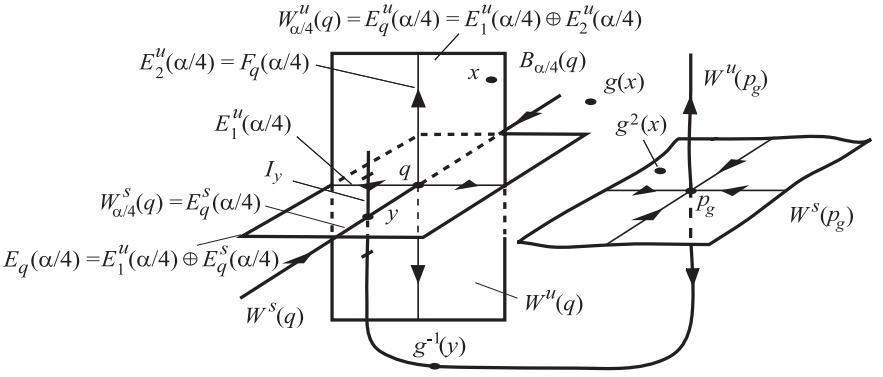


FIGURE 12. The dynamics in the linearized neighborhood.

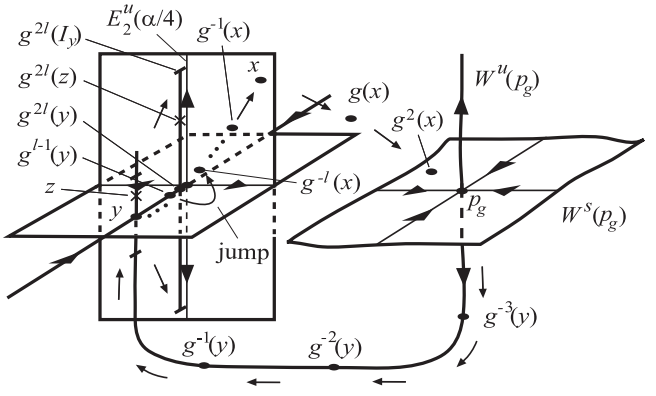


FIGURE 13. An elementary pseudo-orbit of g.

Since $g_{|C_g(p_g)}$ has the shadowing property, $\mathcal{PO}_\delta$ must be ϵ-shadowed by some g-orbit; that is, there exists $z \in M$ (near y) ϵ-shadowing it. Thus $d(g^{2l}(z), x) < \epsilon$. On the other hand, $z \in I_y$ by Lemma 4.1. Since $T_y I_y = F_y$, by the continuity and Dg-invariance of the dominated splitting we will prove that

$$d_H(g^{2l}(I_y) \cap B_{\alpha/4}(q),\ E^u_2(\alpha/4)) < \epsilon,$$

where d_H denotes the Hausdorff metric. Thus $d(x, E^u_2(\alpha/4)) < 2\epsilon$ since $g^{2l}(z) \in g^{2l}(I_y) \cap B_{\alpha/4}(q)$, and this contradicts the choice of ϵ.

5.2. *Formal proof of Proposition 3.2.* Recall assumption (7) on $q \in C_g(p_g) \cap P(g)$, and let $\epsilon_0 = \epsilon_0(q) > 0$ be as in Lemma 3.5. Since $g_{|C_g(p_g)}$ has the shadowing property, by Lemma 3.9 we have

$$x \in W^s(p_g, g) \cap W^u(q, g) \neq \emptyset \quad \text{and} \quad y \in W^s(q, g) \cap W^u(p_g, g) \neq \emptyset.$$

Observe that both x and y are in $C_g(p_g)$, and y is a non-transverse intersection point of $W^s(q, g)$ and $W^u(p_g, g)$ since $\dim E^s_q + \dim E^u_{p_g} = j + \dim M - i < \dim M$.

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5.2.1. *Local linearization.* We perturb g several times as in the proof of Proposition 3.1. In the first perturbation, by Lemma 4.3 we linearize g at q with respect to the exponential coordinates $\exp_q$ without changing its stable and unstable manifolds near x and y, respectively. More precisely, by choosing $0 < \alpha < \epsilon_0$ small enough, we construct $g_1 \in \mathcal{U}(f)$ near g such that:

(b.1) $$g_1(z) = \begin{cases} \exp_q \circ D_q g \circ \exp_q^{-1}(z) & \text{if } z \in B_{\alpha/4}(q), \\ g(z) & \text{if } z \notin B_\alpha(q), \end{cases}$$

and also

$$x \in W^s(p_g, g_1) \cap W^u(q, g_1) \quad \text{and} \quad y \in W^u(p_g, g_1) \cap W^s(q, g_1).$$

Clearly, $g_1(p_g) = p_g$, $g_1(q) = q$,

$$W^s_{\alpha/4}(q, g_1) = \exp_q(E^s_q(\alpha/4)) \quad \text{and} \quad W^u_{\alpha/4}(q, g_1) = \exp_q(E^u_q(\alpha/4)).$$

We may suppose that $y \in \operatorname{int} W^s_{\alpha/4}(q, g_1)$ (respectively $x \in \operatorname{int} W^u_{\alpha/4}(q, g_1)$) by iterating y (respectively x) by g_1 (respectively g_1^{-1}) if necessary.

5.2.2. *Construction of a unstable affine piece.* As in §4.2.2, by perturbing g_1 we construct a unstable affine piece centered at y. Let $I_y \subset T_q M$ be a small piece of the affine space centered at $\exp_q^{-1}(y)$, such that

$$T_y \mathcal{I}_y = T_y W^u(p_g, g_1).$$

Here $\mathcal{I}_y = \exp_q(I_y)$. Then, by perturbing g_1 at $g_1^{-1}(y)$ (reducing the diameter of I_y if necessary) we have the following.

LEMMA 5.1. *There exists $g_2 \in \mathcal{U}(f)$ near g_1 such that $g_2(p_g) = p_g$, $g_2(q) = q$, $y \in W^s(q, g_2)$, and also*

$$\mathcal{I}_y \subset W^u(p_g, g_2) \quad \textit{and} \quad T_y \mathcal{I}_y = T_y W^u(p_g, g_2).$$

Observe that $\{x, y\} \subset C_{g_2}(p_g)$. Moreover, by the uniqueness of a dominated splitting we have the following (recall that there is a dominated splitting on $C_{g_2}(p_g)$ such that $\dim E(g_2) = i$, by assumption).

LEMMA 5.2. *Let $T_{C_{g_2}(p_g)} M = E(g_2) \oplus F(g_2)$ be a dominated splitting on $C_{g_2}(p_g)$ such that $\dim E(g_2) = i$. Then:*

(i) $T_y W^u(p_g, g_2) = F_y(g_2)$;

(ii) $E^s_q \subset E_q(g_2)$ *and* $F_q(g_2) \subset E^u_q$.

Put

$$W^{cs}_{\alpha/4}(q, g_2) = \exp_q(E_q(g_2, \alpha/4)) \quad \text{and} \quad W^{cu}_{\alpha/4}(q, g_2) = \exp_q(F_q(g_2, \alpha/4)).$$

Here $E_q(g_2, \alpha/4)$ (respectively $F_q(g_2, \alpha/4)$) is a closed ball in $E_q(g_2)$ (respectively $F_q(g_2)$) centered at the origin with radius $\alpha/4$. These sets are the so-called local center-stable and center-unstable manifolds of q, respectively (see [**14, 17, 29**]). Finally, with a perturbation of the map at $g_2(x)$, we may assume that

$$x \in W^u_{\alpha/4}(q, g_2) \setminus W^{cu}_{\alpha/4}(q, g_2). \tag{8}$$

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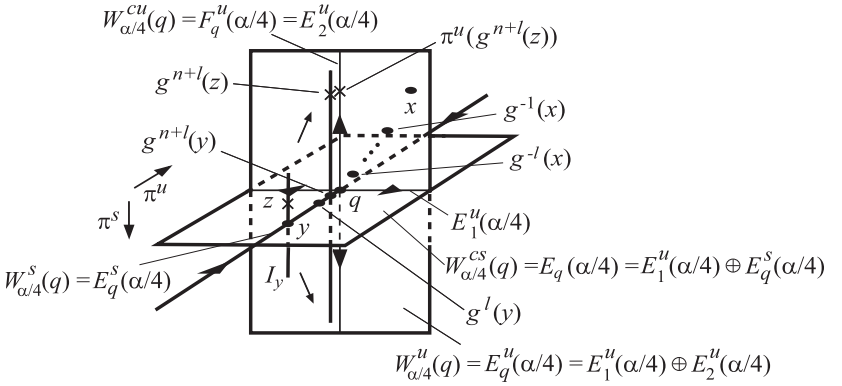


FIGURE 14. Behavior of the shadowing orbit.

5.2.3. *Local identification with the linear part.* By formula (b.1), the local system $g_{2|B_{\alpha/4}(q)}$ is conjugate to the linear part $D_q g$ under the exponential map. As in §4.2.3, by assuming that $B_{\alpha/4}(q) \subset \mathbb{R}^{\dim M}$ and $q = O \in \mathbb{R}^{\dim M}$ we shall identify $g_{2|B_{\alpha/4}(q)}$ with its derivative $D_q g$.

Let us list some notational changes and the properties preserved by the identification. To simplify notation, we denote g_2 by g, $W^s_{\alpha/4}(q, g_2)$ by $W^s_{\alpha/4}(q)$, and so on. Then:
(b.2) $W^s_{\alpha/4}(q) = E^s_q(\alpha/4)$ and $W^u_{\alpha/4}(q) = E^u_q(\alpha/4)$;
(b.3) $W^{cs}_{\alpha/4}(q) = E_q(\alpha/4)$ and $W^{cu}_{\alpha/4}(q) = F_q(\alpha/4)$; and
(b.4) $I_y \subset W^u(p_g)$ and $T_y I_y = F_y$ by Lemmas 5.1 and 5.2(i).

5.2.4. *Projections.* Let $\pi^u : B_{\alpha/4}(q) \to E^u_q$ and $\pi^s : B_{\alpha/4}(q) \to E^s_q$ be the projections parallel to E^s_q and E^u_q, respectively. Put $E^u_1 = E_q \cap E^u_q$ and $E^u_2 = F_q$. Then $E^u_q = E^u_1 \oplus E^u_2$ by Lemma 5.2(ii); see Figure 14. For any $(v^s, v^u_1, v^u_2) \in E^s_q \oplus E^u_1 \oplus E^u_2 = E^s_q \oplus E^u_q$, we define a metric on $B_{\alpha/4}(q)$ ($\subset \mathbb{R}^{\dim M}$) by

$$\|(v^s, v^u_1, v^u_2)\| = \max\{|v^s|, |v^u_1|, |v^u_2|\}.$$

We may suppose that this metric is equivalent to the original metric d on M.

By (b.3) and (b.4), we have the following.

LEMMA 5.3. *$\pi^u(g^n(I_y)) \cap B_{\alpha/4}(q) \to W^{cu}_{\alpha/4}(q)$ and $\pi^s(g^n(I_y)) \cap B_{\alpha/4}(q) \to q$ in the Hausdorff sense as $n \to \infty$.*

Take $N > 0$ such that $g^{-N}(y) \in W^u_{\epsilon_0}(p_g)$, or $y \in g^N(W^u_{\epsilon_0}(p_g))$, and put

$$\epsilon_1 = \min\{\alpha/4, d(x, W^{cu}_{\alpha/4}(q))/2\} > 0 \tag{9}$$

(see (8)). Let us define the unstable connected component of y as follows: for any $\epsilon > 0$, let

$$C^u_\epsilon(y) \text{ be the connected component of } B_\epsilon(y) \cap g^N(W^u_{2\epsilon_0}(p_g)) \text{ containing } y.$$

We may suppose that $C^u_{\epsilon_1}(y) \subset I_y$ and that if $d(g^{-n}(y), g^{-n}(z)) < \epsilon_1$ for all $n \geq 0$, then $z \in C^u_{\epsilon_1}(y)$ by Lemma 4.1, upon reducing ϵ_1 if necessary.

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Recalling that $g_{|C_g(p_g)}$ has the shadowing property, let $0 < \delta = \delta(\epsilon_1) < \epsilon_1$ be the number in the definition of the property. Take $l = l(\delta, \epsilon_1) > 0$ such that $d(g^l(y), g^{-l}(x)) < \delta$ and

$$\max\{d_H(\pi^s(g^{l+n}(I_y)), q), d_H(\pi^u(g^{n+l}(I_y)) \cap B_{\alpha/4}(q), W^{cu}_{\alpha/4}(q))\} < \epsilon_1 \tag{10}$$

for all $n > 0$ (see Lemma 5.3), where d_H is the Hausdorff metric. Then

$$\mathcal{PO}_\delta = \{\ldots, g^{-2}(y), g^{-1}(y), y, g(y), \ldots, g^{l-1}(y), g^{-l}(x), g^{-l+1}(x), \ldots, x, g(x), \ldots\}$$

is an elementary δ-pseudo-orbit of g in $C_g(p_g)$.

5.2.5. *End of the proof of Proposition 3.2.* By the shadowing property of $g_{|C_g(p_g)}$, $\mathcal{PO}_\delta$ must be ϵ_1-shadowed by some g-orbit; that is, there is $z \in M$ near y such that:

- $d(g^{-n}(y), g^{-n}(z)) < \epsilon_1$ for all $n \geq 0$;
- $d(g^n(y), g^n(z)) < \epsilon_1$ for $0 \leq n \leq l-1$; and
- $d(g^{-l+n}(x), g^{l+n}(z)) < \epsilon_1$ for $n \geq 0$.

Hence by the choice of ϵ_1 we have that $z \in C^u_{\epsilon_1}(y) \subset I_y$.

Since $g^n(z) \in B_{\alpha/4}(q)$ for $0 \leq n \leq 2l$, by (10) we have

$$d(\pi^s(g^{2l}(z)), q) < \epsilon_1 \quad \text{and} \quad d(\pi^u(g^{2l}(z)), W^{cu}_{\alpha/4}(q)) < \epsilon_1$$

so that $d(g^{2l}(z), W^{cu}_{\alpha/4}(q)) < \epsilon_1$. On the other hand, we have that $d(x, g^{2l}(z)) < \epsilon_1$. Therefore $d(x, W^{cu}_{\alpha/4}(q)) < 2\epsilon_1$. This contradicts (9), and the proof is complete. □

6. *Proof of Theorem A*

Throughout this section, suppose that $f(p) = p$ is a hyperbolic saddle and that:

- $C_f(p)$ is C^1-stably shadowable.

Let $\mathcal{U}(f)$ be given by Proposition 3.1. The steps of the proof are: (i) for every $g \in \mathcal{U}(f)$, $C_g(p_g) = H_g(p_g)$; (ii) non-existence of tangencies of homoclinic points associated with p_g implies the existence of a dominated splitting on $homo_g(p_g)$; and (iii) extension to the closure by continuity.

Lemma 3.6 asserts (i); thus the main thing to prove in this section is step (ii).

6.1. *The existence of a dominated splitting.* To begin with, for any $g \in \mathcal{U}(f)$, let us define the candidate of a dominated splitting on transverse homoclinic points associated with p_g. Recall that in §3.1 we defined, for any $x \in homo_g(p_g)$,

$$E_x(g) = T_x W^s(p_g, g) \quad \text{and} \quad F_x(g) = T_x W^u(p_g, g).$$

We show that these subspaces become a dominated splitting. Observe that $\dim E_x(g) = \text{index}(p_g)$ and that, by Lemma 3.8, there exists $\gamma > 0$ such that for any $g \in \mathcal{U}(f)$, we have $\angle(E_x(g), F_x(g)) > \gamma$ for any $x \in homo_g(p_g)$.

Let $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ and $\delta > 0$ be as given in Lemma 4.3. We then have the following.

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LEMMA 6.1. *There exists $m > 0$ such that for every $x \in homo_g(p_g)$ (with $g \in \mathcal{U}_0(f)$), there exists $0 < m_1 = m_1(x) \le m$ satisfying*

$$\|D_x g^{m_1}_{|E_x(g)}\| \cdot \|D_{g^{m_1}(x)} g^{-m_1}_{|F_{g^{m_1}(x)}(g)}\| \le \tfrac{1}{2}.$$

Proof. If this is not true, then for any $m > 0$ there are $g_m \in \mathcal{U}_0(f)$ and $x_m \in homo_{g_m}(p_{g_m})$ such that

$$\|D_{x_m} g^n_{m|E_{x_m}(g_m)}\| \cdot \|D_{g_m{}^n(x_m)} g_m^{-n}{}_{|F_{g_m{}^n(x_m)}(g_m)}\| > \tfrac{1}{2}$$

for all $0 < n \le m$. Hence there are vectors $v_m \in F_{x_m}(g_m)$ and $w_m \in E_{x_m}(g_m)$ with $\|v_m\| = \|w_m\| = 1$ such that

$$\frac{\|D_{x_m} g_m^m(w_m)\|}{\|D_{x_m} g_m^m(v_m)\|} > \frac{1}{2}.$$

Thus, by Lemma 4.3, as in the proof of [**34,** Proposition 2], for sufficiently large m we get $g' \in \mathcal{U}(f)$ near g_m such that $x_m \in homo_{g'}(p_{g'})$ and $\angle(E_{x_m}(g'), F_{x_m}(g')) \le \gamma$. This is obtained by multiplying suitable linear maps $L_i : T_{g_m^i(x_m)}M \to T_{g_m^{i+1}(x_m)}M$ that have $\|L_i - D_{g_m^i(x_m)} g_m\| \le \delta$ $(0 \le i \le m-1)$, which keep $F_{g_m^i(x_m)}(g_m)$ unchanged but incline $E_{g_m^i(x_m)}(g_m)$ throughout the g_m-orbit of x_m (see also [**29, 37**]). This contradicts Lemma 3.8, and so the lemma is proved. □

6.2. *End of the proof of Theorem A.* Let $\mathcal{U}_0(f)$ be as above. To reach the conclusion, it is enough to show that for any $g \in \mathcal{U}_0(f)$, $homo_g(p_g)$ admits a dominated splitting $E(g) \oplus F(g)$ with domination constants $C > 0$ and $0 < \lambda < 1$ (which are independent of g).

Let $m > 0$ be as given in Lemma 6.1, and put $\lambda = 2^{-1/m}$. Set

$$C = \sup_{g \in \mathcal{U}_0(f)} \left(\sup_{x \in M} \{\|D_x g^k\| : 0 \le k \le m\} \right) > 0.$$

Then, from Lemma 6.1, we see by a standard argument that the domination property holds for $homo_g(p_g)$ $(g \in \mathcal{U}_0(f))$ with respect to the constants C and λ (see [**34,** End of proof of Theorem 1]). Since $homo_g(p_g)$ is dense in $C_g(p_g)$ by Lemma 3.6, we can extend it by continuity to a dominated splitting on $C_g(p_g)$ (see the proof of [**21,** Proposition 1.3]). Thus Theorem A is proved. □

7. *Proof of Theorem B*

First let us prepare one lemma for diffeomorphisms having the shadowing property on normally hyperbolic invariant submanifolds of M.

Let $V \subset M$ be an invariant submanifold of $f \in \mathrm{Diff}(M)$. We say that V is *normally hyperbolic* [**25**] if there is a splitting $T_V M = TV \oplus N^s \oplus N^u$ such that:

(1) the splitting depends continuously on $x \in V$;
(2) $D_x f(N_x^\sigma) = N^\sigma_{f(x)}$ $(\sigma = s, u)$ for all $x \in V$;
(3) there are constants $C > 0$ and $0 < \lambda < 1$ such that for every triple of unit vectors $v \in T_x V$, $v^s \in N_x^s$ and $v^u \in N_x^u$ $(x \in V)$, we have

$$\|D_x f^n(v^s)\| \le C\lambda^n \|D_x f^n(v)\| \quad \text{and} \quad \|D_x f^n(v^u)\| \ge C^{-1}\lambda^{-n}\|D_x f^n(v)\|$$

for all $n \ge 0$.

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LEMMA 7.1. *Let $\Lambda \subset M$ be a closed invariant set of $f \in \mathrm{Diff}(M)$, and let $V \subset \Lambda$ be a normally hyperbolic f-invariant submanifold of M. Suppose that $f_{|\Lambda}$ has the shadowing property. Then the following holds:*

(i) *if V is an arc, then $f_{|V}$ is not the identity map;*

(ii) *if V is a circle, then $f_{|V}$ is not a rotation map.*

We omit the proof of this lemma and refer the reader to [**33,** Proof of Theorem].

In this section, we suppose hereafter that $f(p) = p$ is a hyperbolic saddle and that:

- $C_f(p)$ is C^1-stably shadowable.

Observe that, by the proof of Theorem A (see §6), there is $\mathcal{U}_0(f)$ such that for any $g \in \mathcal{U}_0(f)$, $C_g(p_g)$ admits a dominated splitting $T_{C_g(p_g)}M = E(g) \oplus F(g)$, and $g_{|C_g(p_g)}$ has the shadowing property. Thus the hypothesis of Corollary 3.3 holds for f.

Before beginning the proof, let us introduce the notion of normalized eigenvalues. For $q \in P(f)$, the *normalized eigenvalues* of $D_q f^{\pi(q)}$ are

$$\{\mu^{1/\pi(q)} : \mu \text{ are eigenvalues of } D_q f^{\pi(q)}\}.$$

7.1. *Rough outline of the proof.* By Corollary 3.3, for any g near f, if $q \in C_g(p_g) \cap P(g)$ is hyperbolic, then $q \sim p_g$. Thus, following [**22,** Lemma II.3], to prove Theorem B it is enough to show that the moduli of the normalized eigenvalues of hyperbolic periodic points $q \sim p_g$ of g are uniformly bounded away from 1 for all g near f (see Proposition 7.3 below).

The proof is by contradiction. If the bound were not uniform, then there must exist g (near f) and $q \in P(g)$ ($q \sim p_g$) for which the weakest normalized eigenvalue λ is close to 1. There are the following possibilities:

(1) λ is complex;

(2) λ is real and

  (2.1) $\mathrm{index}(q) \geq 2$; or

  (2.2) $\mathrm{index}(q) = 1$.

In this rough outline, we consider only the case that λ is real, positive and has multiplicity one, for simplicity. By perturbing the map (and denoting the perturbed map by g again) we can construct a normally hyperbolic $g^{\pi(q)}$-invariant arc $\mathcal{I}_q$ containing q such that $g^{\pi(q)}_{|\mathcal{I}_q}$ is the identity map (see Proposition 7.2 below). Observe that the perturbed map $g_{|C_g(p_g)}$ also has the shadowing property, and thus so does $g^{\pi(q)}_{|C_g(p_g)}$. Moreover, since the relation $q \sim p_g$ is preserved by the above perturbation, it can be seen that $\mathcal{I}_q \subset C_g(p_g)$. Since $\mathcal{I}_q$ is normally hyperbolic and $g^{\pi(q)}_{|\mathcal{I}_q}$ is the identity map, this contradicts Lemma 7.1(i).

7.2. *Non-ss boundary points.* In this paper, we import some concepts used in the proofs of the main results from [**24, 34**]. One of these is the notion of non-ss boundary points. Before starting the formal proof of Theorem B, we introduce non-ss boundary points and prove one preparatory proposition.

Let $q \in P(f)$ be a hyperbolic saddle with $\mathrm{index}(q) \geq 2$, and let $0 < |\lambda_1| \leq \cdots \leq |\lambda_k| < 1$ be the moduli of the normalized eigenvalues of $D_q f^{\pi(q)}_{|E^s_q}$ ($1 \leq k \leq \dim E^s_q$).

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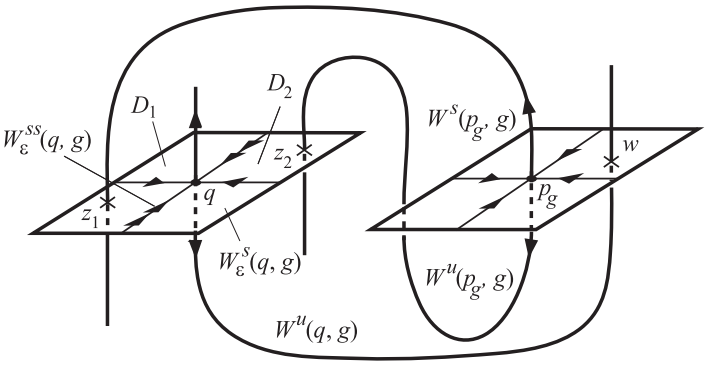


FIGURE 15. Non-ss boundary point.

In this subsection, we consider only the case where the weakest normalized eigenvalue λ_k is real, close to 1 and has multiplicity one. Thus there is $\eta > 0$ satisfying $|\lambda_j| < |\lambda_k| - \eta$ for $j = 1, \ldots, k-1$. It is well-known that there is a *strong stable manifold* $W^{ss}(q)$ of q associated with $\lambda_1, \ldots, \lambda_{k-1}$, which is of codimension one in $W^s(q)$ (see [17]). Thus $W^s_\epsilon(q) \setminus W^{ss}_\epsilon(q)$ has two components for some $\epsilon = \epsilon(q) > 0$. We call such a periodic point an *η-simple periodic point*, or just a *simple periodic point* if it is η-simple for some $\eta > 0$. Moreover, we call $1 - |\lambda_k|$ the *stable gap* of the periodic point.

Let $q \in P(f)$ be a hyperbolic saddle with $\mathrm{index}(q) \geq 2$ such that $q \sim p$. We say that q is a *non-ss boundary point* (see **[34,** p. 474, Definition 7]) provided that:

(P.1) q is η-simple periodic point for some $\eta > 0$; and

(P.2) $W^u(p)$ intersects both components of $W^s_\epsilon(q) \setminus W^{ss}_\epsilon(q)$ transversely.

Notice that $W^u(q)$ also intersects $W^s(p)$ transversely, since $q \sim p$.

Now let us suppose that g is a given diffeomorphism near f and $q \sim p_g$ is a non-ss boundary point of g. Then, by (P.2), $W^u(p_g, g)$ intersects both components of $W^s_\epsilon(q, g) \setminus W^{ss}_\epsilon(q, g)$ (D_1 and D_2, say) *transversely* for some $\epsilon > 0$. In such a case, we set

$$z_i = W^u(p_g, g) \cap D_i \quad \text{for } i = 1, 2. \tag{11}$$

Note that z_1, z_2 and $w = W^u(q, g) \cap W^s(p_g, g)$ are all transverse intersection points; see Figure 15.

For $\delta > 0$, let $\mathcal{U}_\delta(f) = \{g \in \mathrm{Diff}(M) : \rho_{C^1}(f, g) < \delta\}$. Here ρ_{C^1} is the usual C^1-metric on $\mathrm{Diff}(M)$. The following fact is contained in the proof of **[34,** Proposition 3] (see Figure 16(a)).

PROPOSITION 7.2. *For any $\delta > 0$, there exists $\rho = \rho(\delta) > 0$ such that for any $g \in \mathcal{U}_{\delta/2}(f)$, if $q \sim p_g$ is a non-ss boundary point of g whose stable gap is less than ρ, then there are $g_1 \in \mathcal{U}_\delta(f)$ (near g) and a normally hyperbolic $g_1^{\pi(q)}$-invariant arc $\mathcal{I}_q$ with a small diameter centered at q, such that:*

(a) $g_1{}^{l\pi(q)}_{|\mathcal{I}_q} = \mathrm{id}$, *where $l = 1$ or 2;*

(b) $W^u(q, g_1) \cap W^s(p_{g_1}, g_1) \neq \emptyset$ *is a transverse intersection;*

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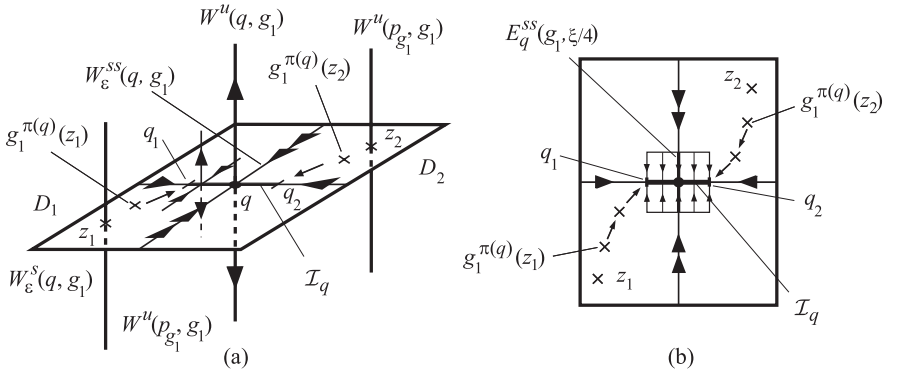


FIGURE 16. The dynamics of g_1 in the neighborhood of q.

(c) *the points z_1 and z_2 in (11) are still transverse intersection points of $W^u(p_{g_1}, g_1)$ and different components of $W^s_\epsilon(q, g_1) \setminus W^{ss}_\epsilon(q, g_1)$; and*

(d) *there are distinct end points q_i $(i = 1, 2)$ of $\mathcal{I}_q$ such that $g_1^{n\pi(q)}(z_i) \to q_i$ as $n \to \infty$ for $i = 1, 2$.*

Proof. Fix any $\delta > 0$, and let $C = \sup_{g \in \mathcal{U}_{\delta/2}(f)}(\sup_{x \in M} \|D_x g\|) > 0$. Take $\rho > 0$ such that $C\rho < \delta/4$. Suppose that $q \sim p_g$ is a non-ss boundary point of $g \in \mathcal{U}_{\delta/2}(f)$ whose stable gap is less than ρ.

As in the proof of [**34,** Proposition 3], fix small mutually disjoint neighborhoods U_i centered at $g^i(q)$ for $i = 0, 1, \ldots, \pi(q) - 1$. To simplify notation, by making use of local charts we identify each U_i with a ball centered at the origin O in $\mathbb{R}^{\dim M}$, so that $g^i(q)$ is identified with O. Also, we shall identify $W^s_\epsilon(g^i(q), g)$ with a ball in $\mathbb{R}^{k'} \subset \mathbb{R}^{\dim M} = \mathbb{R}^{k'} \times \mathbb{R}^{\dim M - k'}$, where $k' = \dim E^s_q(g)$. Let λ_k be the weakest normalized eigenvalue of $D_q g^{\pi(q)}_{|E^s_q(g)}$, and denote by e_1 the eigenvector corresponding to λ_k. Let z_1, z_2 be two points of $W^u(p_g, g)$ in different components of $W^s_\epsilon(q, g) \setminus W^{ss}_\epsilon(q, g)$, and let $\xi > 0$ be a small number (which will be fixed later on).

Now, let $\sigma : \mathbb{R}^{\dim M} \to [0, 1]$ be a C^∞ bump function such that $\sigma(x) = 1$ if $\|x\| \le \xi/4$, $\sigma(x) = 0$ if $\|x\| \ge \xi$, and $|D_x\sigma| < 2/\xi$ for all x. Let A be the matrix in $\mathbb{R}^{\dim M}$ such that

$$A_{|\mathbb{R}^{k'} \times \{0\}^{\dim M - k'}} = \lambda_k^{-1} I \quad \text{and} \quad A_{|\{0\}^{k'} \times \mathbb{R}^{\dim M - k'}} = I,$$

where I is the identity matrix in $\mathbb{R}^{k'}$ or $\mathbb{R}^{\dim M - k'}$. Let $g_1 : M \to M$ be a diffeomorphism defined as

$$g_1(x) = \begin{cases} \sigma(x) A \circ D_{g^i(q)} g(x) + (1 - \sigma(x)) g(x) & \text{if } x \in U_i \text{ for all } i, \\ g(x) & \text{if } x \notin \bigcup_{i=0}^{\pi(q)-1} U_i. \end{cases}$$

Then $g_1^i(q) = g^i(q)$ for all i, and it is not hard to show that $\rho_{C^1}(g_1, g) < \delta/2$ for any arbitrarily small ξ (see the proof of [**16,** Lemma 1.1]).

Let $E^{ss}_q(g_1)$ be the eigenspace of $D_q {g_1}^{\pi(q)}$ corresponding to the local strong stable manifold $W^{ss}_\epsilon(q, g_1)$. Then $g_{1|E^{ss}_q(g_1, \xi/4)}^{\pi(q)} = D_q g_{1|E^{ss}_q(g_1, \xi/4)}^{\pi(q)}$ is contracting. Here $E^{ss}_q(g_1, \xi/4)$ is the closed ball of $E^{ss}_q(g_1)$ centered at the origin with radius $\xi/4$.

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Set $\mathcal{I}_q = \{t \cdot e_1 : -\xi/4 \le t \le \xi/4\}$. Then, for arbitrarily small ξ:

- $\mathcal{I}_q$ is normally hyperbolic;
- $g_1^{l\pi(q)}{}_{|\mathcal{I}_q} = \mathrm{id}$ where $l = 1$ or 2 (corresponding to the sign of λ_k);
- $W^u(q, g_1) \cap W^s(p_{g_1}, g_1) \neq \emptyset$ is a transverse intersection; and
- $g_1^{n\pi(q)}(x) \to \mathcal{I}_q$ as $n \to \infty$ for any point $x \in W^s_\epsilon(q, g_1)$.

Finally, since the neighborhood of q in $W^s_\epsilon(q, g_1)$ is of the form $\mathcal{I}_q \times E^{ss}_q(g_1, \xi/4)$ (see Figure 16(b)), if we fix ξ small enough, then:

- $z_1, z_2 \in W^u(p_{g_1}, g_1)$ are still transverse intersection points in different components of $W^s_\epsilon(q, g_1) \setminus W^{ss}_\epsilon(q, g_1)$; and
- there are distinct end points q_i $(i = 1, 2)$ of $\mathcal{I}_q$ such that $g_1^{n\pi(q)}(z_i) \to q_i$ as $n \to \infty$ for $i = 1, 2$.

Thus the proposition is proved. □

7.3. *End of the proof of Theorem B.* Recall that $f(p) = p$ is a hyperbolic saddle and that $C_f(p)$ is C^1-stably shadowable. Let $\mathcal{U}_0(f)$ be fixed; then by Corollary 3.3 there is $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ such that if $q \in C_g(p_g) \cap P(g)$ is hyperbolic $(g \in \mathcal{V}(f))$, then $q \sim p_g$. Thus, to prove Theorem B, it is enough to show the following proposition.

PROPOSITION 7.3. *Let $C_f(p)$ be C^1-stably shadowable. Then there are $\mathcal{U}_1(f) \subset \mathcal{V}(f)$ and constants $m > 0$, $C > 0$ and $0 < \lambda < 1$ such that for each $g \in \mathcal{U}_1(f)$, if $q \in P(g)$ $(\pi(q) \ge m)$ is hyperbolic with $q \sim p_g$, then*

$$\prod_{i=0}^{k-1} \left\| D_{g^{im}(q)} g^m_{|E^s_{g^{im}(q)}} \right\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \left\| D_{g^{-im}(q)} g^{-m}_{|E^u_{g^{-im}(q)}} \right\| < C\lambda^k,$$

where $k = [\pi(q)/m]$.

Proof. Fix $\delta > 0$ such that $\mathcal{U}_\delta(f) \subset \mathcal{V}(f)$. It is known from [**22,** Lemma II.3] that to prove the proposition, it is enough to show that the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by Dg $(g \in \mathcal{U}_{\delta/2}(f))$ along the hyperbolic periodic points $q \in P(g)$ with $q \sim p_g$ is uniformly hyperbolic (see [**22,** pp. 524–525] for the definition).

The proof is by contradiction. If the above assertion does not hold, then for any $\epsilon > 0$ there must be $g \in \mathcal{U}_{\delta/2}(f)$, a hyperbolic periodic point q of g with $q \sim p_g$, and linear maps $L_i : T_{g^i(q)}M \to T_{g^{i+1}(q)}M$ with $\|L_i - D_{g^i(q)}g\| < \epsilon$ for $i = 0, 1, \ldots, \pi(q) - 1$ such that $\prod_{i=0}^{\pi(q)-1} L_i$ has an eigenvalue of modulus 1. As in the proof of [**34,** Proof of Theorem 2], since ϵ is arbitrary, by Lemma 4.3 we can construct $\tilde{g}$ near f such that q is a hyperbolic periodic point of $\tilde{g}$ with $q \sim p_{\tilde{g}}$, and $D_q\tilde{g}^{\pi(q)}$ has its weakest normalized eigenvalue, λ_k say, with modulus close to 1. Notice that we can take $\tilde{g}$ and $|\lambda_k|$ as close as we wish to g and 1, respectively, and thus $\tilde{g} \in \mathcal{U}_{\delta/2}(f)$.

We may assume without loss of generality that $|\lambda_k| < 1$ and λ_k has multiplicity one. For notational simplicity, we denote $\tilde{g}$ by g. Recall that we have three possibilities (see the second paragraph of §7.1).

Case (1). If λ_k is complex, then with a modification of the map we can construct $g' \in \mathcal{U}_\delta(f)$ near g possessing a normally hyperbolic $g'^{\pi(q)}$-invariant circle $\mathcal{S}_q$ centered at q,

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such that $g'^{\pi(q)}_{|\mathcal{S}_q}$ is conjugate to an irrational rotation map on the unit circle S^1 and also

$$W^u(\mathcal{S}_q, g') \cap W^s(p_{g'}, g') \neq \emptyset, \quad W^s(\mathcal{S}_q, g') \cap W^u(p_{g'}, g') \neq \emptyset$$

(see the proofs of [**34,** Theorem 2] and [**24,** Lemma 4.1]). The next claim follows from this fact.

CLAIM 7.4. $\mathcal{S}_q \subset C_{g'}(p_{g'})$.

To see this, take $x \in W^u(\mathcal{S}_q, g') \cap W^s(p_{g'}, g')$ and $y \in W^s(\mathcal{S}_q, g') \cap W^u(p_{g'}, g')$. Since $g'^{\pi(q)}_{|\mathcal{S}_q}$ is conjugate to an irrational rotation map, the α-limit set of x and the ω-limit set of y are both $\mathcal{S}_q$. On the other hand, $g'^{n\pi(q)}(x) \to p_{g'}$ and $g'^{-n\pi(q)}(y) \to p_{g'}$ as $n \to \infty$. Thus $z \rightsquigarrow p_{g'}$ and $p_{g'} \rightsquigarrow z$ for any $z \in \mathcal{S}_q$, and hence $z \leftrightsquigarrow p_{g'}$ for any $z \in \mathcal{S}_q$, as claimed.

Since $g' \in \mathcal{U}_\delta(f)$, we deduce that $g'_{|C_{g'}(p_{g'})}$ has the shadowing property. However, this contradicts Lemma 7.1(ii) because we know $\mathcal{S}_q \subset C_{g'}(p_{g'})$ by Claim 7.4, and also that $\mathcal{S}_q$ is normally hyperbolic and $g'^{\pi(q)}_{|\mathcal{S}_q}$ is an irrational rotation map. Thus Proposition 7.3 is proved in Case (1). □

Case (2.1). Suppose that λ_k is real and $\mathrm{index}(q) \geq 2$. We may assume further, with a modification of the map, that q is simple and its stable gap is less than $\rho = \rho(\delta) > 0$. This can be done by multiplying suitable linear maps with the derivatives of g throughout the orbit of q, using Lemma 4.3.

In this case we apply Proposition 7.2. If q is not a non-ss boundary point, then using the fact that $q \sim p_g$ we can find another hyperbolic periodic point near q (which we will still denote by q for simplicity) such that it is a non-ss boundary point whose stable gap is less than ρ (see the proofs of [**34,** Theorem 2] and [**24,** Section 4.1]). This result is proved by making use of the so-called *transition matrices* arguments due to [**8**].

Now let $g_1 \in \mathcal{U}_\delta(f)$, $\mathcal{I}_q$, z_i and q_i $(i = 1, 2)$ be as given in Proposition 7.2, satisfying items (a)–(d). Then we have the following.

CLAIM 7.5. $\mathcal{I}_q \subset C_{g_1}(p_{g_1})$.

Indeed, set $w = W^u(q, g_1) \cap W^s(p_{g_1}, g_1)$ by (b). It is easy to see that $q \rightsquigarrow p_{g_1}$, since $g_1^{-n\pi(q)}(w) \to q$ and $g_1^{n\pi(q)}(w) \to p_{g_1}$ as $n \to \infty$. Furthermore, we have $p_{g_1} \rightsquigarrow q_i$ for $i = 1, 2$, since $g_1^{-n\pi(q)}(z_i) \to p_{g_1}$ and $g_1^{n\pi(q)}(z_i) \to q_i$ as $n \to \infty$ for $i = 1, 2$ by (c) and (d). Fix any $x \in \mathcal{I}_q$. Since $g_{1|\mathcal{I}_q}^{l\pi(q)} = \mathrm{id}$ ($l = 1$ or 2) by (a), it is easy to see that $x \leftrightsquigarrow q$ and $x \leftrightsquigarrow q_i$ for $i = 1, 2$. Thus we conclude $x \leftrightsquigarrow p_{g_1}$ as claimed.

As in the previous case, since $g_1 \in \mathcal{U}_\delta(f)$, we deduce that $g_{1|C_{g_1}(p_{g_1})}$ has the shadowing property. But this contradicts Lemma 7.1(i), since we know $\mathcal{I}_q \subset C_{g_1}(p_{g_1})$ by Claim 7.5, $\mathcal{I}_q$ is normally hyperbolic and $g_{1|\mathcal{I}_q}^{l\pi(q)} = \mathrm{id}$ (recall that $g_{1|\mathcal{I}_q}^{l\pi(q)}$ has the shadowing property if and only if $g_{1|\mathcal{I}_q}$ has the shadowing property). Therefore Proposition 7.3 is proved in the Case (2.1). □

Case (2.2). Suppose that λ_k is real and $\mathrm{index}(q) = 1$. Then $W^s(q, g)$ is one-dimensional, and so we denote the two separatrices of $W^s(q, g) \setminus \{q\}$ by $W^{s1}(q, g)$ and $W^{s2}(q, g)$.

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Since $q \sim p_g$, by the same method used above we can find another hyperbolic periodic point near q with index equal to one (which we will still denote by q for simplicity) such that there exist transverse intersection points $z_i \in W^{si}(q, g) \cap W^u(p_g, g)$ for $i = 1, 2$ and $w \in W^u(q, g) \cap W^s(p_g, g) \neq \emptyset$. Thus, as in the proof of Proposition 7.2, by slightly perturbing g to $g_1 \in \mathcal{U}_\delta(f)$ at q without changing its stable manifolds near z_i $(i = 1, 2)$, we can find a normally hyperbolic small arc $\mathcal{I}_q$ centered at q such that:

- $g_{1|\mathcal{I}_q}^{l\pi(q)} = \mathrm{id}$, where $l = 1$ or 2;
- $W^u(q, g_1) \cap W^s(p_{g_1}, g_1) \neq \emptyset$ is a transverse intersection; and
- z_1 and z_2 are still intersection points of $W^u(p_{g_1}, g_1)$ and lie in different components of $W^s(q, g_1) \setminus \{q\}$, while for the boundary points q_i $(i = 1, 2)$ of $\mathcal{I}_q$, we have $g_1^{nl}(z_i) \to q_i$ as $n \to \infty$ for $i = 1, 2$.

From these facts, we can show $\mathcal{I}_q \subset C_{g_1}(p_{g_1})$ in the same manner as the previous case. Thus the shadowing property of $g_{1|C_{g_1}(p_{g_1})}$ contradicts Lemma 7.1(i), because $g_{1|\mathcal{I}_q}^{l\pi(q)}$ is the identity map and $\mathcal{I}_q$ is normally hyperbolic. The proof of Proposition 7.3 is therefore complete. □

8. *Proof of Theorem C*

Throughout this section, suppose that $f(p) = p$ is a hyperbolic saddle and that:

- $C_f(p)$ is C^1-stably shadowable and the $C_f(p)$-germ of f is expansive.

Thus, for $n \in \mathbb{Z}$, $f^n_{|C_f(p)}$ also has the shadowing property and the $C_f(p)$-germ of f^n is expansive. Theorem A, $C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ such that $\dim E = \mathrm{index}(p)$. We show in this section that this splitting is hyperbolic.

Let $m > 0$, $C > 0$ and $0 < \lambda < 1$ be the constants given by Theorem B. We may assume that any periodic point denoted by 'q' satisfies $\pi(q) \geq m$. Clearly, if $p \sim q$ with respect to f, then $p \sim q$ with respect to f^m. For the time being (until the end of the proof of Proposition 8.4), we assume that $m = 1$ in Theorem B for notational simplicity.

8.1. *Local product structure and its application.*

To prove Theorem C, we need two lemmas and one proposition. Let Λ be a closed f-invariant set that admits a dominated splitting $E \oplus F$. Then the local center-stable manifold, $W^{cs}_\epsilon(x)$, and the local center-unstable manifold, $W^{cu}_\epsilon(x)$, exist for all $x \in \Lambda$.

We write $s = \dim E$ and $u = \dim F$, and put $D^j_r = \{x \in \mathbb{R}^j : \|x\| \leq r\}$ $(r > 0)$ for $j = s, u$. Let $Emb_\Lambda(D^j_1, M)$ be the space of C^1-embeddings $\beta : D^j_1 \to M$ such that $\beta(0)$ belongs to Λ endowed with the C^1-topology. The following can be found in [**20,** Proposition 2.3] (see also [**17**]).

LEMMA 8.1. *There exist $\phi^s : \Lambda \to Emb_\Lambda(D^s_1, M)$ and $\phi^u : \Lambda \to Emb_\Lambda(D^u_1, M)$ such that if we define $W^{cs}_\epsilon(x) = \phi^s(x)D^s_\epsilon$ and $W^{cu}_\epsilon(x) = \phi^u(x)D^u_\epsilon$ (where $x \in \Lambda$, $0 < \epsilon < 1$), then the following holds.*

(a) *$T_x W^{cs}_\epsilon(x) = E_x$ and $T_x W^{cu}_\epsilon(x) = F_x$ for all $x \in \Lambda$.*

(b) *For every $0 < \epsilon_1 < 1$, there exists $0 < \epsilon_2 < 1$ such that $f(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f(x))$ and $f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cu}_{\epsilon_1}(f^{-1}(x))$ for all $x \in \Lambda$.*

(c) *For every $0 < \epsilon_1 < 1$, there exists $0 < \delta < 1$ such that if $d(x, y) < \delta$ $(x, y \in \Lambda)$, then $W^{cs}_{\epsilon_1}(x) \cap W^{cu}_{\epsilon_1}(y) \neq \emptyset$ and this intersection is transverse.*

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Let Λ be a closed f-invariant set. We say that Λ has a *local product structure* if for any $\epsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$ (where $x, y \in \Lambda$), then

$$\emptyset \neq W^s_\epsilon(x) \cap W^u_\epsilon(y) \subset \Lambda.$$

The assertions of the next lemma are proved in [**34,** Lemma 6] for C^1-persistently expansive homoclinic classes when f is germ-expansive. By Theorem B and Lemma 8.1, we can prove the same assertions for C^1-stably shadowable chain components.

LEMMA 8.2. *Let $c > 0$ be the constant for the $C_f(p)$-germ expansivity of f. Then the following holds.*

(a) *For any small $\epsilon > 0$, we have $W^{cs}_\epsilon(x) = W^s_\epsilon(x)$ and $W^{cu}_\epsilon(x) = W^u_\epsilon(x)$ for all $x \in C_f(p)$. More precisely, for any $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that $f^n(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f^n(x))$ $(x \in C_f(p))$ for all $n \geq 0$; similarly for $W^{cu}_{\epsilon_2}(x)$.*

(b) *If $\epsilon_1 < c$, then for all $y \in W^{cs}_{\epsilon_2}(x)$ $(x \in C_f(p))$, $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$; similarly for $W^{cu}_{\epsilon_2}(x)$. Here ϵ_2 is given by* (a).

(c) *Moreover, $C_f(p)$ has a local product structure.*

Proof. The method of the proof is the same as that of [**34,** Lemma 6], so I will just outline it. Recall that the set of hyperbolic periodic points homoclinically related to p is dense in $C_f(p)$ (see Remark 3.7). By Theorem B and Lemma 8.1, we can show items (a), (b) and (c) for any $q \in C_f(p) \cap P(f)$ with $q \sim p$ since the $C_f(p)$-germ of f is expansive (see the proof of [**34,** Lemmas 4 and 5]). We then extend these results to $C_f(p)$ by using the density of the above hyperbolic periodic points and the continuity of the local stable and unstable sets (for the details, see [**34,** Lemma 6]). Similar results were proved in [**19**]. □

Remark 8.3. By Lemma 8.2, for any given pseudo-orbit of f in $C_f(p)$, we can find a shadowing point in $C_f(p)$. More precisely, for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^b \subset C_f(p)$ of f $(-\infty \leq a < b \leq \infty)$, there is $y \in C_f(p)$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $a \leq i \leq b-1$ (see [**26**] and [**34,** Lemma 7]).

Since $C_f(p)$ is C^1-stably shadowable, hyperbolic periodic points are dense in $C_f(p)$. We can prove, however, a stronger result which plays a key role in the proof of Theorem C.

PROPOSITION 8.4. *Suppose that $C_f(p)$ is C^1-stably shadowable and that the $C_f(p)$-germ of f is expansive. Then every periodic point $q \in C_f(p) \cap P(f)$ is hyperbolic.*

Proof. Let $q \in C_f(p) \cap P(f)$ be given. The method of the proof is the same as that of [**34,** Proposition 5]. Indeed, by Lemma 8.2(a), for any fixed small $\epsilon > 0$, we have $W^{cs}_\epsilon(x) = W^s_\epsilon(x)$ and $W^{cu}_\epsilon(x) = W^u_\epsilon(x)$ for all $x \in C_f(p)$. Since the set of all hyperbolic periodic points in $C_f(p)$ that are homoclinically related to p is dense, we can pick a hyperbolic periodic point $q_1 \in C_f(p)$ as close as we wish to q, such that:

- $W^s(p) \cap W^u(q_1) \neq \emptyset$ and $W^u(p) \cap W^s(q_1) \neq \emptyset$; and
- $W^s_\epsilon(q) \cap W^u(q_1) \neq \emptyset$ and $W^u_\epsilon(q) \cap W^s(q_1) \neq \emptyset$ (see Lemma 8.1(c)).

Since these intersections are all transverse, by applying the λ-lemma to q_1 we have

$$W^s_\epsilon(q) \cap W^u(p) \neq \emptyset \quad \text{and} \quad W^u_\epsilon(q) \cap W^s(p) \neq \emptyset. \tag{12}$$

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Observe that by Lemma 8.2(a) and (b), the modulus of any eigenvalue of $D_q f^{\pi(q)}_{|T_q W^s_\epsilon(q)}$ is ≤ 1, and similarly for $D_q f^{-\pi(q)}_{|T_q W^u_\epsilon(q)}$.

We now show that q is hyperbolic by contradiction. If this were not true, then by slightly perturbing f throughout the orbit of q (mimicking the proof of Proposition 7.2) we can get a diffeomorphism g arbitrarily near f such that:

(i) q is a *hyperbolic* periodic point of g possessing the weakest normalized eigenvalue whose modulus is arbitrarily close to 1; and

(ii) $W^\sigma_\epsilon(q, g)$ is arbitrarily C^1-close to $W^\sigma_\epsilon(q) = W^{c\sigma}_\epsilon(q)$ $(\sigma = s, u)$, so that $q \sim p_g$ by (12).

On the other hand, recall that, under the assumption that $C_f(p)$ is C^1-stably shadowable, we have shown in the proof of Proposition 7.3 that the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by Dg along the hyperbolic periodic points $q \in C_g(p_g)$ such that $q \sim p_g$ is uniformly hyperbolic for all g near f. Thus the (i) and (ii) above contradict what we have shown in Proposition 7.3. □

8.2. *End of the proof of Theorem C.* Suppose that $C_f(p)$ is C^1-stably shadowable and that the $C_f(p)$-germ of f is expansive. By Theorem A, $C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ such that $\dim E = \mathrm{index}(p)$. We now show its hyperbolicity.

First, let me emphasize that Theorem B, together with Proposition 8.4, implies the existence of constants $m > 0$, $C > 0$ and $0 < \lambda < 1$ such that for any $q \in C_f(p) \cap P(f)$ $(\pi(q) \geq m)$,

$$\prod_{i=0}^{k-1} \left\| D_{f^{im}(q)} f^m_{|E^s_{f^{im}(q)}} \right\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \left\| D_{f^{-im}(q)} f^{-m}_{|E^u_{f^{-im}(q)}} \right\| < C\lambda^k,$$

where $k = [\pi(q)/m]$. Observe that, by the uniqueness of the dominated splitting, $E_q = E^s_q$ and $F_q = E^u_q$ for any $q \in C_f(p) \cap P(f)$ (see Remark 3.4).

For notational simplicity we put $g = f^m$. Before starting the proof, we prepare two lemmas. The first lemma, which is easily checked by the continuity of the dominated splitting, is a simple consequence of the above fact (cf. [**20,** Proposition 2.4]).

LEMMA 8.5. *With the above notation, for any $0 < \lambda < \gamma < 1$, there exists $\bar\epsilon > 0$ such that for any $q \in C_f(p) \cap P(f)$ and $x \in C_f(p)$:*

(a) *if $d(g^i(x), g^i(q)) < \bar\epsilon$ for all $0 \leq i \leq k-1$, then we have*

$$\prod_{i=0}^{k-1} \left\| D_{g^i(x)} g_{|E_{g^i(x)}} \right\| < C\gamma^k;$$

(b) *if $d(g^{-i}(x), g^{-i}(q)) \leq \bar\epsilon$ for all $0 \leq i \leq k-1$, then we have*

$$\prod_{i=0}^{k-1} \left\| D_{g^{-i}(x)} g^{-1}_{|F_{g^{-i}(x)}} \right\| < C\gamma^k.$$

Here $k = [\pi(q)/m]$.

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Recalling that the $C_f(p)$-germ of g is expansive and denoting by $c > 0$ the expansivity constant, for $0 < \epsilon < c/2$ let $0 < \delta = \delta(\epsilon) < \epsilon$ be the number in the definition of the shadowing property for $g_{|C_f(p)}$. The next lemma is well-known; see [**20,** Proposition 1.1] and [**4**].

LEMMA 8.6. *For every periodic δ-pseudo-orbit of g in $C_f(p)$ with period l, there exists a periodic point ϵ-shadowing the periodic δ-pseudo-orbit whose period is a divisor of l.*

We are now in a position to prove the theorem. By Lemma 8.6, following the proof of [**34,** Theorem 3] we can show that $C_f(p)$ is hyperbolic since $C_f(p)$ has a local product structure by Lemma 8.2(c). However, here I will give another proof based on [**20,** Proof of Theorem C] and [**19**]. This involves showing that $T_{C_f(p)}M = E \oplus F$ is hyperbolic, i.e. there are constants $K > 0$ and $0 < \mu < 1$ such that $\|D_x g^n_{|E_x}\| \le K\mu^n$ and $\|D_x g^{-n}_{|F_x}\| \le K\mu^n$ for all $x \in C_f(p)$ and $n \ge 0$. The proof is by contradiction.

Denote by $\mathcal{S}$ the family of compact f-invariant subsets of $C_f(p)$ which are not hyperbolic. It is easy to see that if $\{\Lambda_\alpha : \alpha \in \mathcal{A}\} \subset \mathcal{S}$ satisfies $\Lambda_{\alpha'} \subset \Lambda_{\alpha''}$ or $\Lambda_{\alpha''} \subset \Lambda_{\alpha'}$ for all $\alpha', \alpha'' \in \mathcal{A}$, then $\bigcap_\alpha \Lambda_\alpha \in \mathcal{S}$. Hence, by Zorn's lemma, there exists a minimal element Λ of $\mathcal{S}$, i.e. a $\Lambda \in \mathcal{S}$ such that $\Lambda' \in \mathcal{S}$ and $\Lambda' \subset \Lambda$ implies $\Lambda' = \Lambda$. It is proved in [**20**] that $\Lambda \in \mathcal{S}$ implies that $\{\|Dg^{-n}(v)\| : n \ge 0\}$ is bounded for some $x \in \Lambda$ and $v \in E_x$ $(v \ne 0)$, or $\{\|Dg^n(v)\| : n \ge 0\}$ is bounded for some $x \in \Lambda$ and $v \in F_x$ $(v \ne 0)$ (see also [**19,** pp. 171–172]). We consider the first case.

CLAIM 8.7. *The minimal element satisfies $\Lambda = \alpha(x)$.*

Observe that $\alpha(x) \in \mathcal{S}$, because if $\alpha(x)$ is hyperbolic, then $\lim_{n\to\infty} \|D_x g^{-n}(v)\| = \infty$). Thus $\alpha(x) = \Lambda$.

If $x \in C_f(p) \cap P(f)$, then $\alpha(x)$ is the orbit $\mathcal{O}_g(x)$ of x which is hyperbolic. This gives a contradiction, so we can assume that x is not periodic. Since $x \in \Lambda$, we see that $x \in \alpha(x)$. Thus, for any $\delta > 0$, there exists $l > 0$ such that $d(g^{-l}(x), x) < \delta$. Therefore

$$\{\ldots, g^{-l}(x), g^{-l+1}(x), \ldots, g^{-2}(x), g^{-1}(x), g^{-l}(x), g^{-l+1}(x), \ldots\}$$

is a periodic δ-pseudo-orbit of g. By Lemma 8.2(c), $g_{|C_f(p)}$ has the local product structure. Thus, for any $\epsilon > 0$, we can find $q \in C_f(p) \cap P(f)$ such that $d(g^{-i}(x), g^{-i}(q)) < \epsilon$ for $0 \le i \le \pi(q)$ by Lemma 8.6. In general, $\pi(q)$ is less than or equal to the length of the above periodic δ-pseudo-orbit of g.

Let $\bar{\epsilon}$ be given by Lemma 8.5 for some fixed $\lambda < \gamma < 1$, and put $z = g^{-\pi(q)}(x)$. Since ϵ is arbitrary, we may assume that $d(g^i(z), g^i(q)) < \bar{\epsilon}$ for $0 \le i \le \pi(q)$. Thus, by Lemma 8.5,

$$\|D_z g^{\pi(q)}_{|E_z}\| \le \prod_{i=0}^{\pi(q)-1} \|D_{g^i(z)} g_{|E_{g^i(z)}}\| < C\gamma^{\pi(q)}. \tag{13}$$

On the other hand, the existence of $x \in \Lambda$ and $v \in E_x$ implies that there is a constant $K = K(v) > 0$ such that $\|D_x g^{-i}(v)\| \le K\|v\|$ for all $i \ge 0$. So

$$\|D_z g^{\pi(q)}_{|E_z}\| \ge \frac{\|v\|}{\|D_x g^{-\pi(q)}(v)\|} \ge \frac{1}{K} > 0.$$

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Since x is not periodic, by taking ϵ small enough (and hence by taking δ small enough), we can suppose that $\pi(q)$ is arbitrarily large so that $C\gamma^{\pi(q)} < 1/K$. This contradicts (13) and completes the proof of Theorem C. □

9. *Proof of Theorem D*

As mentioned in §1, the 'if' part of Theorem D is proved in [**2,** Proposition 2.9]. We give here a more basic proof that takes a conventional approach.

9.1. *End of the proof of Theorem D: the 'if' part.*

This follows directly from the next proposition, and the result is independent of the dimension of M.

PROPOSITION 9.1. *Let C_f be a chain component of $f \in \mathrm{Diff}(M)$. If C_f is hyperbolic, then it is locally maximal.*

Proof. Since C_f is hyperbolic, $f_{|C_f}$ has the shadowing property and the C_f-germ of f is expansive. First we show the following claim.

CLAIM 9.2. *For any $\epsilon > 0$, there is a $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}}$ of f in C_f, there is a $y \in C_f$ satisfying $d(f^i(y), x_i) < \epsilon$ for $i \in \mathbb{Z}$; that is, the shadowing point y can be taken from C_f.*

To prove the claim, it is enough to show that $f_{|C_f}$ has a local product structure (see §8.1 and Remark 8.3). Since C_f is hyperbolic, for any $\epsilon > 0$ there is a $\delta > 0$ such that if $d(x, y) < \delta$ (with $x, y \in C_f$), then $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ have a non-empty transverse intersection. Fix any $x, y \in C_f$ with $d(x, y) < \delta$, and let

$$z = W^s_\epsilon(x) \cap W^u_\epsilon(y) \quad \text{and} \quad w = W^u_\epsilon(x) \cap W^s_\epsilon(y).$$

We show that $z, w \in C_f$.

For any $\eta > 0$, there is $n > 0$ such that $\max\{d(f^n(x), f^n(z)), d(f^{-n}(y), f^{-n}(z))\} < \eta$. Since $f^{\pm}(C_f) = C_f$, we have $f^n(x), f^{-n}(y) \in C_f$. Thus $f^n(x) \rightsquigarrow x$ and $x \rightsquigarrow f^{-n}(y)$, i.e. there is a η-pseudo-orbit $\{x_i\}_{i=0}^{n_\eta}$ with $x_0 = f^n(x)$, $x_{n_\eta} = x$, and a η-pseudo-orbit $\{y_i\}_{i=0}^{m_\eta}$ with $y_0 = x$, $y_{m_\eta} = f^{-n}(y)$. Since η is arbitrary, $z \leftsquigarrow x$ so that $z \in C_f$. Similar reasoning shows that $w \in C_f$ and thus $f_{|C_f}$ has a local product structure, proving the claim.

Now fix $0 < \epsilon \le c/2$ (where $c > 0$ is the expansivity constant), and let $0 < \delta < \epsilon$ be the number in the definition of the shadowing property for $f_{|C_f}$. Furthermore, choose $0 < \delta' < \delta/2$ such that $d(x, y) < \delta'$ implies $d(f(x), f(y)) < \delta/2$. We will show that $C_f = \bigcap_{i\in\mathbb{Z}} f^i(B_{\delta'}(C_f))$. The $\subset$ part is obvious. To show the converse, we need to demonstrate that any $x \in \bigcap_{i\in\mathbb{Z}} f^i(B_{\delta'}(C_f))$ is in C_f. For each $i \in \mathbb{Z}$, pick $x_i \in C_f$ such that $d(f^i(x), x_i) < \delta'$. It is easy to see that $\{x_i\}_{i\in\mathbb{Z}}$ is a δ-pseudo-orbit of $f_{|C_f}$ by the choice of δ'. By the shadowing property of $f_{|C_f}$, together with Claim 9.2, there is a $y \in C_f$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Thus

$$d(f^i(x), f^i(y)) \le d(f^i(x), x_i) + d(x_i, f^i(y)) < \delta' + \epsilon < c$$

for all $i \in \mathbb{Z}$, so that $x = y \in C_f$. □

Hereafter, let $M = M^2$ be a surface, and suppose that:

- $f(p) = p$ is a hyperbolic saddle and $C_f(p)$ is C^1-stably shadowable.

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9.2. *Dynamics on the local center-stable and center-unstable manifolds.* Before starting the proof of the 'only if' part, we prepare two lemmas. Let m, C, and λ be given by Theorem B, and write $g = f^m$ for convenience. Since there is a dominated splitting $T_{C_f(p)}M^2 = E \oplus F$ by Theorem A, if we fix $\epsilon_1 > 0$, then there is a $\delta > 0$ such that $x, y \in C_f(p)$ and $d(x, y) \le \delta$ implies that $W^{cs}_{\epsilon_1}(x)$ transversely intersects $W^{cu}_{\epsilon_1}(y)$ at one and only one point by Lemma 8.1(c). Under this condition, put

$$[x, y] = W^{cs}_{\epsilon_1}(x) \cap W^{cu}_{\epsilon_1}(y).$$

Moreover, if $x \in C_f(p)$ and $y \in W^{cs}_{\epsilon_1}(x)$, we let $d_s(x, y) = \inf\{\epsilon : y \in W^{cs}_{\epsilon}(x)\}$. Also, for $x \in C_f(p)$ and $y \in W^{cu}_{\epsilon_1}(x)$, define $d_u(x, y) = \inf\{\epsilon : y \in W^{cu}_{\epsilon}(x)\}$. We may assume further that $d(x, y) < \delta$ implies

$$g([x, y]) = [g(x), g(y)] \quad \text{and} \quad g^{-1}([x, y]) = [g^{-1}(x), g^{-1}(y)]. \tag{14}$$

The following lemma was proved in [**20,** Lemma 4.1].

LEMMA 9.3. *There exist $K > 0$, $0 < \mu < 1$ and $\epsilon_2 > 0$ such that if $x_i \in C_f(p)$ $(i = 1, 2)$, $l > 0$ and $y = [x_1, x_2]$ satisfy $d_s(g^n(x_1), g^n(y)) \le \epsilon_2$ and $d_u(g^n(x_2), g^n(y)) \le \epsilon_2$ for all $0 \le n \le l$, then*

$$\frac{d_s(g^n(x_1), g^n(y))}{d_s(x_1, y)} \le K\mu^n \frac{d_u(g^n(x_2), g^n(y))}{d_u(x_2, y)}$$

for all $0 \le n \le l$.

If $x \in C_f(p)$ and $y \in W^{c\sigma}_{\epsilon_1}(x)$ $(\sigma = s, u)$, then let $\lambda^{c\sigma}(x, y)$ be the compact arc in $W^{c\sigma}_{\epsilon_1}(x)$ with boundary $\{x\} \cup \{y\}$. For $w \in \lambda^{cs}(x, y)$ (respectively $w \in \lambda^{cu}(x, y)$) define $\tilde{E}_w = T_w W^{cs}_{\epsilon_1}(x)$ (respectively $\tilde{F}_w = T_w W^{cu}_{\epsilon_1}(x)$). The following lemma is an easy consequence of Theorem B (the result does not depend on the dimension of the manifold).

LEMMA 9.4. *For any $0 < \lambda < \gamma < 1$, there exists $\epsilon_3 > 0$ such that for any $x \in C_f(p)$ and $y \in W^{c\sigma}_{\epsilon_1}(x)$ $(\sigma = s, u)$:*

(a) *if there is a hyperbolic periodic point $q \in C_f(p) \cap P(f)$ $(\pi(q) \ge m)$ and a $w \in \lambda^{cs}(x, y)$ such that $d(g^i(w), g^i(q)) \le \epsilon_3$ for all $0 \le i \le k-1$, then we have $\prod_{i=0}^{k-1} \| D_{g^i(w)} g_{|\tilde{E}_{g^i(w)}} \| < C\gamma^k$;*

(b) *if there is a hyperbolic periodic point $q \in C_f(p) \cap P(f)$ $(\pi(q) \ge m)$ and a $w \in \lambda^{cu}(x, y)$ such that $d(g^{-i}(w), g^{-i}(q)) \le \epsilon_3$ for all $0 \le i \le k-1$, then we have $\prod_{i=0}^{k-1} \| D_{g^{-i}(w)} g^{-1}_{|\tilde{F}_{g^{-i}(w)}} \| < C\gamma^k$.*

Here $k = [\pi(q)/m]$.

Proof. For any $\epsilon > 0$, there is a $\delta > 0$ such that for $w \in \lambda^{cs}(x, y)$, if $d(w, q) < \delta$ then $\| D_w g_{|\tilde{E}_w} \| \le e^{\epsilon} \| D_q g_{|E^s_q} \|$. Thus assertion (a) follows from Theorem B in a straightforward manner. The proof of assertion (b) is analogous. □

9.3. *End of the proof of Theorem D: the 'only if' part.* Let $C_f(p)$ be C^1-stably shadowable and locally maximal. Recall that $g = f^m$. To obtain the conclusion, it suffices to show that the $C_f(p)$-germ of f is expansive, by virtue of Theorem C. Since $C_f(p)$ is

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locally maximal, if we prove that $g_{|C_f(p)}$ is expansive, then the $C_f(p)$-germ of f will be expansive (because $g_{|C_f(p)} = f^m_{|C_f(p)}$ is expansive if and only if $f_{|C_f(p)}$ is expansive).

By Remark 3.7, hyperbolic periodic points q ($\pi(q) \geq m$) are dense in $C_f(p)$. Thus we can show that $g_{|C_f(p)}$ is expansive in the same way as the proof of [**20,** Theorem A], which is outlined here for completeness (see [**20,** pp. 390–392] for the details).

Fix $\epsilon_1 > 0$, and let $\delta > 0$ be as in (14). Let $0 < \epsilon_2 \leq \epsilon_1$ be the number given by Lemma 9.3, and let $0 < \epsilon_3 \leq \epsilon_2$ be the number given by Lemma 9.4 (for some fixed $\lambda < \gamma < 1$). Fix $0 < \delta_1 = \delta_1(\epsilon_3) \leq \delta$ small enough such that $d_s(x, y) \leq \delta_1$ (where $x, y \in C_f(p)$) implies that $d_s(x, [x, y]) \leq \epsilon_3/2$ and $d_u(y, [x, y]) \leq \epsilon_3/2$.

Now, taking any points $x, y \in C_f(p)$ such that $d(g^n(x), g^n(y)) \leq \delta_1$ for all $n \in \mathbb{Z}$, we show that $x = y$ and thus that δ_1 is an expansive constant for $g_{|C_f(p)}$. If $x \neq y$, then $y \notin W^{cs}_{\epsilon_1}(x)$ or $y \notin W^{cu}_{\epsilon_1}(x)$. We consider the former case (the other case is similar). Let $z = [x, y]$; then, by (14), $z \neq y$ and $g^n(z) = [g^n(x), g^n(y)]$ for all $n \in \mathbb{Z}$. Thus

$$d_u(g^n(y), g^n(z)) \leq \epsilon_3/2 < \epsilon_2 \quad \text{for all } n \in \mathbb{Z} \tag{15}$$

by the choice of δ_1. Since $C_f(p)$ is locally maximal, by reducing ϵ_2 if necessary, we have the following.

CLAIM 9.5. $\lambda^{cu}(y, z) \subset C_f(p)$.

Fix $\epsilon_4 > 0$ such that $d_s(v, x_1) \leq \epsilon_4$ implies $d_s(g(v), g(x_1)) \leq \epsilon_2$ for $v \in W^{cs}_{\epsilon_1}(x_1)$ ($x_1 \in C_f(p)$). By Claim 9.5 we can pick a hyperbolic periodic point $q \in C_f(p) \cap P(f)$ near $\lambda^{cu}(y, z)$ and, in particular, near z, which satisfies

$$K\epsilon_2 \cdot \frac{d_s(q, [q, y])}{d_u(y, [q, y])} \leq \epsilon_4. \tag{16}$$

Here $K > 0$ is the constant given by Lemma 9.3. We may suppose that $\pi(q) \geq m$ is arbitrarily large because transversal homoclinic points associated with p are dense in $C_f(p)$ (see Remark 3.7).

Obviously, $d_u(g^n(y), g^n([q, y])) \leq \epsilon_2$ for all $n \geq 0$ by (15). Since δ_1 is small enough, by the choice of ϵ_4 and (16), it follows inductively from Lemma 9.3 that

$$d_s(g^n(q), g^n([q, y])) \leq K\mu^n\epsilon_2 \cdot \frac{d_s(q, [q, y])}{d_u(y, [q, y])}$$

for all $n \geq 0$. Thus there is $n' > 0$ such that if $n \geq n'$, then

$$d_s(g^n(q), g^n([q, y])) \leq \epsilon_3/2. \tag{17}$$

By combining (15) with (17) we see that $n \geq n'$ implies

$$d_s(g^n(q), g^n(w)) \leq \epsilon_3 \quad \text{for any } w \in \lambda^{cu}(y, z).$$

Observe that the above n' does not depend on the choice of the hyperbolic periodic point q in the inequality (16).

Finally, put $k = [\pi(q)/m]$ and apply Lemma 9.4 to the hyperbolic periodic point $g^{n'}(q)$. Then $\|D_w g^k_{|\tilde{F}_w}\| \geq C^{-1}\gamma^{-k} > 1$ for any $w \in \lambda^{cu}(g^{n'}(y), g^{n'}(z))$. This contradicts the fact that $g^k(\lambda^{cu}(g^{n'}(y), g^{n'}(z))) = \lambda^{cu}(g^k(g^{n'}(y)), g^k(g^{n'}(z)))$ remains bounded when $k \to \infty$. Therefore $g_{|C_f(p)}$ is expansive, and the proof is complete. □

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