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FURTHER ARITHMETIC PROPERTIES OF OVERCUBIC PARTITION TRIPLES

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Abstract

We prove several new congruences for the overcubic partition triples function, using both elementary techniques and the theory of modular forms. These extend the recent list of such congruences given by Nayaka, Dharmendra and Kumar ['Divisibility properties for overcubic partition triples', *Integers* **24** (2024), Article no. a80, 9 pages]. We also generalise overcubic partition triples to overcubic partition k-tuples and prove arithmetic properties for these partitions.

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1. Introduction

An integer partition of n is a sequence of nonincreasing parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\sum_{i=0}^{k} \lambda_k = n$. We denote by $p(n)$ the number of partitions of any nonnegative integer n. For instance, there are 7 partitions of 5, namely

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1 \quad \text{and} \quad 1+1+1+1+1,$$

and hence $p(5) = 7$. Euler proved that the generating function of $p(n)$ is given by

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{\prod_{i\geq 1}(1-q^i)} := \frac{1}{f_1},$$

where for $k \geq 1$, we define the short-hand notation $f_k := \prod_{i\geq 1}(1-q^{ki})$. For a general survey of the theory of partitions, we refer to the books of Andrews [1] and Johnson [8].

Since the time of Euler, mathematicians have studied numerous special subsets as well as generalisations of the set of partitions and a rich theory has grown up. One such generalisation is the *cubic partition* function, first studied by Chan [4, 5]. A cubic

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partition of n is a partition of n in which the even parts can appear in two colours. If $a(n)$ denotes the number of such partitions of n, then its generating function is given by

$$\sum_{n\geq 0} a(n)q^n = \frac{1}{f_1 f_2}.$$

Shortly after Chan's work, Kim [9] studied the overpartition analogue of the cubic partition, called the *overcubic partition* function. The number of overcubic partitions of n, denoted by $\overline{a}(n)$, counts the number of overlined cubic partitions, that is, the cubic partitions where the first instance of each part is allowed to be overlined. The generating function is given by

$$\sum_{n\geq 0} \overline{a}(n)q^n = \frac{f_4}{f_1^2 f_2}. \tag{1.1}$$

Zhao and Zhong [19] subsequently studied the number of *cubic partition pairs*, denoted by $b(n)$, with the generating function

$$\sum_{n\geq 0} b(n)q^n = \frac{1}{f_1^2 f_2^2}.$$

The overpartition version $\overline{b}(n)$ of this function counting the number of overcubic partition pairs of n was studied by Kim [10]; its generating function is given by

$$\sum_{n\geq 0} \overline{b}(n)q^n = \frac{f_4^2}{f_1^4 f_2^2}.$$

We can extend this definition even further, as was done recently by Nayaka *et al.* [11]. They denoted by $\overline{bt}(n)$ the number of overcubic partition triples of n and gave the generating function

$$\sum_{n\geq 0} \overline{bt}(n)q^n = \frac{f_4^3}{f_1^6 f_2^3}. \tag{1.2}$$

One of the fundamental questions that arises in the theory of integer partitions is whether $p(n)$ or any interesting subclass of partitions satisfy nice arithmetic properties. For $p(n)$, the answer is found in Ramanujan's congruences:

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Following this cue from Ramanujan, several mathematicians have studied such Ramanujan-type congruences for many sub-classes of partitions as well as for generalisations of partitions, including for $a(n), \overline{a}(n), b(n), \overline{b}(n)$ and $\overline{bt}(n)$. The main goal of this paper is to extend the list of congruences for $\overline{bt}(n)$ given by Nayaka *et al.* [11]. We also extend the definition of overcubic partition triples to overcubic partition k-tuples and explore arithmetic properties of this class of partitions.

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Before stating our main results, we note that Nayaka *et al.* [11, (46)] showed that $\overline{bt}(2n+1) \equiv 0 \pmod 2$ for all $n \geq 1$. In fact, for all $n \geq 1$,

$$\overline{bt}(n) \equiv 0 \pmod 2. \tag{1.3}$$

This follows easily from the binomial theorem by observing that

$$\sum_{n\geq 0} \overline{bt}(n)q^n = \prod_{i\geq 1}\left(\frac{1+q^{2i}}{1+q^{2i}-2q^i}\right)^3 = \prod_{i\geq 1}\left(1+2\frac{q^i}{1+q^{2i}-2q^i}\right)^3 \equiv 1 \pmod 2.$$

We also note the following congruence, which we will use without commentary in what follows: for a prime p and positive integers k and l,

$$f_k^{p^l} \equiv f_{pk}^{p^{l-1}} \pmod{p^l}.$$

We now state our first result.

THEOREM 1.1. *For all $n \geq 0$,*

$$\overline{bt}(4n+3) \equiv 0 \pmod 4, \tag{1.4}$$
$$\overline{bt}(8n+5) \equiv 0 \pmod{32}, \tag{1.5}$$
$$\overline{bt}(8n+6) \equiv 0 \pmod 4,$$
$$\overline{bt}(8n+7) \equiv 0 \pmod{64}, \tag{1.6}$$
$$\overline{bt}(16n+10) \equiv 0 \pmod{32}, \tag{1.7}$$
$$\overline{bt}(16n+12) \equiv 0 \pmod 4,$$
$$\overline{bt}(16n+14) \equiv 0 \pmod{64},$$
$$\overline{bt}(32n+20) \equiv 0 \pmod{32}, \tag{1.8}$$
$$\overline{bt}(32n+24) \equiv 0 \pmod 4,$$
$$\overline{bt}(32n+28) \equiv 0 \pmod{64}.$$

REMARK 1.2. Some of our congruences are stronger than those found by Nayaka, Dharmendra and Kumar. For instance, they proved that for all $n \geq 0$,

$$\overline{bt}(8n+5) \equiv 0 \pmod 8, \quad \overline{bt}(8n+7) \equiv 0 \pmod{32}, \qquad ([11, \text{Theorem 1}])$$

$$\overline{bt}(16n+10) \equiv 0 \pmod{16}, \quad \overline{bt}(16n+14) \equiv 0 \pmod{16}, \qquad ([11, \text{Theorem 4}])$$

$$\overline{bt}(32n+20) \equiv 0 \pmod{16}, \quad \overline{bt}(32n+28) \equiv 0 \pmod{16}. \qquad ([11, \text{Theorem 5}])$$

There are numerous other congruences modulo powers of 2 and multiples of 3 for $\overline{bt}(n)$. We prove two such congruences here to give a flavour.

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THEOREM 1.3. *For all $n \geq 0$,*

$$\overline{bt}(72n + 21) \equiv 0 \pmod{128},$$
$$\overline{bt}(72n + 69) \equiv 0 \pmod{384}.$$

We prove Theorems 1.1 and 1.3 in Section 2 using Smoot's implementation [18] of Radu's algorithm [12, 13] based on the theory of modular forms.

The main purpose of proving Theorem 1.1 was to 'guess' the following theorem.

THEOREM 1.4. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{bt}(2^{\alpha}(4n + 3)) \equiv 0 \pmod{4}, \tag{1.9}$$
$$\overline{bt}(2^{\alpha}(8n + 5)) \equiv 0 \pmod{32}. \tag{1.10}$$

We prove Theorem 1.4 in Section 3 using elementary means. We can go even further, but we leave the following as an open problem.

CONJECTURE 1.5. For all $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{bt}(2^{\alpha}(8n + 7)) \equiv 0 \pmod{64}.$$

The main purpose of proving Theorem 1.3 was to 'guess' the following conjecture.

CONJECTURE 1.6. For all $n \geq 0$ and $\alpha \geq 0$,

$$\overline{bt}(144n + 42) \equiv 0 \pmod{384},$$
$$\overline{bt}(2^{\alpha}(72n + 21)) \equiv 0 \pmod{128},$$
$$\overline{bt}(2^{\alpha}(72n + 69)) \equiv 0 \pmod{128}.$$

As the reader has probably guessed, there is nothing special in taking overcubic partition pairs or triples. We can define $\overline{b}_k(n)$ to be the number of overcubic partition k-tuples, with the generating function

$$\sum_{n\geq 0} \overline{b}_k(n)q^n = \frac{f_4^k}{f_1^{2k} f_2^k}. \tag{1.11}$$

Then, $\overline{b}_1(n) = \overline{a}(n), \overline{b}_2(n) = \overline{b}(n)$ and $\overline{b}_3(n) = \overline{bt}(n)$. Similar to the $\overline{bt}(n)$ function, there seem to be many congruences that the $\overline{b}_k(n)$ function satisfies modulo powers of 2. We just state a few of them here. We begin with the following easy result.

THEOREM 1.7. *For all $n \geq 0$ and $k \geq 1$ with $n, k \in \mathbb{Z}$,*

$$\overline{b}_{2k+1}(n) \equiv \overline{a}(n) \pmod{4}.$$

PROOF. We have

$$\sum_{n\geq 0} \overline{b}_{2k+1}(n)q^n = \frac{f_4^{2k+1}}{f_1^{2(2k+1)} f_2^{2k+1}} = \frac{f_4^{2k} f_4}{f_1^{4k} f_2^{2k} f_1^2 f_2} \equiv \frac{f_4}{f_1^2 f_2} \pmod{4}.$$

This completes the proof, via (1.1). □

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Theorem 1.7 can be used to prove an infinite family of Ramanujan-like congruences modulo 4 for the functions $\overline{b}_{2k+1}$. We do not explore this aspect here.

Our next two theorems give further congruences modulo powers of 2 for the overcubic partition k-tuples function.

THEOREM 1.8. *For all $n \geq 0$, $k \geq 0$ with $n, k \in \mathbb{Z}$ and $p \geq 3$ prime, and all quadratic nonresidues r modulo p with $1 \leq r \leq p-1$,*

$$\overline{b}_{2k+1}(2pn + R) \equiv 0 \pmod 4,$$

where

$$R = \begin{cases} r & \text{if } r \text{ is odd,} \\ p + r & \text{if } r \text{ is even.} \end{cases}$$

PROOF. From [16, Theorem 2.5], for all $n \geq 1, \overline{a}(n) \equiv 0 \pmod 4$ if and only if n is neither a square nor twice a square. So it is enough for us to show that $2pn + R$ as defined above is never a square and never twice a square, thanks to Theorem 1.7. Clearly, $2pn + R$ is always odd by definition, so it cannot be twice a square. Next, from the definition of R, we see that $2pn + R \equiv r \pmod p$. Since r is defined to be a quadratic nonresidue modulo p, we know that r cannot be congruent to a square modulo p. Thus, $2pn + R$ cannot equal a square. This concludes the proof. □

THEOREM 1.9. *For all $n \geq 0$ and $k \geq 0$,*

$$\begin{aligned}
\overline{b}_{2k+1}(8n + 1) &\equiv 0 \pmod 2, \\
\overline{b}_{2k+1}(8n + 2) &\equiv 0 \pmod 2, \\
\overline{b}_{2k+1}(8n + 3) &\equiv 0 \pmod 4, \\
\overline{b}_{2k+1}(8n + 4) &\equiv 0 \pmod 2, \\
\overline{b}_{2k+1}(8n + 5) &\equiv 0 \pmod 8, \\
\overline{b}_{2k+1}(8n + 6) &\equiv 0 \pmod 4, \\
\overline{b}_{2k+1}(8n + 7) &\equiv 0 \pmod{16}.
\end{aligned}$$

We give a sketch proof of this theorem in Section 5. In fact, it is easy to verify the modulo 2 congruences, as we did with (1.3).

Along with the study of Ramanujan-type congruences, it is also interesting to explore the distribution of the coefficients of a formal power series modulo M. Given an integral power series $A(q) := \sum_{n=0}^{\infty} a(n)q^n$ and an integer r with $0 \leq r \leq M$, the arithmetic density $\delta_r(A, M; X)$ is defined by

$$\delta_r(A, M; X) = \frac{\#\{n \leq X : a(n) \equiv r \pmod M\}}{X}.$$

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An integral power series A is called *lacunary modulo* M if

$$\lim_{X\to\infty} \delta_0(A, M; X) = 1,$$

which means that almost all the coefficients of A are divisible by M. It turns out that the partition function $\overline{bt}(n)$ is almost always divisible by 2^k for $k \geq 1$. We prove the following result in Section 4.

THEOREM 1.10. *For $\ell \geq 1$, let $G(q) = \sum_{n=0}^{\infty} \overline{b}_\ell(n)q^n$. Then, for every positive integer k,*

$$\lim_{X\to\infty} \delta_0(G, 2^k; X) = 1.$$

2. Proofs of Theorems 1.1 and 1.3

In this section, we prove Theorems 1.1 and 1.3 using Smoot's implementation [18] of Radu's algorithm [12, 13]. The algorithm takes as input the generating function

$$\sum_{n=0}^{\infty} a_r(n)q^n = \prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta},$$

and positive integers m and N, another positive integer M, and a sequence $(r_\delta)_{\delta|M}$ indexed by the positive divisors δ of M. With this input, Radu's algorithm tries to produce a set $P_{m,j}(j) \subseteq \{0, 1, \ldots, m-1\}$ which contains j and is uniquely defined by m, $(r_\delta)_{\delta|M}$ and j. Then, it decides if there exists a sequence $(s_\delta)_{\delta|N}$ such that

$$q^\alpha \prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{s_\delta} \cdot \prod_{j'\in P_{m,j}(j)} \sum_{n=0}^{\infty} a(mn+j')q^n$$

is a modular function with certain restrictions on its behaviour on the boundary of the upper half-plane $\mathbb{H}$. Smoot [18] implemented this algorithm in Mathematica and we use his `RaduRK` package, which requires the software package `4ti2`.

It is natural to guess that $N = m$ (which corresponds to the congruence subgroup $\Gamma_0(N)$), but this is not always the case, although N and m are usually closely related to one another. The determination of the correct value of N is an important problem for the usage of `RaduRK` and it depends on the Δ^* criterion. It is easy to check the minimum N which satisfies this criterion by running `minN[M, r, m, j]`. The generating function of $\overline{bt}(n)$ given in (1.2) corresponds to $M = 4$ and $r = \{-6, -3, 3\}$.

PROOF OF THEOREM 1.1. Since the proofs of all the congruences listed in Theorem 1.1 are similar, we only prove (1.6) here. The rest of the output can be obtained by visiting https://manjilsaikia.in/publ/mathematica/Overcubic_Triples.nb. We use (1.2) and calculate `minN[4,{-6,-3,3},8,7]`. This gives $N = 8$, which is easily handled on a modest laptop. Radu's algorithm now gives a straightforward proof of (1.6). Here, we give the output of `RK`.

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```
In[1] := RaduRK[8,4,{-6,-3,3},8,7]
```

$$\prod_{\delta\,|\,\mathtt{M}}(\mathtt{q}^{\delta};\mathtt{q}^{\delta})_{\infty}^{\mathtt{r}_{\delta}} = \sum_{\mathtt{n}=\mathtt{0}}^{\infty}\mathtt{a}(\mathtt{n})\,\mathtt{q}^{\mathtt{n}}$$

$$\boxed{\mathtt{f}_1(\mathtt{q})\cdot\prod_{\mathtt{j}'\in P_{\mathtt{m},\mathtt{r}}(\mathtt{j})}\sum_{\mathtt{n}=\mathtt{0}}^{\infty}\mathtt{a}(\mathtt{mn}+\mathtt{j}')\,\mathtt{q}^{\mathtt{n}} = \sum_{\mathtt{g}\in\mathtt{AB}}\mathtt{g}\cdot\mathtt{p}_{\mathtt{g}}(\mathtt{t})}$$

```
Modular Curve: X₀(N)
```

```
Out[2] =
```

N :	8
$\{M,(r_\delta)_{\delta\|M}\}$:	$\{4,\{-6,-3,3\}\}$
m :	8
$P_{m,r}$(j) :	{7}
f_1(q) :	$\dfrac{(q;q)_\infty^{69}(q^4;q^4)_\infty^{30}}{q^{15}(q^2;q^2)_\infty^{29}(q^8;q^8)_\infty^{64}}$
t :	$\dfrac{(q^4;q^4)_\infty^{12}}{q(q^2;q^2)_\infty^{4}(q^8;q^8)_\infty^{8}}$
AB :	{1}
$\{p_g$(t): g ∈ $AB\}$	$\{9792t^{15}+6606336t^{14}+905825280t^{13}$ $+46058225664t^{12}+1124900028416t^{11}$ $+15177685794816t^{10}+122507156520960t^{9}$ $+616578030764032t^{8}+1960114504335360t^{7}$ $+3885487563472896t^{6}+4607590516391936t^{5}$ $+3018471877115904t^{4}+949826648801280t^{3}$ $+110835926040576t^{2}+2628519985152t\}$
Common Factor:	64

The interpretation of this output is as follows. The first entry in the procedure call `RK[8, 4, {-6, -3, 3}, 8, 7]` corresponds to specifying $N = 8$, which fixes the space of modular functions

$$M(\Gamma_0(N)) := \text{the algebra of modular functions for } \Gamma_0(N).$$

The second and third entries of the procedure call `RK[8, 4, {-6, -3, 3}, 8, 7]` give the assignment $\{M,(r_\delta)_{\delta|M}\} = \{4,(-6,-3,3)\}$, which corresponds to specifying $(r_\delta)_{\delta|M} = (r_1,r_2,r_4) = (-6,-3,3)$ so that

$$\sum_{n\geq 0}\overline{bt}(n)q^n = \prod_{\delta|M}(q^{\delta};q^{\delta})_{\infty}^{r_\delta} = \frac{f_4^3}{f_1^6 f_2^3}.$$

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The last two entries of the procedure call `RK[8, 4, {-6, -3, 3}, 8, 7]` correspond to the assignment $m = 8$ and $j = 7$, which means that we want the generating function

$$\sum_{n\geq 0} \overline{bt}(n)(mn + j)q^n = \sum_{n\geq 0} \overline{bt}(n)(8n + 7)q^n.$$

So, $P_{m,r}(j) = P_{8,r}(7)$ with $r = (-6, -3, 3)$. The output $P_{m,r}(j) := P_{8,(-6,3,3)}(7) = \{7\}$ means that there exists an infinite product

$$f_1(q) = \frac{(q;q)_\infty^{69}(q^4;q^4)_\infty^{30}}{q^{15}(q^2;q^2)_\infty^{29}(q^8;q^8)_\infty^{64}}$$

such that

$$f_1(q) \sum_{n\geq 0} \overline{bt}(n)(8n + 7)q^n \in M(\Gamma_0(8)).$$

Finally, the output

$$t = \frac{(q^4;q^4)_\infty^{12}}{q(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}, \quad AB = \{1\} \quad \text{and} \quad \{p_g(\text{t}): \text{g} \in AB\},$$

presents a solution to the question of finding a modular function $t \in M(\Gamma_0(8))$ and polynomials $p_g(t)$ such that

$$f_1(q) \sum_{n\geq 0} \overline{bt}(8n + 7)q^n = \sum_{g\in AB} p_g(t) \cdot g.$$

In this specific case, we see that the singleton entry in the set $\{p_g(t) : g \in AB\}$ has the common factor 64, and thus proves (1.6). □

REMARK 2.1. Some recent applications of this method can be found in [2, 14].

PROOF OF THEOREM 1.3. Since the proof of Theorem 1.3 is similar to the proof of Theorem 1.1, we refer to the output file available at https://manjilsaikia.in/publ/mathematica/Overcubic_Triples_72.nb. □

3. Proof of Theorem 1.4

We will use the following 2-dissections (see, for example, [3, Lemma 2]) in the proof of Theorem 1.4.

LEMMA 3.1. *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{3.1}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{3.2}$$

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$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{3.3}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{3.4}$$

PROOF OF (1.9). Nayaka *et al.* [11, (45)] showed

$$\sum_{n\geq 0} \overline{bt}(n)q^n = \frac{f_4^3 f_8^{15}}{f_2^{18} f_{16}^6} + 6q\frac{f_4^5 f_8^9}{f_2^{18} f_{16}^2} + 12q^2\frac{f_4^7 f_8^3 f_{16}^2}{f_2^{18}} + 8q^3\frac{f_4^9 f_{16}^6}{f_2^{18} f_8^3}. \tag{3.5}$$

From (3.5),

$$\sum_{n\geq 0} \overline{bt}(2n)q^n \equiv \frac{f_2^3 f_4^{15}}{f_1^{18} f_8^6} \pmod 4. \tag{3.6}$$

Using (1.2) and (3.6), we obtain, for all $n \geq 0$,

$$\overline{bt}(2n) \equiv \overline{bt}(n) \pmod 4. \tag{3.7}$$

Combining (3.7) with (1.4) yields (1.9). □

REMARK 3.2. It can be shown that $\overline{bt}(n) \equiv \overline{a}(n) \pmod 4$. With this and a result of Sellers [16, Corollary 2.6] for $\overline{a}(n)$, we have an alternative proof of (1.9).

PROOF OF (1.10). We re-write (1.2) as

$$\sum_{n\geq 0} \overline{bt}(n)q^n = \frac{f_4^3}{f_2^3} \cdot \frac{1}{(f_1^4)^2} \cdot f_1^2.$$

Employing (3.1), (3.4) and then extracting the even-powered terms in q gives

$$\begin{aligned}\sum_{n\geq 0} \overline{bt}(2n)q^n &= \frac{f_2^{29}}{f_1^{30} f_4^3 f_8^2} + 16q\frac{f_4^{13} f_2^5}{f_1^{22} f_8^2} - 16q\frac{f_8^2 f_2^{19}}{f_1^{26} f_4} \\ &\equiv \frac{f_1^2 f_2^{13}}{f_4^3 f_8^2} + 16qf_1^{24} - 16qf_1^{24} \equiv \frac{f_1^2 f_2^{13}}{f_4^3 f_8^2} \pmod{32}.\end{aligned}$$

Extracting the even-powered terms in q by using (3.1) gives

$$\sum_{n\geq 0} \overline{bt}(4n)q^n \equiv \frac{f_1^{14} f_4^3}{f_2^5 f_8^2} \pmod{32}. \tag{3.8}$$

Again, re-writing (1.2) as

$$\sum_{n\geq 0} \overline{bt}(n)q^n = \frac{f_4^3}{f_2^3} \cdot \frac{1}{f_1^4} \cdot \frac{1}{f_1^2},$$

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employing (3.2), (3.4) and then extracting the even-powered terms of q,

$$\sum_{n\geq 0}\overline{bt}(2n)q^n = \frac{f_4f_2^{17}}{f_1^{22}f_8^2} + 8q\frac{f_4^3f_8^2f_2^7}{f_1^{18}} \equiv \frac{f_2f_4f_1^{10}}{f_8^2} + 8qf_2f_4f_1^{18} \pmod{32}$$
$$\equiv \frac{f_2f_4}{f_8^2}\cdot(f_1^4)^2\cdot f_1^2 + 8qf_2f_4\cdot(f_1^4)^4\cdot f_1^2 \pmod{32}.$$

Employing (3.1), (3.3) and further extracting the even-powered terms of q,

$$\sum_{n\geq 0}\overline{bt}(4n)q^n \equiv \frac{f_2^{19}}{f_1^2f_4^5f_8^2} + 16q\frac{f_1^2f_8^2f_2^9}{f_4^3} + 16q\frac{f_1^6f_4^{11}}{f_2^5f_8^2} - 16q\frac{f_8^2f_2^{41}}{f_1^6f_4^{17}}$$
$$\equiv \frac{f_2^{19}}{f_1^2f_4^5f_8^2} + 16qf_1^{24} + 16qf_1^{24} - 16qf_1^{24}$$
$$\equiv \frac{f_2^{19}}{f_1^2f_4^5f_8^2} + 16qf_{16}f_8 \pmod{32}.$$

Finally, employing (3.2) and extracting the terms involving q^{2n},

$$\sum_{n\geq 0}\overline{bt}(8n)q^n \equiv \frac{f_1^{14}f_4^3}{f_2^5f_8^2} \pmod{32}. \tag{3.9}$$

From (3.8) and (3.9),

$$\overline{bt}(4n) \equiv \overline{bt}(8n) \pmod{32}. \tag{3.10}$$

Combining (1.5), (1.7), (1.8) and (3.10) completes the proof. □

4. Proof of Theorem 1.10

We first recall some fundamental ideas from the theory of modular forms. Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi iz}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an *eta-quotient* if it can be expressed as a finite product of the form

$$f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta},$$

where N is a positive integer and each r_δ is an integer.

Define

$$G(\tau) := \frac{\eta(\delta_1\tau)^{r_1}\eta(\gamma_1\tau)^{r_2}\cdots\eta(\gamma_u\tau)^{r_u}}{\eta(\gamma_1\tau)^{s_1}\eta(\gamma_2\tau)^{s_2}\cdots\eta(\gamma_t\tau)^{s_t}}, \tag{4.1}$$

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where r_i, s_i, δ_i and γ_i are positive integers with $\delta_1, \ldots, \delta_u, \gamma_1, \ldots, \gamma_t$ distinct and $u, t \geq 0$. The weight of $G(\tau)$ is given by

$$\frac{1}{2}\Big(\sum_{i=1}^{u} r_i - \sum_{i=1}^{t} s_i\Big).$$

Also define $\mathcal{D}_G := \gcd(\delta_1, \delta_2, \ldots, \delta_u)$. The following result by Cotron *et al.* will be useful in proving Theorem 1.10.

THEOREM 4.1 [6, Theorem 1.1]. *Suppose $G(\tau)$ is an eta-quotient of the form (4.1) with integer weight. If p is a prime such that p^a divides $\mathcal{D}_G$ and*

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \gamma_i s_i}{\sum_{i=1}^{u} r_i/\delta_i}},$$

then $G(\tau)$ is lacunary modulo p^j for any positive integer j. Moreover, there exists a positive constant α, depending on p and j, such that the number of integers $n \leq X$ with p^j not dividing $b(n)$ is $O(X/\log^{\alpha} X)$.

PROOF OF THEOREM 1.10. From (1.11),

$$\sum_{n\geq 0} \overline{b}_\ell(n) q^n = \frac{f_4^\ell}{f_1^{2\ell} f_2^\ell} = \frac{\eta^\ell(4z)}{\eta^{2\ell}(z)\eta^\ell(2z)}.$$

Following the notation used in (4.1) and the paragraph following it,

$$\delta_1 = 4, \quad r_1 = \ell, \quad \gamma_1 = 1, \quad s_1 = 2\ell, \quad \gamma_2 = 2 \quad \text{and} \quad s_2 = \ell.$$

Also $\mathcal{D}_G = 4$ and the weight is $-\ell \in \mathbb{Z}$. Next, we see that $2^2 \mid 4$ and

$$2^2 \geq \sqrt{\frac{2\ell + 2\ell}{\ell/4}} = \sqrt{16} = 4.$$

Choosing $p = 2$ and $a = 2$ in Theorem 4.1 completes the proof. □

5. Sketch of the proof of Theorem 1.9

From the work of Sellers [16, Corollary 2.3],

$$\sum_{n\geq 0} \overline{b}_t(n) q^n = \Big(\varphi(q) \prod_{i\geq 1} \varphi(q^{2^i})^{3\cdot 2^{i-1}}\Big)^t,$$

where

$$\varphi(q) := 1 + 2\sum_{n\geq 0} q^{n^2} = \varphi(q^4) + 2q\psi(q^8),$$

with $\psi(q) := \sum_{n\geq 0} q^{n(n+1)/2}$. Since $\big(\prod_{i\geq 3} \varphi(q^{2^i})\big)^{3\cdot 2^{i-1}\cdot t}$ is a function of q^8, it is enough to do the 8-dissection of the first three terms. Also, as the highest modulus involved

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in the theorem is 16 and the other moduli are divisors of 16, we will prove our result modulo 16 if we consider only modulo 16 in this 8-dissection. Re-write

$$\sum_{n\geq 0}\overline{b}_t(n)q^n=\Big(\sum_{j=0}^{7}a_{t,j}q^jF_{t,j}(q^8)\Big)\Big(\prod_{i\geq 3}\varphi(q^{2^i})\Big)^{3\cdot 2^{i-1}\cdot t},$$

where $F_{t,j}(q^8)$ is a function of q^8 whose power series representation has integer coefficients. It suffices to prove the congruences

$$\begin{aligned}
a_{t,1}&\equiv 0 \pmod 2,\\
a_{t,2}&\equiv 0 \pmod 2,\\
a_{t,3}&\equiv 0 \pmod 4,\\
a_{t,4}&\equiv 0 \pmod 2,\\
a_{t,5}&\equiv 0 \pmod 8,\\
a_{t,6}&\equiv 0 \pmod 4,\\
a_{t,7}&\equiv 0 \pmod{16}.
\end{aligned}$$

We can do this by induction following exactly the pattern of the proofs of [17, Theorem 2.2] and [15, Lemma 6.1], so we omit the details here. The Mathematica file is available at https://manjilsaikia.in/publ/mathematica/overcubictuples.nb.

6. Concluding remarks

Overpartition k-tuples with odd parts are close functions to the $\overline{b}_k(n)$ function. We denote by $\overline{OPT}_k(n)$ the number of overpartition k-tuples of n with odd parts. The generating function is given by

$$\sum_{n\geq 0}\overline{OPT}_k(n)=\frac{f_2^{3k}}{f_1^{2k}f_4^k}.$$

In [7, 15], the authors and their collaborators studied arithmetic properties of this function. It seems that some of the techniques used in these papers translate directly for the $\overline{b}_k(n)$ function. In particular, in [7], there are congruences modulo powers of 2 and 3 for the $\overline{OPT}_{2k+1}(n)$ function (with $k\geq 1$), which we believe may lead to similar results for the $\overline{b}_k(n)$ function. Additionally, in [15], there are infinite families of congruences modulo small powers of 2 for $\overline{OPT}_{2k+1}(n)$ (with $k\geq 1$) and $\overline{OPT}_4(n)$. We believe similar results hold for $\overline{b}_k(n)$.

We have just scratched the surface for congruences modulo powers of 2. A systematic study will unearth several more. For instance, we believe that Theorem 1.9 can be strengthened by taking appropriate values of k. The techniques that we have used in this paper seem unsuitable to find more general results.

An elementary proof of Theorem 1.1 is not difficult to obtain. We took the algorithmic approach for the sake of brevity. The modulo 4 congruences are not difficult to see and, in fact, follow from elementary dissection formulae. However, the

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modulo 32 and 64 congruences need more work. An elementary proof of Theorem 1.3 would be even more involved and would probably give some hints on how to prove Conjecture 1.6.

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