Ergod. Th. & Dynam. Sys. (2020), **40**, 3296–3309 © *Cambridge University Press, 2019*
doi:10.1017/etds.2019.40

Refined scales of weak-mixing dynamical systems: typical behaviour

SILAS L. CARVALHO † and CÉSAR R. DE OLIVEIRA ‡

† *Departamento de Matemática, UFMG, Belo Horizonte, MG, 30161-970 Brazil*

‡ *Departamento de Matemática, UFSCar, São Carlos, SP, 13560-970 Brazil*

(*Received* 6 *December* 2017 *and accepted in revised form* 15 *May* 2019)

Abstract. We study sets of measure-preserving transformations on Lebesgue spaces with continuous measures taking into account extreme scales of variations of weak mixing. It is shown that the generic dynamical behaviour depends on subsequences of time going to infinity. We also present corresponding generic sets of (probability) invariant measures with respect to topological shifts over finite alphabets and Axiom A diffeomorphisms over topologically mixing basic sets.

Key words: scales of mixing, Koopman operator, Halmos lemma, Rohlin lemma

2010 Mathematics Subject Classification: 37A25 (Primary); 28D05, 54E52 (Secondary)

1. *Introduction and results*

Let G be the complete topological group of automorphisms (i.e., measure-preserving and invertible transformations) of a Lebesgue space $(X, \mathcal{A}, m)$ with continuous measure m (i.e., a non-atomic probability space obtained at the completion of a Borel measure on a complete separable metric space) endowed with the weak topology ($T_j \to T$ weakly in G if, for each $A \in \mathcal{A}$, $m(T_j A \,\Delta\, T A) \to 0$) [**8**]. From technical and historical perspectives, the properties of ergodicity, weak mixing and strong mixing play a key role in the convergence to equilibrium of some classes of dynamical systems. An important answer to the question of prevalence of any of such properties was first given by Halmos [**7**, **8**] (see [**1**] for an alternative proof), who has shown that the subset of G whose elements are weak mixing contains a dense G_δ set in G, which is usually called generic in the Baire sense, and complemented by Rohlin [**15**], who has shown that the class of strong-mixing transformations is meagre.

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

One important characterization of weak mixing is in terms of the induced unitary operator (also known as the Koopman operator) $U_T : \mathrm{L}^2_m(X) \to \mathrm{L}^2_m(X)$, $(U_T f)(x) := f(Tx)$: the automorphism T is weak mixing if, and only if,

$$\lim_{L\to\infty} \frac{1}{L} \sum_{k=0}^{L-1} |(f, U_T^k f)|^2 = 0 \quad \text{for all } f \in [1]^\perp, \tag{1.1}$$

where $[1]^\perp$ is the orthogonal complement of the constant functions in $\mathrm{L}^2_m(X)$ and $(f, g) = \int_X \bar{f}\, g\, dm$ denotes the inner product in this space (the dynamics of the space of constant functions is trivial since $U_T 1 = 1$; see [**17**] for details). Regarding the spectral properties of this operator, Knill [**11**] has shown that a generic subset of G has purely singular continuous spectrum on $[1]^\perp$.

Our goal in this work is to refine Halmos' result in a quantitative way; replace $1/L$ by $1/L^\alpha$, $0 < \alpha < 1$, on the left-hand side of (1.1), with special attention to both extreme alternatives, that is, α close to the lower bound 0 and also close to the upper bound 1. Another look at this is through the consideration of the rates of convergence

$$\frac{1}{L} \sum_{k=0}^{L-1} |(f, U_T^k f)|^2 \sim \frac{1}{L^{1-\alpha}}.$$

Thus, we introduce, for each $\alpha \in [0, 1]$, each $f \in [1]^\perp$ and each $T \in G$,

$$\begin{aligned} \mathcal{M}_\alpha(T;\, f) &:= \liminf_{L\to\infty} \frac{1}{L^\alpha} \sum_{k=0}^{L-1} |(f, U_T^k f)|^2, \\ \mathcal{M}^\alpha(T;\, f) &:= \limsup_{L\to\infty} \frac{1}{L^{1-\alpha}} \sum_{k=0}^{L-1} |(f, U_T^k f)|^2. \end{aligned} \tag{1.2}$$

Note the different powers of L in $\mathcal{M}_\alpha(T;\, f)$ and $\mathcal{M}^\alpha(T;\, f)$. This is because we are mainly interested in small values of α in both cases and, in this situation, $\mathcal{M}^\alpha(T;\, f) = \infty$ probes some kind of criticality of the weak-mixing property (1.1), whereas $\mathcal{M}_\alpha(T;\, f) < \infty$ probes a convergence rate of $L^{-1} \sum_{k=0}^{L} |(f, U_T^k f)|^2$ to zero (so, fast decaying of $|(f, U_T^k f)|^2$), both in terms of suitable subsequences of time L going to infinity. In this direction, we have the following result.

THEOREM 1.1. *There exists a generic set $\mathcal{G} \subset G$ of weak-mixing automorphisms such that, for each $T \in \mathcal{G}$, the set*

$$\{f \in [1]^\perp \mid \mathcal{M}_\alpha(T;\, f) = 0 \text{ and } \mathcal{M}^\alpha(T;\, f) = \infty, \forall 0 < \alpha < 1\} \tag{1.3}$$

is generic in $[1]^\perp$.

We remark that one can combine Theorem 1.1 with Rohlin's result mentioned above and conclude that the elements of the generic set $\mathcal{G}$ are not strong mixing.

In fact, we have a result which is stronger than (1.3): if we denote the set in (1.3) by $\mathcal{Z}^T$, it is possible to replace $\mathcal{M}_\alpha(T;\, f) = 0$, $f \in \mathcal{Z}^T$, by

$$\mathcal{M}_\alpha(T;\, f, g) := \liminf_{L\to\infty} \frac{1}{L^\alpha} \sum_{k=0}^{L-1} |(f, U_T^k g)|^2 = 0 \quad \text{for all } f \in \mathcal{Z}^T, \text{ for all } g \in [1]^\perp,$$

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

and $\mathcal{M}^{\alpha}(T;\, f) = \infty$, $f \in \mathcal{Z}^T$, by

$$\mathcal{M}^{\alpha}(T;\, f, g) := \limsup_{L\to\infty} \frac{1}{L^{1-\alpha}} \sum_{k=0}^{L-1} |(f, U_T^k g)|^2$$
$$= \infty \quad \text{for all } f \in \mathcal{Z}^T, \text{ for all } g \in \mathcal{H}_f^T \setminus \{0\},$$

where $\mathcal{H}_f^T := \overline{\{U_T^k f \mid k \in \mathbb{Z}\}}$, in conformity with one of the descriptions of weak mixing that is also physically attractive, i.e., $\mathcal{M}^0(T;\, f, g) = 0$ for all $f, g \in [1]^{\perp}$. The usual Halmos' proof that $\mathcal{M}^0(T;\, f)$ implies the stronger version $\mathcal{M}^0(T;\, f, g) = 0$ uses a characterization of weak mixing in terms of sets of density zero in $\mathbb{N}$ (see, for instance, [**17**, §1.7]); however, such characterization does not apply for $\alpha \neq 0$. We have opted to keep $\mathcal{M}_{\alpha}(T;\, f)$ and $\mathcal{M}^{\alpha}(T;\, f)$ in (1.3) and in the main body of the work to maintain our line of reasoning as clear as possible, and have indicated in the Appendix the main technical modifications which are necessary to carry out such generalization.

Remark 1.2. If $T_i : (X_i, \mathcal{A}_i, m_i) \to (X_i, \mathcal{A}_i, m_i)$, $i = 1, 2$, are automorphisms and $R : (X_1, \mathcal{A}_1, m_1) \to (X_2, \mathcal{A}_2, m_2)$ is a measure isomorphism (conjugation), that is, a bijective (mod 0) transformation with $R * m_1 = m_2$, then $T_2 = R \circ T_1 \circ R^{-1}$ and $U_{T_2} = U_R^{-1} \circ U_{T_1} \circ U_R$. It then follows that $\mathcal{M}_{\alpha}(T_2;\, f) = \mathcal{M}_{\alpha}(T_1;\, U_R f)$ and $\mathcal{M}^{\alpha}(T_2;\, f) = \mathcal{M}^{\alpha}(T_1;\, U_R f)$ and so the conclusions in Theorem 1.1 are invariant under measure isomorphisms.

Remark 1.3.

(1) If one replaces $(f, U_T^k f)$ by $(f, U_T^k f) - (1,\, f)(f, 1)$ in the definitions of $\mathcal{M}_{\alpha}(T;\, f)$ and $\mathcal{M}^{\alpha}(T;\, f)$, then the set in (1.3) may indicate a dense G_{δ} subset of $\mathrm{L}_m^2(X)$ (instead of just $[1]^{\perp}$).

(2) Note that if $\mathcal{M}_{\alpha}(T;\, f) = 0$, then $\mathcal{M}_{\alpha'}(T;\, f) = 0$ and if $\mathcal{M}^{\alpha}(T;\, f) = \infty$, then $\mathcal{M}^{\alpha'}(T;\, f) = \infty$, with both statements valid for all $0 \leq \alpha \leq \alpha' \leq 1$.

There are some results about the speed of convergence of the ergodic averages $(1/L)\sum_{n=0}^{L-1}(U_T^n f)(x)$ (say, $f \in [1]^{\perp}$); for ergodic systems, Birkhoff's theorem asserts that it converges to zero almost surely. By choosing specific functions, Krengel [**12**] has shown that such convergence can be arbitrarily slow (see also [**10**]) and del Junco and Rosenblatt [**6**] have shown that this phenomenon is generic.

Intuitively, from the spectral point of view, if $T \in \mathcal{G}$ and f belongs to (1.3), with α close to 0, then there exists a subsequence $L_{n_k} \to \infty$ such that the dynamics of f under T resembles a dynamics of an automorphism with absolutely continuous spectrum, which is, in particular, strong mixing, whereas for another subsequence $L_{n_j} \to \infty$, the dynamics of f under T resembles a dynamics of a periodic automorphism, whose spectrum is pure point.

We also have corresponding results about generic sets of probability invariant measures with respect to a topological shift over finite alphabets and, as a consequence, about Axiom A diffeomorphisms over topologically mixing basic sets.

Let $(A^{\mathbb{Z}}, S)$ be a topological shift over the finite alphabet $A = \{1, \ldots, n\}$, $n \geq 2$, and consider the space $M(A^{\mathbb{Z}})$ of shift-invariant Borel probability measures, endowed with

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

the weak*-topology. Let, for some $T \in \mathcal{G}$, $\mathcal{P} = (P_1, \ldots, P_l)$ be a (partition) generator of cardinality $\Delta(T, \mathcal{P}) = l$; such generator exists if T is ergodic (a condition automatically performed by every $T \in \mathcal{G}$) and satisfies $h(T) < \infty$, by Krieger's generator theorem (more precisely, $e^{h(T)} \leq \Delta(T, \mathcal{P}) \leq e^{h(T)+1}$; see [**13**] for details). One obtains a mapping $U : G \to X$ if one lets $(Ux)_i = k$, if $T^i x \in P_k$, $1 \leq k \leq l$, $i \in \mathbb{Z}$, for m-almost every $x \in X$.

Setting, for each Borel set $A \subset A^{\mathbb{Z}}$, $(U * m)(A) = m(U^{-1}A)$, one has defined a shift-invariant measure $\nu = U * \mu$ on $A^{\mathbb{Z}}$. Now, by the definition of a generator, (X, T, m) and $(A^{\mathbb{Z}}, S, \nu)$ are isomorphic in the sense of measure theory (see Remark 1.2).

Let, for each $p \geq 2$, $G_p := \{T \in G \mid \exists \mathcal{P}$ be such that $\Delta(T; \mathcal{P}) = p\}$ (a non-empty set, by the discussion above); it follows from Theorem 1.1 that, for each $T \in G_p \cap \mathcal{G}$ (which is a generic subset of G_p, a locally compact Hausdorff space), the set $\{f \in [1]^\perp \mid \mathcal{M}_\alpha(T; f) = 0$ and $\mathcal{M}^\alpha(T; f) = \infty, \forall 0 < \alpha < 1\}$ is a generic subset of $[1]^\perp$. Combining this with the measure isomorphism between (X, T, m) and $(A^{\mathbb{Z}}, S, \nu)$, and with Remark 1.2, one has the following result.

COROLLARY 1.4. *There exists a generic set $\mathcal{A} \subset M(A^{\mathbb{Z}})$ such that, for each $\mu \in M(A^{\mathbb{Z}})$, the set*

$$\{f \in [1_\mu]^\perp \mid \mathcal{M}_\alpha(S; f) = 0 \text{ and } \mathcal{M}^\alpha(S; f) = \infty, \ \forall\, 0 < \alpha < 1\}$$

is generic in $[1_\mu]^\perp := \{f \in \mathrm{L}^2_\mu(X) \mid \int f \, d\mu = 0\}$.

COROLLARY 1.5. *Let T be an Axiom A diffeomorphism and Ω an infinite basic set such that the restriction $T|_\Omega$ is topologically mixing. Then there exists a generic set $\mathcal{A} \subset M(\Omega)$ such that, for each $\mu \in \mathcal{A}$,*

$$\{f \in [1_\mu]^\perp \mid \mathcal{M}_\alpha(T; f) = 0 \text{ and } \mathcal{M}^\alpha(T; f) = \infty, \ \forall\, 0 < \alpha < 1\}$$

is $[1_\mu]^\perp$.

1.1. *Strategy of proof of Theorem 1.1.* First, we discuss the global picture of the proof. We need to show that some suitable sets are G_δ and dense. In order to obtain these G_δ sets, we need, for each fixed $f \in [1]^\perp$, the (semi-)continuity of the limit superior and limit inferior of the correlation function $C_f^T(L)$ with respect to $T \in G$ (or continuity of $f \mapsto C_f^T(L)$). We show that such limits are directly related to the (upper/lower) correlation dimensions of spectral measures, $D_2^\pm(\mu_f^T)$, and apply consecutively Theorem 1.6 to make conclusions about $\mathcal{M}_\alpha(T; f)$ and $\mathcal{M}^\alpha(T; f)$; these relations are also important since such dimensions take values in the interval $[0, 1]$ (this is always true for Borel finite measures on $\mathbb{R}$; see [**3**]). On the other hand, we use Rohlin's result on the density of periodic automorphisms [**8**, **15**] and Halmos' conjugacy lemma [**7**, **8**] to prove directly that, for each $\delta \in (0, 1)$, $\{T \in G \mid \mathcal{M}^\alpha(T; f) = \infty$ for all $\alpha < \delta\}$ and $\{T \in G \mid M_\alpha(T; f) = 0$ for all $\alpha < \delta\}$ are both dense in G for f in a dense G_δ subset $\mathcal{B}$ of $[1]^\perp$; finally, we let $\delta_n = 1/n$ and then take intersections over n to conclude the proof. In what follows, we present more details of this short overview of our strategy.

We consider first a fixed non-zero vector $f \in [1]^\perp$ and then explore the relation between decaying rates of

$$C_f^T(L) := \frac{1}{L} \sum_{k=0}^{L-1} |(f, U_T^k f)|^2 \tag{1.4}$$

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

and the correlation dimensions $D_2^{\pm}(\mu_f^T)$ of the spectral measures μ_f^T of the induced operator U_T associated with f. Such quantities are given (see [**3**, **4**] for details) by

$$D_2^-(\mu_f^T) := \liminf_{\varepsilon\to 0} \frac{\ln I_{\mu_f^T}(\varepsilon)}{\ln \varepsilon}, \quad D_2^+(\mu_f^T) := \limsup_{\varepsilon\to 0} \frac{\ln I_{\mu_f^T}(\varepsilon)}{\ln \varepsilon}, \tag{1.5}$$

where

$$I_{\mu_f^T}(\varepsilon) := \int_{[0,2\pi)} [\mu_f^T((x-\varepsilon, x+\varepsilon))]\, d\mu_f^T(x). \tag{1.6}$$

As a matter of fact, the actual spectral measure μ_f^T is the push-forward, with respect to the mapping $[0, 2\pi) \ni x \mapsto e^{ix}$ to the unit circle $\partial\mathbb{D}$, of the measure presented in equation (1.6); throughout the text, we refer to the measure in (1.6) as the spectral measure of U_T and f, with the usual notation μ_f^T (it is clear that the spectral measure and its pull-back have the same correlation dimensions for each $T \in G$ and each $f \in \mathrm{L}_m^2(X)$).

These correspondences between the lower and the upper rates of convergence of $C_f^T(L)$ and $D_2^{\pm}(\mu_f^T)$ were proved in [**2**] for continuous unitary groups generated by self-adjoint operators; their use in the determination of G_δ sets of self-adjoint operators with particular spectral properties was first proposed in [**4**]. In the present context, we obtain the following result.

THEOREM 1.6. *Let $T \in G$. Fix a non-zero $f \in [1]^\perp$ and $\alpha \in [0, 1)$. Then:*

(1) $\mathcal{M}_\alpha(T;\, f) = 0 \implies D_2^+(\mu_f^T) \geq 1 - \alpha$;

(2) $\mathcal{M}^\alpha(T;\, f) = \infty \implies D_2^-(\mu_f^T) \leq \alpha$;

(3) $D_2^+(\mu_f^T) \geq 1 - \alpha \implies \mathcal{M}_{\alpha'}(T;\, f) = 0$ *for all* $\alpha' \in (\alpha, 1]$;

(4) $D_2^-(\mu_f^T) \leq \alpha \implies \mathcal{M}^{\alpha'}(T;\, f) = \infty$ *for all* $\alpha' \in (\alpha, 1]$.

The proof of Theorem 1.6 is presented in §2. Note that, for technical reasons related to the involved limits, the conclusions in items (3) and (4) are restricted to $\alpha' > \alpha$.

As already mentioned, in §3 we use Rohlin's result on the density of periodic automorphisms and Halmos' conjugacy lemma to get suitable dense subsets of $\mathcal{M}^\alpha$ and $\mathcal{M}_\alpha$. Next we show that for each $T \in G$, $\{f \in [1]^\perp \mid D_2^-(\mu_f^T) = 0 \text{ and } D_2^+(\mu_f^T) = 1\}$ is a G_δ set (Proposition 4.1); we then combine the previous density results for $\mathcal{M}^\alpha$ and $\mathcal{M}_\alpha$ with Theorem 1.6 (items (1) and (2)) to prove that there exists a G_δ set $\mathcal{G} \subset G$ of weak-mixing automorphisms such that, for each $T \in \mathcal{G}$, $\{f \in [1]^\perp \mid D_2^-(\mu_f^T) = 0$ and $D_2^+(\mu_f^T) = 1\}$ is generic in $[1]^\perp$; this result is presented in §4 (Theorem 4.3), which is finally combined with Theorem 1.6 (items (3) and (4)) in order to conclude the proof of Theorem 1.1.

Actually, the fact that $\{f \in [1]^\perp \mid D_2^-(\mu_f^T) = 0$ and $D_2^+(\mu_f^T) = 1\}$ is a generic set in $[1]^\perp$ for each $T \in \mathcal{G}$ is interesting in its own right and can be seen as a refinement of [**11**, Theorem 1.2]. We also emphasize that Theorem 1.6 is another compelling result in its own right, since it is one of the few results that connect dynamical properties of (X, T, m) with spectral properties of T; it refines, in some sense, the statement that (X, T, m) is weak mixing if, and only if, T has continuous spectrum (see [**17**, Theorem 1.26]). Since we cannot access information about $D_2^{\pm}(\mu_f^T)$ directly from f and T (we do not have explicit laws for these spectral measures), the proof that, for each

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

$T \in G$, the set $\{f \in [1]^{\perp} \mid D_2^-(\mu_f^T) < \alpha \text{ and } D_2^+(\mu_f^T) > 1-\alpha, \forall \alpha \in (0,1)\}$ is dense comes from its dynamical counterpart (that is, $\{f \in [1]^{\perp} \mid \mathcal{M}^{\alpha}(T; f) = \infty$ and $\mathcal{M}_{\alpha}(T; f) = 0, \forall \alpha \in (0,1)\}$).

2. *Correlation dimensions and scales of weak mixing*

The relation between correlation dimensions and the long-time behaviour of dynamics is presented in Proposition 2.2 below, which is an adapted version of [**2**, Theorem 2.2] (see also [**9**] for an alternative proof of this relation in [**2**]) to our setting; in fact, all the results in this section hold true in any probability space. For the proof of Proposition 2.2, we need the following lemma.

LEMMA 2.1. *Let $T \in G$. Given a non-zero $f \in \mathrm{L}_m^2(X)$, for each $L \in \mathbb{N}$, one has*

$$A_f^T(1/L) := \sum_{k\geq 0} e^{-2k/L} |(f, U_T^k f)|^2 = \frac{e^{2/L}}{2\pi} \int_0^{2\pi} |(f, (U_T - e^{1/L+i\theta})^{-1} f)|^2 d\theta$$
$$= \int_{[0,2\pi)^2} \frac{1 - e^{-2/L}\cos(x-y)}{1 - 2e^{-2/L}\cos(x-y) + e^{-4/L}} \, d\mu_f^T(x)\, d\mu_f^T(y).$$

Proof. The first equality is [**5**, Lemma 3.16]. Since, for each $L \in \mathbb{N}$,

$$|(f, (U_T - e^{1/L+i\theta})^{-1} f)|^2 = \int_{[0,2\pi)^2} \frac{1}{e^{ix} - e^{1/L+i\theta}} \cdot \frac{1}{e^{-iy} - e^{1/L-i\theta}} \, d\mu_f^T(x)\, d\mu_f^T(y)$$

(by the change of variables formula; recall that μ_f^T is the pull-back, with respect to the mapping $[0, 2\pi) \ni x \mapsto e^{ix} \in \partial\mathbb{D}$, of the spectral measure of U_T associated with f), one has

$$\frac{e^{2/L}}{2\pi} \int_0^{2\pi} |(f, (U_T - e^{1/L+i\theta})^{-1} f)|^2 \, d\theta$$
$$= \int_{[0,2\pi)^2} \left(-\frac{1}{2\pi i b} \int_{\gamma} \frac{dz}{(z-a)(z-b^{-1})} \right) d\mu_f^T(x)\, d\mu_f^T(y),$$

where $\gamma = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$, $a = e^{ix-1/L}$, $b = e^{-iy-1/L}$ and $z = e^{i\theta}$. In fact,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{a - e^{i\theta}} \cdot \frac{1}{b - e^{-i\theta}} d\theta = -\frac{1}{2\pi i b} \int_{\gamma} \frac{dz}{(z-a)(z-b^{-1})} = \frac{1}{1-ab},$$

so

$$A_f^T(1/L) = \int_{[0,2\pi)^2} \frac{1}{1 - e^{i(x-y)-2/L}} \, d\mu_f^T(x) d\mu_f^T(y)$$
$$= \int_{[0,2\pi)^2} \frac{1 - e^{-2/L}\cos(x-y)}{1 - 2e^{-2/L}\cos(x-y) + e^{-4/L}} \, d\mu_f^T(x) d\mu_f^T(y),$$

given that $A_f^T(1/L)$ is a real number. □

PROPOSITION 2.2. *Let $T \in G$. For each non-zero $f \in \mathrm{L}_m^2(X)$,*

$$\liminf_{L\to\infty} \frac{\ln C_f^T(L)}{\ln L} = -D_2^+(\mu_f^T), \quad \limsup_{L\to\infty} \frac{\ln C_f^T(L)}{\ln L} = -D_2^-(\mu_f^T),$$

where $C_f^T(L)$ and $D_2^{\pm}(\mu_f^T)$ are given by (1.4) and (1.5), respectively.

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

Proof. We just present the proof of the first identity. Let $\nu > 0$. Then

$$A_f^T(1/(2L)) = \sum_{k=0}^{[L^{1+\nu}]-1} e^{-k/L}|(f, U_T^k f)|^2 + \sum_{k\geq[L^{1+\nu}]} e^{-k/L}|(f, U_T^k f)|^2$$
$$\leq \sum_{k=0}^{[L^{1+\nu}]-1} |(f, U_T^k f)|^2 + \frac{e^{-[L^{1+\nu}]/L}}{1-e^{-1/L}}\|f\|^4,$$

where $[x]$ stands for the integer part of $x \in \mathbb{R}$. Since

$$\lim_{L\to\infty} \frac{e^{-[L^{1+\nu}]/L}}{[L^{1+\nu}](1-e^{-1/L})} = 0,$$

there exists an $L_1 = L_1(\nu) \in \mathbb{N}$ such that, for each $L > L_1$, $A_f^T(1/(2L)) < 2L^{1+\nu}C_f^T([L^{1+\nu}])$ (if not, there would exist (L_n), $L_n \to \infty$, such that

$$\frac{e^{-[L_n^{1+\nu}]/L_n}\|f\|^4}{(1-e^{-1/L_n})} \geq 2L_n^{1+\nu}C_f^T([L_n^{1+\nu}])$$

and then $\liminf_{L\to\infty}(\ln C_f^T(L))/(\ln L) = -\infty$).

For each $L \in \mathbb{N}$ and each $x \in [0, 2\pi)$, let $y \in (x - 1/L, x + 1/L) \cap [0, 2\pi)$. Then there exists an $L_2 \in \mathbb{N}$ (which does not depend on x) such that, for each $L > L_2$,

$$\frac{1-e^{-2/L}\cos(x-y)}{1-2e^{-2/L}\cos(x-y)+e^{-4/L}} \geq \frac{1}{1-e^{-2/L}} \cdot \frac{(1-e^{-2/L})^2}{1-2e^{-2/L}\cos(1/L)+e^{-4/L}}$$
$$> \frac{1}{2(1-e^{-2/L})}.$$

Hence, for each $L > L_2$ and each fixed $x \in [0, 2\pi)$,

$$\int_{(x-1/L,x+1/L)} \frac{1-e^{-2/L}\cos(x-y)}{1-2e^{-2/L}\cos(x-y)+e^{-4/L}}\, d\mu_f^T(y) > \frac{\mu_f^T((x-1/L, x+1/L))}{2(1-e^{-2/L})},$$

from which it follows, together with Lemma 2.1, that

$$A_f^T(1/L) > \frac{1}{2(1-e^{-2/L})} I_{\mu_f^T}(1/L).$$

By combining the previous estimates, it follows that, for each $L > \max\{L_1, L_2\}$,

$$I_{\mu_f^T}(1/(2L)) < 4L^{1+\nu}(1-e^{-1/L})C_f^T([L^{1+\nu}])$$

and therefore

$$\liminf_{L\to\infty} \frac{\ln I_{\mu_f^T}(1/L)}{\ln L} \leq \nu + (1+\nu)\liminf_{L\to\infty} \frac{\ln C_f^T(L)}{\ln L}.$$

Since $\nu > 0$ is arbitrary, one gets

$$\liminf_{L\to\infty} \frac{\ln C_f^T(L)}{\ln L} \leq -D_2^+(\mu_f^T).$$

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

Before we proceed, recall that $0 \le D_2^+ \le 1$ (see [**3**]). Now, we follow the calculations presented in the proof of [**11**, Theorem 3.6] and get, for each $\nu > 0$ and each $L \in \mathbb{N}$,

$$\begin{aligned} C_f^T(L) &\le e\sqrt{\pi} \int_{[0,2\pi)^2} e^{-(x-y)^2L^2/4}\, d\mu_f^T(x) d\mu_f^T(y) \\ &= e\sqrt{\pi}\left(\int_{[0,2\pi)} d\mu_f^T(x) \int_{y\in(x-L^{-(1-\nu)},x+L^{-(1-\nu)})} e^{-(x-y)^2L^2/4}\, d\mu_f^T(y)\right) \\ &\quad + e\sqrt{\pi}\left(\int_{[0,2\pi)} d\mu_f^T(x) \int_{y\notin(x-L^{-(1-\nu)},x+L^{-(1-\nu)})} e^{-(x-y)^2L^2/4}\, d\mu_f^T(y)\right) \\ &\le e\sqrt{\pi}(I_{\mu_f^T}(2L^{-(1-\nu)}) + \|f\|^2\, e^{-L^{2\nu}/4}); \end{aligned}$$

taking L sufficiently large (so that $\|f\|^2\, e^{-L^{2\nu}/4} \le I_{\mu_f^T}(2L^{-(1-\nu)})$; if not, $D_2^+(\mu_f^T) = +\infty$, which is an absurd), one gets

$$C_f^T(L) \le 2e\sqrt{\pi}\, I_{\mu_f^T}(2L^{-(1-\nu)}),$$

from which it follows that

$$\liminf_{L\to\infty} \frac{\ln C_f^T(L)}{\ln L} \ge -D_2^+(\mu_f^T). \qquad \square$$

Proof. (Theorem 1.6) The result is a direct consequence of equation (1.2) and Proposition 2.2. $\square$

3. *Density*

Recall that a transformation $T \in G$ is antiperiodic [**8**, p. 70] if it is almost nowhere periodic, i.e., the set of periodic points of T has m-measure zero.

LEMMA 3.1. *Let $T \in G$ and fix $\alpha \in (0, 1]$. Then:*

(a) *if $A^T := \{h \in [1]^\perp \mid \mathcal{M}_\alpha(T; h) = 0\}$ is dense, then T is antiperiodic;*

(b) *if T is periodic, then $\mathcal{M}^\alpha(T; h) = \infty$ for all non-zero $h \in [1]^\perp$.*

Proof. (a) Suppose that T is not antiperiodic; thus, there exist $k_0 \in \mathbb{N}$ and $B \in \mathcal{A}$ with $0 < m(B) < 1$ such that, for each $x \in B$, $T^{k_0}x = x$. Let $h_B(x) = N(\chi_B(x) - m(B))$ with $N = (m(B) - m(B)^2)^{-1/2}$, so that $(h_B, h_B) = 1$ and $h_B \in [1]^\perp$.

Let $f \in [1]^\perp$ be such that $\|f - h_B\| < 1/(40k_0)$ and, for each $L > k_0$, write $L = nk_0 + r$ with $0 \le r < k_0$, so that $L \to \infty$ if, and only if, $n \to \infty$. Then

$$\begin{aligned} \frac{1}{L^\alpha} \sum_{k=0}^{nk_0+r-1} |(f, U_T^k f)|^2 &\ge \frac{1}{L^\alpha} \sum_{k=0}^{nk_0+r-1} |(h_B, U_T^k h_B)|^2 - \frac{L^{1-\alpha}}{4k_0} \\ &= \frac{n^{1-\alpha}}{(k_0 + r/n)^\alpha}\left(1 - \frac{k_0 + r/n}{4k_0}\right) > \frac{n^{1-\alpha}}{2^{1+\alpha}k_0^\alpha} \end{aligned}$$

and so $\mathcal{M}_\alpha(T; f) > 0$ (note that $\mathcal{M}_\alpha(T; f) = \infty$ if $\alpha < 1$). Hence, A^T is not dense in $[1]^\perp$.

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

(b) Since T is periodic, there is $k_0 \in \mathbb{N}$ so that for m-almost every $x \in X$, $T^{k_0}x = x$ (which results in $U_{T^{k_0}}h = h$ for all $h \in \mathrm{L}^2_m(X)$). Then, using the same reasoning as in part (a), one gets

$$\frac{1}{L^{1-\alpha}} \sum_{k=0}^{nk_0+r-1} |(h, U_T^k h)|^2 \geq \frac{n}{k_0^{1-\alpha}\,(n+1)^{1-\alpha}} \|h\|^2$$

and so $\mathcal{M}^\alpha(T; h) = \infty$. □

PROPOSITION 3.2. *Fix $\alpha \in (0, 1]$. Then:*

(1) $\{T \in G \mid \mathcal{M}^\alpha(T; h) = \infty, \forall\, 0 \neq h \in [1]^\perp\}$ *is dense in G;*

(2) *there exists a dense G_δ set $\mathcal{B} \subset [1]^\perp$, independent of α, such that, for each $h \in \mathcal{B}$,*

$$\{T \in G \mid \mathcal{M}_\alpha(T; h) = 0\}$$

is dense in G.

Proof. (1) The result is a direct consequence of Rohlin's lemma [**8**, p. 75], which states that the set of periodic automorphisms is dense in G (in fact, this holds in the uniform topology on G as well), and Lemma 3.1(b).

(2) Let, for each $T \in G$,

$$B^T := \left\{ h \in [1]^\perp \mid \sup_L \sum_{k=0}^{L-1} |(h, U_T^k h)|^2 < \infty \right\}$$

and let A^T be as in the statement of Lemma 3.1(a). Since $\mathcal{M}_\alpha(T; h) \leq \mathcal{M}^{1-\alpha}(T; h) = 0$ for all $h \in B^T$ and all $\alpha \in (0, 1]$, it follows that $B^T \subset A^T$.

By the second remark after [**14**, Theorem 5.3], B^T is dense in $\{f \in [1]^\perp \mid \mu_f^{U_T}$ and is absolutely continuous$\}$. Pick $R \in G$ so that U_R has purely absolutely continuous spectrum on $[1]^\perp$ (let R be, for instance, a K-automorphism [**16**]); it then follows that B^R is dense in $[1]^\perp$, so A^R is dense in $[1]^\perp$ as well.

Claim. For each $S \in G$, the set $A^{S^{-1}RS}$ is a dense G_δ in $[1]^\perp$.

Fix $S \in G$ and $f \in A^{S^{-1}RS}$. Since $U_{S^{-1}}^{-1} = U_S$, one has $A^{S^{-1}RS} = U_S^{-1}(A^R)$; in fact,

$$(f, (U_{S^{-1}} U_R U_S)^k f) = (f, U_{S^{-1}} U_R^k\, U_S f) = (U_S f, U_R^k\, U_S f) = (h, U_R^k\, h)$$

with $h = U_S f \in A^R$ and therefore $\mathcal{M}_\alpha(S^{-1}RS;\ f) = 0$ if, and only if, $\mathcal{M}_\alpha(R; h) = 0$. The density of $A^{S^{-1}RS}$ follows from the density of A^R and from the fact that U_S^{-1} is a unitary operator with $U_S^{-1}[1]^\perp = [1]^\perp$.

Now, given that the mapping $[1]^\perp \ni f \mapsto \inf_L (1/L^{1-\beta}) \sum_{k=0}^{L-1} |(f, U_T^k f)|^2$ is upper semi-continuous (with the norm topology in $\mathrm{L}^2_m(X)$) for each $\beta \in [0, 1]$ and each $T \in G$, it follows that $A^{S^{-1}RS}$ is a G_δ set in $[1]^\perp$. This concludes the verification of the Claim.

Let $\Lambda := \{S^{-1}RS \mid S \in G\}$ be the conjugation class of R; by the Claim and Lemma 3.1(a), each element of Λ is antiperiodic. It then follows from Halmos' conjugacy lemma [**8**, p. 77] that Λ is dense.

Take any dense countable collection in Λ, say $\{S_j^{-1}RS_j\}$ (such collection exists since G is separable), and define $\mathcal{B} := \bigcap_j A^{S_j^{-1}RS_j}$; since, by the Claim, each $A^{S_j^{-1}RS_j}$ is a dense G_δ subset of $[1]^\perp$, it follows that $\mathcal{B}$ satisfies the properties required in the statement of the proposition. □

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

4. *Generic sets*

PROPOSITION 4.1. *Fix* $\alpha \in (0, 1)$. *Then:*

(a) *for each* $0 \neq f \in [1]^{\perp}$, *both of the sets*

$$C_f^-(\alpha) := \{T \in G \mid D_2^-(\mu_f^T) \leq \alpha\}, \quad C_f^+(\alpha) := \{T \in G \mid D_2^+(\mu_f^T) \geq 1 - \alpha\}$$

are G_δ *in* G;

(b) *for each* $T \in G$, *both of the sets*

$$E_T^-(\alpha) := \{f \in [1]^{\perp} \mid D_2^-(\mu_f^T) \leq \alpha\}, \quad E_T^+(\alpha) := \{f \in [1]^{\perp} \mid D_2^+(\mu_f^T) \geq 1 - \alpha\}$$

are G_δ *in* $[1]^{\perp}$.

Proof. Since the proofs of both items are similar, we will only discuss item (a). By combining adapted versions of [**4**, Lemmas 3.3 and 4.6], one has

$$C_f^-(\alpha) = \bigcap_{l=l_0}^{\infty} A_{\alpha+1/l}^f, \quad C_f^+(\alpha) = \bigcap_{k=k_0}^{\infty} B_{1-\alpha-1/k}^f \tag{4.1}$$

with $l_0 := [3/(\min\{\alpha, 1-\alpha\})] + 1$, $k_0 := [3/(1-\alpha)] + 1$, where, for each $\beta > 0$,

$$A_\beta^f := \bigcap_{n=1}^{\infty} \{T \in G \mid \text{ for each } m, \exists L > m \text{ with } L^\beta C_f^T(L) > n\},$$

$$B_\beta^f := \bigcap_{n=1}^{\infty} \{T \in G \mid \text{ for each } m, \exists L > m \text{ with } L^\beta\, C_f^T(L) < 1/n\}.$$

Since weak convergence in G results in strong resolvent convergence for the induced operators (indeed, $U_{T_n}\chi_A \to U_T\chi_A$ for each $A \in \mathcal{A}$; this convergence is also valid for simple functions, which are dense in $\mathrm{L}_m^2(X)$), it follows that for each $L \in \mathbb{N}$, the mapping $G \ni T \mapsto C_f^T(L)$ is continuous. Therefore, one has for each $\alpha \in (0, 1)$, $n \geq 1$, $l \geq l_0$, that

$$\{T \in G \mid L^{\alpha+1/l} C_f^T(L) > n\}$$

is an open subset of G. The fact that $C_f^-(\alpha)$ is a G_δ is now a consequence of relation (4.1). The proof that $C_f^+(\alpha)$ is a G_δ, for each $\alpha \in (0, 1)$, is completely analogous. □

In what follows, $\mathcal{B}$ is the dense G_δ subset of $[1]^{\perp}$ presented in the statement of Proposition 3.2.

PROPOSITION 4.2. *Let* $f \in \mathcal{B}$. *Then* $C_f := \{T \in G \mid D_2^-(\mu_f^T) = 0$ *and* $D_2^+(\mu_f^T) = 1\}$ *is a dense* G_δ *subset of* G.

Proof. It follows from Theorem 1.6 (items (1) and (2)) and Proposition 3.2 that, for each $\alpha \in (0, 1)$, both sets $\{T \in G \mid D_2^+(\mu_f^T) \geq 1 - \alpha\}$, $\{T \in G \mid D_2^-(\mu_f^T) \leq \alpha\}$ are dense; therefore, by Proposition 4.1(a), $C_f^{\pm}(\alpha)$ are dense G_δ subsets of G.

Since

$$C_f = \bigcap_{m>1} (C_f^+(1/m) \cap C_f^-(1/m)),$$

the result follows. □

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

THEOREM 4.3. *There exists a dense G_δ set $\mathcal{G} \subset G$ of weak-mixing automorphisms such that, for each $T \in \mathcal{G}$, the set*

$$\{f \in [1]^\perp \mid D_2^-(\mu_f^T) = 0 \text{ and } D_2^+(\mu_f^T) = 1\}$$

is generic in $[1]^\perp$.

Proof. Let, for each $S \in G$, $E_S := \{f \in [1]^\perp \mid D_2^-(\mu_f^S) = 0 \text{ and } D_2^+(\mu_f^S) = 1\}$, let C_f be as in Proposition 4.2 and let $\mathcal{F}$ be a countable dense subset of $\mathcal{B}$ (such subset exists since $\mathcal{B} \subset [1]^\perp$ is separable); then it follows from Proposition 4.2 that $\mathcal{A} := \bigcap_{f\in\mathcal{F}} C_f$ is a dense G_δ subset of G.

Fix $T \in \mathcal{A}$. It follows from Proposition 4.1(b) and the equality

$$E_T = \bigcap_{m>1} (E_T^+(1/m) \cap E_T^-(1/m))$$

that E_T is a G_δ subset of $[1]^\perp$; it is, in fact, a dense G_δ subset of $[1]^\perp$, given that $\mathcal{F} \subset E_T$ for each $T \in \mathcal{A}$.

Let $\mathcal{G}^{wm} := \{T \in G \mid T \text{ be weak mixing}\}$; since $\mathcal{G}^{wm}$ is a dense G_δ set [**7**, **8**], just set $\mathcal{G} := \mathcal{A} \cap \mathcal{G}^{wm}$ and we are done. □

Proof. (Theorem 1.1) Let $\mathcal{G} := \cap_{f\in\mathcal{F}}\{T \in G \mid T$ be weak mixing, $D_2^-(\mu_f^T) = 0$ and $D_2^+(\mu_f^T) = 1\}$, where $\mathcal{F}$ is any countable dense subset of $\mathcal{B}$. Given $T \in \mathcal{G}$, it is direct from Theorem 1.6 (items (3) and (4)) and Theorem 4.3 that

$$\{f \in [1]^\perp \mid \mathcal{M}_\alpha(T;\, f) = 0 \text{ and } \mathcal{M}^\alpha(T;\, f) = \infty, \forall 0 < \alpha < 1\}$$

is generic in $[1]^\perp$. □

Acknowledgements. We thank the anonymous referee for invaluable suggestions regarding the statements of our main results and the organization of the manuscript. SLC thanks the partial support by FAPEMIG (Universal Project CEX-APQ-03398-17). CRdO thanks the partial support by CNPq (Projects 41004/2014-8 and 303503/2018-1).

A. *Appendix*

The main technical difficulties in carrying out the proofs from $\mathcal{M}_\alpha(T;\, f)$ and $\mathcal{M}^\alpha(T;\, f)$ to $\mathcal{M}_\alpha(T;\, f, g)$ and $\mathcal{M}^\alpha(T;\, f, g)$ is discussed in Proposition A.1. By employing this proposition, an interested reader will be able to fill up the gaps necessary to generalize our main results. We emphasize again that the usual strategy of Halmos (that is also employed in other contexts in which the concept of weak mixing is presented), of replacing the usual Cesàro convergence in the definition of weak mixing by convergence up to a subset of density zero, does not apply to our setting and so one needs different ideas in the proof of Proposition A.1.

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

PROPOSITION A.1. *Fix $T \in G$ and $\alpha \in [0, 1)$. Then:*

(1) *if $D_2^+(\mu_f^T) \geq 1 - \alpha$ for all non-zero $f \in [1]^\perp$, then*

$$\liminf_{L\to\infty} \frac{1}{L^{\alpha'}} \sum_{k=0}^{L-1} |(f, U_T^k g)|^2 = 0$$

for all $\alpha' \in (\alpha, 1]$, $0 \neq g \in [1]^\perp$;

(2) *if $D_2^-(\mu_f^T) \leq \alpha$ for all non-zero $f \in [1]^\perp$, then*

$$\limsup_{L\to\infty} (1/L^{1-\alpha'}) \sum_{k=0}^{L-1} |(f, U_T^k g)|^2 = \infty$$

for all $\alpha' \in (\alpha, 1]$, $0 \neq g \in \mathcal{H}_f^T$.

Proof. (1) Suppose that, for each non-zero $f \in [1]^\perp$, $D_2^+(\mu_f^T) \geq 1 - \alpha$. Since $(f, U_T^k g) = 0$ for all $k \in \mathbb{Z}$ if $g \notin \mathcal{H}_f^T$, we just need to consider the case where $0 \neq g \in \mathcal{H}_f^T$. Fix $0 < \delta \leq 1 - \alpha$ and let ν satisfy the equation $2\nu(1 - \alpha) = \delta$. Since, for each $L \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{L} \sum_{k=0}^{L-1} |(f, U_T^k g)|^2 &\leq e\sqrt{\pi} \int_{[0,2\pi)^2} |F_g(e^{ix})||F_g(e^{iy})| e^{-(x-y)^2 L^2/4}\, d\mu_f^T(x) d\mu_f^T(y) \\ &\leq e\sqrt{\pi}\|g\|^2 (I_{\mu_f^T}((\sqrt{2}L)^{-1+\nu}) + \|f\|^2\, e^{-L^{2\nu}/2})\end{aligned}$$

(see the proof of [**11**, Theorem 3.6]), one has, for sufficiently large L,

$$\frac{1}{L} \sum_{k=0}^{L-1} |(f, U_T^k g)|^2 \leq 2e\sqrt{\pi}\|g\|^2 I_{\mu_f^T}((\sqrt{2}L)^{-1+\nu}).$$

Now, it follows from equation (1.5) that there exists a sequence (L_n), $L_n \to \infty$, such that

$$I_{\mu_f^T}((\sqrt{2}L_n)^{-1+\nu}) < 2\, L_n^{-1+\alpha+\nu(1-\alpha)}.$$

Combining the previous results, one concludes that (possibly taking a subsequence of (L_n))

$$\frac{1}{L_n^{\alpha+\delta}} \sum_{k=0}^{L_n-1} |(f, U_T^k g)|^2 \leq 4e\sqrt{\pi}\|g\|^2 L_n^{-\delta/2},$$

from which it follows that

$$\liminf_{L\to\infty} \frac{1}{L^{\alpha+\delta}} \sum_{k=0}^{L-1} |(f, U_T^k g)|^2 = 0.$$

(2) Suppose that, for each non-zero $f \in [1]^\perp$, $D_2^-(\mu_f^T) \leq \alpha$. Let $0 \neq g \in \mathcal{H}_f^T$ and define, for each $L \in \mathbb{N}$,

$$A_{f,g}^T(1/L) = \sum_{k\geq 0} e^{-2k/L} |(f, U_T^k g)|^2.$$

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

Let B be an arbitrary Borel subset of $[0, 2\pi)$ and let $h \in \mathrm{L}^2_m(X)$,

$$h = (\chi_B \circ s^{-1})(U_T) f,$$

where $s : [0, 2\pi) \to \partial\mathbb{D}$, $s(x) = e^{ix}$ (such h exists since $L^2_m(X)$ and $L^2_{\mu_f^T}$ are isomorphic; note that $h = 1$ if $B = [0, 2\pi)$). Following the arguments presented in the proof of Lemma 2.1, one has, for each $L \in \mathbb{N}$ and each B,

$$\begin{aligned} A^T_{f,hg}(1/L) &= \int_{B^2} \frac{\Re(F_g(e^{ix})\overline{F_g(e^{iy})})(1 - e^{-2/L}\cos(x-y))}{1 - 2e^{-2/L}\cos(x-y) + e^{-4/L}}\, d\mu_f^T(x) d\mu_f^T(y) \\ &\quad - \int_{B^2} \frac{\Im(F_g(e^{ix})\overline{F_g(e^{iy})})\, e^{-2/L}\sin(x-y)}{1 - 2e^{-2/L}\cos(x-y) + e^{-4/L}}\, d\mu_f^T(x) d\mu_f^T(y) \\ &=: \int_{B^2} H(x, y, L)\, d\mu_f^T(x) d\mu_f^T(y). \end{aligned} \tag{A.1}$$

Since, for each $L \in \mathbb{N}$ and each B, $A^T_{f,hg}(1/L)$ is non-negative, one concludes that the mapping H defined in relation (A.1) is μ_f^T-almost everywhere non-negative in $[0, 2\pi)$. Thus, given arbitrary Borel subsets $B, C \subset [0, 2\pi)$ and $L \in \mathbb{N}$,

$$A^T_{f,g}(1/L) \geq \int_B \int_C H(x, y, L)\, d\mu_f^T(x) d\mu_f^T(y). \tag{A.2}$$

Now, since μ_f^T is a Radon measure, it follows from Lusin's theorem that there exist a positive constant D and a compact set of positive μ_f^T-measure, A, such that $(F_g \circ e^{i\cdot}) \restriction A$ is a (non-constant) uniformly continuous function and, for each $x \in A$, $|F_g(e^{ix})|^2 > D$. Therefore, it follows that for sufficiently large L, each $x \in A$ and each $y \in (x - 1/L, x + 1/L) \cap A$, $\Re(F_g(e^{ix})\overline{F_g(e^{iy})}) > |F_g(e^{ix})|^2/2 > D/2$ and $|\Im(F_g(e^{ix})\overline{F_g(e^{iy})})| < D/8$.

By letting in (A.2) $B = A$, $C = C(x) = (x - 1/L, x + 1/L) \cap A$ for each $x \in A$, and by the fact that for each sufficiently large L, each $x \in [0, 2\pi)$ and each $y \in (x - 1/L, x + 1/L)$,

$$\begin{aligned} \frac{1 - e^{-2/L}\cos(x-y)}{1 - 2e^{-2/L}\cos(x-y) + e^{-4/L}} &\geq \frac{1}{2(1 - e^{-2/L})}, \\ \frac{e^{-2/L}\sin(x-y)}{1 - 2e^{-2/L}\cos(x-y) + e^{-4/L}} &\leq \frac{3}{2(1 - e^{-2/L})}, \end{aligned}$$

one gets

$$A^T_{f,g}(1/L) \geq \frac{D}{16(1 - e^{-2/L})} I_{\mu_{\tilde{f}}^T}(1/L); \tag{A.3}$$

here, $\tilde{f}(x) := f(x)$ if $x \in A$, $f(x) = 0$ otherwise (note that A is chosen in order to have $\tilde{f} \in [1]^\perp$).

Fix $0 < \delta \leq 1 - \alpha$ and let $\nu = \delta/(2(1 - \alpha))$. It follows from the hypothesis applied to $\tilde{f}$ and Definition (1.5) that there exists a sequence (L_n), $L_n \to \infty$, such that

$$I_{\mu_{\tilde{f}}^T}((2L_n)^{-(1-\nu)}) > \frac{L_n^{-\alpha(1-\nu)}}{2}. \tag{A.4}$$

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press

Finally, it follows from the arguments presented in the first part of the proof of Proposition 2.2 and relations (A.3) and (A.4) that, for n sufficiently large,

$$\frac{1}{L_n}\sum_{k=0}^{L_n-1}|(f, U_T^k g)|^2 > \frac{A_{f,g}^T((2L_n)^{-1+\nu})}{2L_n} > \frac{D\, I_{\mu_{\bar{f}}^T}((2L_n)^{-1+\nu})}{32L_n(1-e^{-2/(2L_n)^{1-\nu}})}$$
$$> \frac{D\, L_n^{-1-\alpha+\alpha\nu}}{64(1-e^{-2/(2L_n)^{1-\nu}})} > C_1\, L_n^{-\alpha-(1-\alpha)\nu},$$

where C_1 is a positive constant; hence,

$$\limsup_{L\to\infty}\frac{1}{L^{1-\alpha-\delta}}\sum_{k=0}^{L-1}|(f, U_T^k g)|^2 = \infty. \qquad \square$$

References

[**1**] S. Alpern. New proofs that weak mixing is generic. *Invent. Math.* **32** (1976), 263–279.
[**2**] J.-M. Barbaroux, J.-M. Combes and R. Montcho. Remarks on the relation between quantum dynamics and fractal spectra. *J. Math. Anal. Appl.* **213** (1997), 698–722.
[**3**] J.-M. Barbaroux, F. Germinet and S. Tcheremchantsev. Generalized fractal dimensions: equivalence and basic properties. *J. Math. Pures Appl.* **80** (2001), 977–1012.
[**4**] S. L. Carvalho and C. R. de Oliveira. Correlation dimension wonderland theorems. *J. Math. Phys.* **57** (2016), 063501, 19 pp.
[**5**] D. Damanik, J. Fillman and R. Vance. Dynamics of unitary operators. *J. Fractal Geom.* **1** (2014), 391–425.
[**6**] A. del Junco and J. Rosenblatt. Counterexamples in ergodic theory and number theory. *Math. Ann.* **245** (1979), 185–197.
[**7**] P. Halmos. In general a measure preserving transformation is mixing. *Ann. of Math. (2)* **45** (1944), 786–792.
[**8**] P. Halmos. *Lectures on Ergodic Theory*. The Mathematical Society of Japan, Tokyo, 1956.
[**9**] M. Holschneider. Fractal wavelet dimensions and localization. *Comm. Math. Phys.* **160** (1994), 457–473.
[**10**] S. Kakutani and K. Petersen. The speed of convergence in the ergodic theorem. *Monatsh. Math.* **91** (1981), 11–18.
[**11**] O. Knill. Singular continuous spectrum and quantitative rates of weak mixing. *Discrete Contin. Dyn. Syst.* **4** (1998), 33–42.
[**12**] U. Krengel. On the speed of convergence in the ergodic theorem. *Monatsh. Math.* **86** (1978), 3–6.
[**13**] W. Krieger. On entropy and generators of measure-preserving transformations. *Trans. Amer. Math. Soc.* **149** (1970), 354–464. Erratum *Trans. Amer. Math. Soc.* **168** (1972), 519.
[**14**] Y. Last. Quantum dynamics and decomposition of singular continuous spectra. *J. Funct. Anal.* **142** (2001), 406–445.
[**15**] V. Rohlin. A 'general' measure-preserving transformation is not mixing. *Dokl. Akad. Nauk SSSR (N.S.)* **60** (1948), 349–351 (in Russian).
[**16**] V. Rohlin. Lectures on the entropy theory of transformations with invariant measure. *Russian Math. Surveys* **22** (1967), 1–52.
[**17**] P. Walters. *An Introduction to Ergodic Theory*. Springer, New York, 1982.

https://doi.org/10.1017/etds.2019.40 Published online by Cambridge University Press