Five mathematical gobbets

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Introduction

At university I was intrigued by a historian friend and his constant quest for gobbets – short, apposite quotations that he avidly collected for possible use in future essays. It strikes me that there might be analogous nuggets of mathematics and I offer the following five short, self-contained items both for amusement and for possible classroom use. Each contains a surprising element and I have deliberately left a few loose ends to encourage further exploration.

1. *An everyday encounter with the inequality of the means*

This gobbet aims to explain the empirical observation that, if you try to free a car with its wheels spinning fast enough when stuck in a muddy field, then you get splattered with mud on your front at a height above that of the wheels.

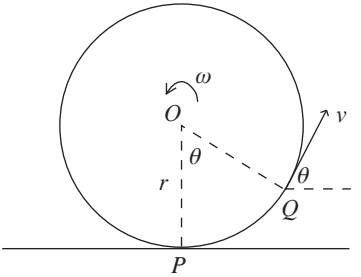


FIGURE 1: A wheel, centre O, with a particle of mud flying off at Q

As shown in Figure 1, suppose a particle of mud flies off the wheel tangentially at Q with (fixed) speed $v = r\omega$ where ω is the (fixed) angular velocity of the wheel and $\angle POQ = \theta$. From the usual projectile model, the greatest height reached by the mud above Q is $\dfrac{v^2 \sin^2\theta}{2g}$, so the greatest height reached above the ground at P is $h(\theta) = r - r\cos\theta + \dfrac{v^2\sin^2\theta}{2g}$. But

$$h(\theta) = r + \frac{v^2}{2g} - \frac{v^2}{2g}\left(\cos^2\theta + \frac{2gr}{v^2}\cos\theta\right) = r + \frac{v^2}{2g} + \frac{gr^2}{2v^2} - \frac{v^2}{2g}\left(\cos\theta + \frac{gr}{v^2}\right)^2.$$

So, for $0 \leqslant \theta \leqslant \pi$:

- If $v^2 \leqslant rg$, the (endpoint) maximum of $h(\theta)$ is attained when $\theta = \pi$ and $h(\pi) = 2r$;
- If $v^2 > rg$, the (stationary) maximum of $h(\theta)$ occurs when $\cos\theta = -\dfrac{gr}{v^2}$ and $h_{\max} = r + \dfrac{v^2}{2g} + \dfrac{gr^2}{2v^2}$.

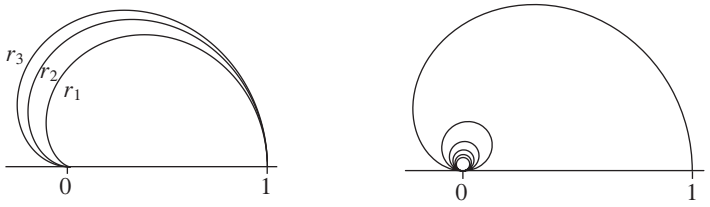
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In the latter case, the inequality of arithmetic and geometric means then guarantees that, because $\frac{v^2}{2g} \neq \frac{gr^2}{2v^2}$, $h_{\max} > r + \sqrt{\frac{v^2}{2g}\cdot\frac{gr^2}{2v^2}} = 2r$. So, if the wheel is spinning fast enough ($v^2 > rg$ or $\omega^2 > \frac{g}{r}$), the splattering mud reaches above the height of the wheels!

This problem, in different but equivalent contexts, can be found in several mechanics books such as [1] and has the distinction of having appeared as a question in entrance examinations for both Oxford [2] and Cambridge [3].

2. *An (unusual) graph is worth a thousand words*

This gobbet centres on the polar graphs depicted in Figure 2.



(a) The graphs $r_1 = \cos^2 \frac{1}{2}\theta$, $r_2 = \frac{\sin\theta}{\theta}$, $r_3 = \cos\frac{1}{2}\theta$ for $0 < \theta < \pi$

(b) The graph $r = \frac{\sin\theta}{\theta}$ for $0 < \theta < \infty$

FIGURE 2

In Figure 2(a), we see that $\cos^2\frac{\theta}{2} < \frac{\sin\theta}{\theta} < \cos\frac{\theta}{2}$ for $0 < \theta < \pi$ so that $\cos^2\frac{\theta}{2} < \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{\theta} < \cos\frac{\theta}{2}$ or $\sin\frac{1}{2}\theta < \frac{1}{2}\theta < \tan\frac{1}{2}\theta$, a manifestation of the inequalities usually met in connection with small-angle approximations.

Looking more closely, the radial gaps between the curves suggest that $r_2 - r_1 \approx 2(r_3 - r_2)$ or $3r_2 \approx r_1 + 2r_3$ leading to

$$\frac{6\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{\theta} \approx \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + 2\right) \text{ or } \frac{\theta}{2} \approx \frac{3\sin\frac{\theta}{2}}{2 + \cos\frac{\theta}{2}}.$$

This is a reflection of the (tight) Cusa-Huygens' inequality discussed in [4], where the Maclaurin expansion of the right-hand side starts $\frac{1}{2}\theta - \frac{\theta^5}{5760} + \ldots$. Notice that with $\theta = \frac{1}{2}\pi$, we have the approximation $\frac{\pi}{4} \approx \frac{3\sin\frac{1}{4}\pi}{2 + \cos\frac{1}{4}\pi}$ or $\pi \approx \frac{1}{7}(24\sqrt{2} - 12) \approx 3.134$. Using $\theta = \frac{1}{3}\pi$ gives $\pi \approx \frac{1}{13}(72 - 18\sqrt{3}) \approx 3.140$, but my favourite value to use is $\theta = \frac{1}{6}\pi$ and those exotic surd values on the calculator

$$\sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4} \text{ and } \cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4},$$

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from which

$$\pi \approx \frac{36(\sqrt{6}-\sqrt{2})}{8+\sqrt{6}+\sqrt{2}} \approx 3.14151.$$

Finally, we consider Figure 2(b) showing loops of the curve $r = \frac{\sin\theta}{\theta}$ corresponding to the intervals $0 < \theta \leqslant \pi$, $\pi < \theta \leqslant 2\pi$, $2\pi < \theta \leqslant 3\pi$, … with both positive and negative values of r plotted; that the first 'loop' is open reflects the fact that, as $\theta \to 0$, $\frac{\sin\theta}{\theta} \to 1$ rather than 0.

Zooming in, we see that the loops are increasingly circular because, for $2k\pi < \theta \leqslant (2k+1)\pi$, the loop of $r = \frac{\sin\theta}{\theta}$ is trapped between the circles $r = \frac{\sin\theta}{(2k+1)\pi}$ and $r = \frac{\sin\theta}{2k\pi}$ (with a similar argument for the loops with negative r when $(2k-1)\pi < \theta \leqslant 2k\pi$). The area of each loop is given by $\frac{1}{2}\int_{k\pi}^{(k+1)\pi} \frac{\sin^2\theta}{\theta^2} d\theta$, an integral with no closed form. But the total area of the loops, including the first one with $0 < \theta \leqslant \pi$, is (by parts, followed by the substitution $t = 2\theta$):

$$\int_0^\infty \frac{\sin^2\theta}{2\theta^2} d\theta = \left[-\frac{\sin^2\theta}{2\theta^2}\right]_0^\infty + \frac{1}{2}\int_0^\infty \frac{2\sin\theta\cos\theta}{\theta} d\theta = \frac{1}{2}\int_0^\infty \frac{\sin 2\theta}{\theta} d\theta = \frac{1}{2}\int_0^\infty \frac{\sin t}{t} dt,$$

a well-known integral with value $\frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$.

Since $\frac{1}{2}\int_0^\pi r_3^2\, d\theta = \frac{1}{2}\int_0^\pi \cos^2\frac{\theta}{2}\, d\theta = \frac{\pi}{4}$, we have the striking observation that the total area of the closed loops of $r_2 = \frac{\sin\theta}{\theta}$ (for $\pi \leqslant \theta < \infty$) exactly corresponds to the area between r_2 and r_3 in Figure 2(a).

It can seem weird to students that we can find the total area of the loops without knowing the individual areas, but this is actually the norm: most indefinite integrals cannot be done, but we might strike lucky with a definite integral if the limits are right. It was this subtlety that led to G. H. Hardy's famous opinion at the end of [5], 'This integral $\left[\int_0^\infty \frac{\sin x}{x} dx\right]$ (and all the other standard definite integrals) lie quite outside the legitimate range of school mathematics.'

3. *Reciprocal power series*

A mistake which is sometimes seen from students is exemplified by the following attempt to find the Maclaurin series for $\sec x$:

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots} = 1 - 2x^2 + 24x^4 - \ldots .$$

This provokes the natural question: do there exist non-zero coefficients a_0, a_1, a_2, … such that

$$\frac{1}{a_0 + a_1x + a_2x^2 + \ldots} = \frac{1}{a_0} + \frac{1}{a_1}x + \frac{1}{a_2}x^2 + \ldots .$$

We shall call this special property of a power series "Property R". The power series for $\sec x$, which starts $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots$, does not have Property R.

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Property R may be rewritten as

$$1 = \left(a_0 + a_1x + a_2x^2 + \ldots\right)\left(\frac{1}{a_0} + \frac{1}{a_1}x + \frac{1}{a_2}x^2 + \ldots\right).$$

Equating successive coefficients of x^n in turn:

$$(n = 1) \qquad \frac{a_0}{a_1} + \frac{a_1}{a_0} = 0, \text{ or } a_0^2 + a_1^2 = 0.$$

We see immediately that there are no examples with the coefficients a_0, a_1, a_2, … all real, but $a_0 = 1, a_1 = i$ gets us started.

$$(n = 2) \qquad \frac{a_0}{a_2} + 1 + \frac{a_2}{a_0} = 0$$

so, with $a_0 = 1$, we have $a_2^2 + a_2 + 1 = 0$ and so we may choose

$$a_2 = \frac{-1 + \sqrt{3}i}{2} = e^{\frac{2}{3}\pi i}.$$

$$(n = 3) \qquad \frac{a_0}{a_3} + \frac{a_1}{a_2} + \frac{a_2}{a_1} + \frac{a_3}{a_0} = 0$$

which, with a_0, a_1, a_2 as above, simplifies to $a_3^2 + \sqrt{3}a_3 + 1 = 0$. So we may choose $a_3 = \dfrac{-\sqrt{3} + i}{2} = e^{\frac{5}{6}\pi i}$.

Reviewing the values of a_1, a_2, a_3, we see that $a_n = \alpha^{n+2}$ with $\alpha = e^{\frac{1}{6}\pi i}$, so we may boldly conjecture that the series $f(x) = 1 + \sum_{n=1}^{\infty} \alpha^{n+2}x^n$ has Property R. Equating coefficients as above means we have to verify that $\dfrac{a_0}{a_n} + \displaystyle\sum_{k=1}^{n-1} \frac{a_k}{a_{n-k}} + \frac{a_n}{a_0} = 0$. But the left-hand side is

$$\alpha^{n+2} + \alpha^{-n-2} + \sum_{k=1}^{n-1} \alpha^{2k-n} = \alpha^{n+2} + \alpha^{-n-2} + \alpha^{2-n}.\frac{\alpha^{2n-2} - 1}{\alpha^2 - 1}$$

$$= \frac{(\alpha^n - \alpha^{-n-2})(\alpha^4 - \alpha^2 + 1)}{\alpha^2 - 1} = 0,$$

because $\alpha^4 - \alpha^2 + 1 = \dfrac{\alpha^6 + 1}{\alpha^2 + 1} = 0$ since $\alpha = e^{\frac{1}{6}\pi i}$.

A closed form for $f(x)$ is

$$f(x) = 1 + \sum_{n=1}^{\infty} a^{n+2}x^n = 1 + \frac{\alpha^3 x}{1 - \alpha x} = \frac{1 + (\alpha^3 - \alpha)x}{1 - \alpha x} = \frac{1 - \alpha^{-1}x}{1 - \alpha x},$$

which we can alternatively use to verify Property R directly. For, by the construction of $f(x)$, we know that $f(x) = \dfrac{1 - \alpha^{-1}x}{1 - \alpha x} = 1 + \displaystyle\sum_{n=1}^{\infty} a_n x^n$

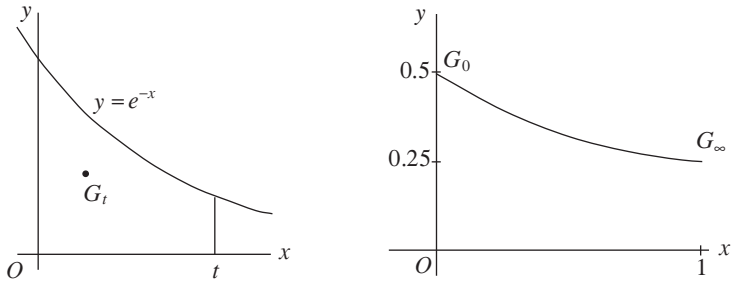
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where $a_n = \alpha^{n+2}$. Since $\dfrac{1}{f(x)} = \dfrac{1 - \alpha x}{1 - \alpha^{-1}x} = 1 + \sum_{n=1}^{\infty} b_n x^n$, we may replace α by α^{-1} to see that $b_n = \alpha^{-n-2} = \dfrac{1}{a_n}$ as required.

Obviously we have made several choices during this calculation: it would be interesting to know whether there are other functions with Property R which are not just non-zero scalar multiples of the Möbius transformation $\dfrac{1 - \alpha^{-1}x}{1 - \alpha x}$.

4. *The locus of a centre of mass*

Let G_t denote the centre of mass of the uniform lamina shown in Figure 3(a), bounded by the curve $y = e^{-x}$, the ordinates $x = 0$, $x = t$ and the x-axis.



(a) Showing G_t, the centre of mass of the lamina

(b) The locus of G_t for $0 \leqslant t < \infty$

FIGURE 3

Calculating the coordinates of $G_t(X, Y)$ in the usual way requires the values of the three integrals:

$$\int_0^t y\,dx = \int_0^t e^{-x}dx = 1 - e^{-t},$$

$$\int_0^t y^2\,dx = \int_0^t e^{-2x}dx = \frac{1}{2}\left(1 - e^{-2t}\right),$$

$$\int_0^t xy\,dx = \int_0^t xe^{-x}dx = \left[-xe^{-x} - e^{-x}\right]_0^t = 1 - e^{-t} - te^{-t},$$

giving

$$X = \frac{1 - e^{-t} - te^{-t}}{1 - e^{-t}} = 1 - \frac{te^{-t}}{1 - e^{-t}} \text{ and } Y = \frac{\frac{1}{4}\left(1 - e^{-2t}\right)}{1 - e^{-t}} = \tfrac{1}{4}\left(1 + e^{-t}\right).$$

In Figure 3(b), as t varies from 0 to ∞, the position of G_t varies from $G_0\left(0, \frac{1}{2}\right)$ to $G_\infty\left(1, \frac{1}{4}\right)$ and the curve joining these two points is clearly decreasing. The Cartesian equation linking X and Y is found by eliminating e^{-t}

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$$X = 1 + \frac{(4Y - 1)\ln(4Y - 1)}{1 - (4Y - 1)} = 1 + \frac{(4Y - 1)\ln(4Y - 1)}{2 - 4Y}.$$

So far, so standard. But what about the area, A, under the curve in Figure 3(b)? This is given by

$$\begin{aligned}
A &= \frac{1}{4} + \int_{\frac{1}{4}}^{\frac{1}{2}} \left(1 + \frac{(4Y - 1)\ln(4Y - 1)}{2 - 4Y}\right) dY \\
&= \frac{1}{2} + \frac{1}{4}\int_0^1 \frac{(1 - u)\ln(1 - u)}{u} du, \text{ on setting } u = 2 - 4Y \\
&= \frac{1}{2} - \frac{1}{4}\int_0^1 \ln(1 - u)\, du + \frac{1}{4}\int_0^1 \frac{\ln(1 - u)}{u} du \\
&= \frac{1}{2} + \frac{1}{4} - \frac{1}{4}\int_0^1 \frac{1}{u}\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \ldots\right) du, \text{ by expanding } \ln(1 - u) \\
&= \frac{3}{4} - \frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \\
&= \frac{3}{4} - \frac{\pi^2}{24}, \text{ using the solution of the Basel problem, } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\end{aligned}$$

An unexpectedly pretty result!

5. *Doubly improper integrals*

A natural question when teaching about improper integrals is whether there is a convergent integral where infinity intrudes both as a limit of integration and as a singularity. This was the theme of a recent interesting article [6], which inspired this gobbet. So much of teaching is about making decisions or resolving dilemmas in the heat of the moment. Which of the following would provide the best experience in the classroom? What do you want to revise or consolidate? What do you want to make fresh connections with? For simplicity, we will only consider continuous functions with domain $(0, \infty)$ and with a singularity located at $x = 0$.

(a) *Cobbling together an example*

In my experience, students are often wary of split definitions, but we can splice together functions with the required behaviour at 0 and ∞, for example

$$f(x) = \left\{ \begin{array}{ll} x^{-\frac{1}{2}}, & 0 < x < 1 \\ e^{1-x}, & 1 \leqslant x < \infty \end{array} \right\}$$

with $\int_a^A f(x)dx = \left[2\sqrt{x}\right]_a^1 + \left[-e^{1-x}\right]_1^A = 3 - 2\sqrt{a} - e^{1-A} \to 3$, as $a \to 0^+, A \to \infty$.

(b) *An example with built-in integration practice*

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Consider $f(x) = \dfrac{1}{\sqrt{x}(1+x)}$ with the substitution $x = t^2$:

$$\int_a^A f(x)dx = \int_{\sqrt{a}}^{\sqrt{A}} \frac{1}{t(1+t^2)} 2t\,dt = 2\left[\tan^{-1}\sqrt{A} - \tan^{-1}\sqrt{a}\right] \to \pi \text{ as } a \to 0^+,\ A \to \infty.$$

(c) *An example like (b), but with pedigree!*

The gamma function for arguments between 0 and 1 provides a similar but more advanced version of (b). For example, $\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$, arising from the substitution

$$x = \frac{1}{2}t^2 \ : \ \int_a^A x^{-\frac{1}{2}} e^{-x} dx = \sqrt{2}\int_{\sqrt{2a}}^{\sqrt{2A}} \frac{1}{t} e^{-\frac{1}{2}t^2} t\,dt = \sqrt{2}\int_{\sqrt{2a}}^{\sqrt{2A}} e^{-\frac{1}{2}t^2} dt \to \sqrt{\pi}$$

as $a \to 0^+, A \to \infty$, using the known value of the probability integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\,dt = 1.$$

(d) *Saving work and a general principle*

It clearly saves work if the graph of $y = f(x)$ is symmetrical in $y = x$, for then convergence of the integral of $f(x)$ at ∞ implies (and is implied by) its convergence at 0. Equivalently, if $f(x)$ is strictly decreasing, we require $f = f^{-1}$. We could construct such examples by splicing as in (a). For example

(i) $$f(x) = \begin{cases} x^{-\frac{1}{2}} & 0 < x < 1 \\ \frac{1}{x^2} & 1 \leqslant x < \infty \end{cases},$$

(ii) $$f(x) = \begin{cases} -\ln x & 0 < x < 0{\cdot}56714 \\ e^{-x} & 0{\cdot}56714 \leqslant x < \infty \end{cases},$$

(with an associated discussion about where the 0·56714 comes from and how to calculate it!).

But more elegant examples come from the observation that if g has an inverse and $f = f^{-1}$, then $g^{-1}fg$ satisfies $g^{-1}fg = (g^{-1}fg)^{-1}$ and so produces another example.

(iii) With $f(x) = 1 - x$ and $g(x) = e^{-x}$, we have $g^{-1}fg(x) = -\ln(1 - e^{-x})$. The integral of this converges at ∞ since $-\ln(1 - e^{-x}) \approx e^{-x}$, and it thus automatically converges at $x = 0$ as well. But there is an historical bonus if we also evaluate the integral:

$$\int_a^A -\ln(1 - e^{-x})\,dx = \int_a^A \left(\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(e^{-na} - e^{-nA}\right) \to \sum_{n=1}^{\infty} \frac{1}{n^2},$$

as $a \to 0^+, A \to \infty$; which, as in our fourth gobbet, gives the ever-welcome

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opportunity to mention Euler, the Basel problem and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$!

(iv) For a more ambitious example, take $f(x) = \frac{1-x}{1+x}$ and $g(x) = e^{-x}$ to obtain $g^{-1}fg(x) = -\ln\left(\frac{1-e^{-x}}{1+e^{-x}}\right) = \ln\left(\frac{e^x-1}{e^x+1}\right)$, which is the example discussed in [6]. It can also be expressed more compactly as $y = \ln\coth\frac{x}{2}$. And a similar calculation to that in (iii), which is given in [6], shows that

$$-\int_0^{\infty} \ln\left(\frac{1-e^{-x}}{1+e^{-x}}\right)dx = \frac{\pi^2}{4}.$$

(v) With $f(x) = 1 - x$ and $g(x) = \tanh x$, we get $g^{-1}fg(x) = \tanh^{-1}(1 - \tanh x)$. Here the integral converges at ∞ since, for large x,

$$\tanh^{-1}\left(1 - \frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = \tanh^{-1}\left(\frac{2}{e^{2x}+1}\right) \approx \frac{2}{e^{2x}+1} \approx 2e^{-2x}.$$

But this time $\int_0^{\infty} \tanh^{-1}(1 - \tanh x)\,dx \approx 0.896077$ is a recalcitrant integral for which WolframAlpha gives the closed form $\frac{1}{24}\left[6\,\mathrm{Li}_2\left(-\frac{1}{3}\right) + 2\pi^2 + 2(\ln 3)^2\right]$, involving the digamma function $\mathrm{Li}_2(z)$.

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