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Effective approximation to complex algebraic numbers by quadratic numbers

Prajeet Bajpai and Yann Bugeaud

Abstract. We establish an effective improvement on the Liouville inequality for approximation to complex nonreal algebraic numbers by quadratic complex algebraic numbers.

1 Introduction

Throughout this paper, the (naïve) height $H(\alpha)$ of an algebraic number α is the (naïve) height $H(P)$ of its minimal defining polynomial $P(X)$ over $\mathbb{Z}$, that is, the maximum of the absolute values of the coefficients of $P(X)$. The present note is a follow-up of [1], where we investigated the effective approximation to complex algebraic numbers by algebraic numbers of degree at most 4. Our previous results were restricted to totally complex numbers (that is, complex algebraic numbers having no real Galois conjugates). Here, we focus our attention on quadratic approximation to complex nonreal algebraic numbers having at least one real Galois conjugate.

Let ξ be a complex nonreal algebraic number of degree $d \geq 2$. By a Liouville-type argument (see e.g. [8] or below the statement of Theorem 1.1), there exists an effectively computable, positive number $c_1(\xi)$ such that

$$|\xi - \alpha| > c_1(\xi) H(\alpha)^{-\frac{d}{2}}, \quad \text{for every quadratic complex number } \alpha \neq \xi. \tag{1.1}$$

When $d = 2, 3$, this is best possible; see e.g. [5, Proposition 10.2]. For $d \geq 4$, it follows from [5, Corollary 2.4] (see also [1, Lemma 4.7]) that (1.1) is best possible (up to the value of $c_1(\xi)$) if and only if $\mathbb{Q}(\xi)$ is a quartic CM-field (a number field K is a CM-field if it is a quadratic extension K/F such that the image of every complex embedding of F is contained in $\mathbb{R}$, but there is no complex embedding of K whose image is contained in $\mathbb{R}$). In the case where $\mathbb{Q}(\xi)$ is not a quartic CM-field, (1.1) can be considerably improved by means of the Schmidt Subspace Theorem. Namely, for every $\varepsilon > 0$, [5, Corollary 2.4] asserts that there exists a positive number $c_2(\xi, \varepsilon)$ such that

$$|\xi - \alpha| > c_2(\xi, \varepsilon) H(\alpha)^{-\frac{3}{2}-\varepsilon}, \quad \text{for every quadratic complex number } \alpha \neq \xi. \tag{1.2}$$

The current techniques do not yield an explicit value for $c_2(\xi, \varepsilon)$. Hence, there is a large gap between the effective statement (1.1) and the ineffective statement (1.2). In the

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present note, we apply Baker's theory of linear forms in logarithms to get an effective improvement of (1.1). We establish the

Theorem 1.1 *Let ξ be a complex nonreal algebraic number of degree $d \geq 4$. If $d = 4$, assume furthermore that $\mathbb{Q}(\xi)$ is not a CM-field. Then, there exist effectively computable positive real numbers $\kappa(\xi)$ and $c(\xi)$ such that*

$$|\xi - \alpha| > c(\xi) H(\alpha)^{-\frac{d}{2}+\kappa(\xi)}, \quad \text{for every quadratic complex number } \alpha.$$

Under the additional assumption that ξ is totally complex, Theorem 1.1 has been proved in [1]. Both proofs use the same method, the main difference being a simple observation that we overlooked in [1]. Let us explain it below.

The question of effective approximation to algebraic numbers by algebraic numbers of bounded degree is deeply connected to the effective resolution of norm-form equations. Indeed, let ξ be an algebraic number of degree $d \geq 3$. Set $K = \mathbb{Q}(\xi)$. Put $\delta = 1$ if ξ is real and $\delta = 2$ otherwise. Let n be a positive integer with $n \leq d - 2$. Let α be an algebraic number of degree n. Then, its minimal defining polynomial $P_\alpha(X) := x_0 + x_1 X + \ldots + x_n X^n$ satisfies

$$|P_\alpha(\xi)| = |x_0 + x_1\xi + \ldots + x_n\xi^n| \ll_{\xi,n} |\xi - \alpha|\, H(P_\alpha)$$

and

$$1 \leq |\mathrm{Norm}_{K/\mathbb{Q}}(P_\alpha(\xi))| \ll_{\xi,n} |P_\alpha(\xi)|^\delta \cdot H(\alpha)^{d-\delta} \ll_{\xi,n} |\xi - \alpha|^\delta\, H(\alpha)^d, \tag{1.3}$$

since $|P_\alpha(\xi)| = |P_\alpha(\overline{\xi})|$ if ξ is nonreal with complex conjugate $\overline{\xi}$. Here and below, the constants implicit in $\ll_{x,y,\ldots}$ and in $\gg_{x,y,\ldots}$ are positive, effectively computable and depend at most on $x, y, \ldots$ This gives immediately that

$$|\xi - \alpha| \gg_{\xi,n} H(\alpha)^{-d}, \quad \text{if } \xi \text{ is real,}$$

and

$$|\xi - \alpha| \gg_{\xi,n} H(\alpha)^{-d/2}, \quad \text{if } \xi \text{ is complex nonreal.}$$

By (1.3), these lower bounds for $|\xi - \alpha|$ obtained by a Liouville-type argument can be improved (at the level of the exponent of $H(\alpha)$) as soon as we can establish a lower bound for $|\mathrm{Norm}_{K/\mathbb{Q}}(P_\alpha(\xi))|$ of the form $H(P_\alpha)^c$, valid for every integer polynomial $P_\alpha(X)$ of height at least H_0, for some positive, effectively computable real numbers c and H_0, depending only on ξ and n. For $n = 1$ (and ξ real, otherwise there is nothing to do), this has been done by Feldman [7] (see also [3]), who applied the theory of linear forms in logarithms, first developed by Alan Baker [2].

Recently, under suitable assumptions on ξ, we showed in [1] that there exist positive, effectively computable, real numbers $\delta(\xi)$ and $H_0(\xi)$ such that

$$|\mathrm{Norm}_{K/\mathbb{Q}}(P(\xi))| \geq H(P)^{\delta(\xi)}, \tag{1.4}$$

holds for every integer polynomial $P(X)$ of height at least $H_0(\xi)$ and degree at most 2 (resp., 3, 4). This implies that no roots of $P(X)$ is very close to ξ and allowed us to

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obtain various effective results for approximation to ξ by algebraic numbers of degree at most 2 (resp., 3, 4) of the form

$$|\xi - \alpha| \gg_\xi H(\alpha)^{-d/2+\kappa(\xi)},$$

for some positive, effectively computable, number $\kappa(\xi)$.

It is important to observe that if this method can be applied to ξ, then it applies as well to *every* Galois conjugate of ξ. This remark explains why we required in [1] the algebraic number ξ to be totally complex: no effective result better than Liouville's inequality is known for approximation to real algebraic numbers by quadratic numbers, and this question is likely to be very difficult. However, to get an effective improvement over Liouville's inequality, we do not need to prove (1.4) for *all* integer polynomials $P_\alpha(X)$. Indeed, if there is a Galois conjugate ξ_0 of ξ such that $|P_\alpha(\xi_0)| < |P_\alpha(\xi)|$, then the trivial upper bound $|P_\alpha(\xi_0)| \ll_{\xi,n} H(P_\alpha)$ used in (1.3) can be replaced by $|P_\alpha(\xi_0)| < |P_\alpha(\xi)|$, and we eventually obtain

$$1 \ll_{\xi,n} |\xi - \alpha|^3 H(\alpha)^d.$$

This provides an effective improvement on Liouville's inequality. A version of this observation has already been used at the end of the proof of [1, Theorem 2.3].

The next section gathers two estimates from the theory of linear forms in logarithms. Theorem 1.1 is established in Section 3.

Throughout the paper, we let h denote the logarithmic Weil height and we set $h_*(\cdot) = \max\{h(\cdot), 1\}$ and $\log_*(\cdot) = \max\{\log(\cdot), 1\}$.

2 Auxiliary results

We recall first a classical estimate for linear forms in complex logarithms of algebraic numbers.

Theorem 2.1 *Let $n \geq 1$ be an integer. Let $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$ be nonzero algebraic numbers. Let D denote the degree of the algebraic number field generated by $\alpha_1, \ldots, \alpha_{n+1}$ over $\mathbb{Q}$. Let $b_1, \ldots, b_n$ be nonzero integers and set*

$$B = \max\{|b_1|, \ldots, |b_n|\}.$$

If $\alpha_1^{b_1} \ldots \alpha_n^{b_n} \alpha_{n+1} \neq 1$, then we have

$$\log|\alpha_1^{b_1} \ldots \alpha_n^{b_n} \alpha_{n+1} - 1| > -c(n, D)\, h_*(\alpha_1) \cdots h_*(\alpha_{n+1})\, \log_*\Big(\frac{B}{h_*(\alpha_{n+1})}\Big).$$

Proof See [4] or [10]. ∎

We also need the following auxiliary result on multiplicative dependence relations between algebraic numbers. A version of it was originally proved by Loxton and van der Poorten [9].

Theorem 2.2 *Let $m \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_m$ be multiplicatively dependent nonzero algebraic numbers. Let $\log \alpha_1, \ldots, \log \alpha_m$ be any determination of their*

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logarithms. Let D be the degree of the number field generated by $\alpha_1, \dots, \alpha_m$ over $\mathbb{Q}$. For $1 \le j \le m$, let A_j be a real number satisfying

$$\log A_j \ge \max\Big\{h(\alpha_j), \frac{|\log \alpha_j|}{D}, 1\Big\}.$$

Then there exist rational integers $n_1, \dots, n_m$, not all of which are zero, such that

$$n_1 \log \alpha_1 + \cdots + n_m \log \alpha_m = 0, \quad \alpha_1^{n_1} \cdots \alpha_m^{n_m} = 1,$$

and

$$|n_k| \le \big(11(m-1)D^3\big)^{m-1} \frac{(\log A_1)\cdots(\log A_m)}{\log A_k}, \quad \text{for } 1 \le k \le m.$$

Proof See [4, Theorem 10.5]. ∎

3 Proof of Theorem 1.1

Set $K = \mathbb{Q}(\xi)$. Let α be a quadratic complex (nonreal) algebraic number. Let $P_\alpha(X) = x_0 + x_1 X + x_2 X^2$ denote its minimal defining polynomial over $\mathbb{Z}$. Set

$$\mathrm{Norm}_{K/\mathbb{Q}}(x_0 + x_1\xi + x_2\xi^2) = m,$$

and

$$\mathbf{x} = x_0 + x_1\xi + x_2\xi^2 = P_\alpha(\xi).$$

Let $\sigma_1, \dots, \sigma_d$ denote the d embeddings of K into $\mathbb{C}$ numbered in such a way that

$$|\sigma_1(\mathbf{x})| \ge |\sigma_2(\mathbf{x})| \ge \cdots \ge |\sigma_{d-1}(\mathbf{x})| \ge |\sigma_d(\mathbf{x})|.$$

For later use, recall that

$$d^{-1} \log|\sigma_1(\mathbf{x})| \le h(\mathbf{x}) \le \log|\sigma_1(\mathbf{x})|.$$

and observe that

$$\sigma_1(\mathbf{x}) \cdot \sigma_2(\mathbf{x}) \cdots \sigma_d(\mathbf{x}) = m.$$

By [1, Lemma 4.5] (which follows from the proof of [6, Proposition 4.3.12]), there exists a unit u in K such that, putting $\mu = \mathbf{x}/u$, we have

$$C_1^{-1}|m|^{1/d} \le |\sigma(\mu)| \le C_1|m|^{1/d},$$

for every complex embedding σ of K, with a constant $C_1 = C_1(K) \ge 1$.

The next lemma is a reformulation of [1, Theorem 4.6].

Lemma 3.1 *Keep the above notation. Assume there are effectively computable positive real numbers κ_1 and h_0, depending only on ξ, such that*

$$h(\mathbf{x}) \le \kappa_1 h_*(\mu),$$

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holds when $h(\mathbf{x}) > h_0$. Then, there exist effectively computable, positive c, κ, depending only on ξ, such that

$$|\xi - \alpha| > c\, H(\alpha)^{-\frac{d}{2}+\kappa}.$$

Our aim is to prove the existence of κ_1 and h_0 as in Lemma 3.1. Thus, without any loss of generality, we assume that

$$h(\mathbf{x}) \geq 2h_*(\mu), \tag{3.1}$$

and also

$$|\sigma_d(\mathbf{x})| < 1.$$

Indeed, otherwise we get at once $|\sigma_1(\mathbf{x})| \leq |m|$ and we obtain

$$dh_*(\mu) \gg \log_* |m| \geq \log |\sigma_1(\mathbf{x})| \geq h(\mathbf{x}).$$

Here and below, the constants implicit in $\gg$ and in $\ll$ are positive, effectively computable and depend at most on ξ.

Since the totally complex case has been fully addressed in [1], we assume that K is not totally complex. In particular, its unit group has rank 2.

Case 1. Assume first that the embeddings can be numbered in such a way that $\sigma_d(\mathbf{x}) = \mathbf{x} = P_\alpha(\xi)$. It is a complex nonreal number. Without any loss of generality, we assume that $\sigma_{d-1}(\mathbf{x})$ is its complex conjugate. The proof then goes as in [1], except for Subsubcase $(\star)$ below.

Subcase $d = 4$. It follows from [1, Lemma 5.1] that there are nonzero a_0, a_1, a_2 in the Galois closure of K such that

$$\sigma_1(\mathbf{x}) + a_0\sigma_2(\mathbf{x}) + a_1\sigma_3(\mathbf{x}) + a_2\sigma_4(\mathbf{x}) = 0.$$

Upon dividing by $\sigma_1(\mathbf{x})$ and recalling that $\mathbf{x} = \mu u$, we get

$$|1 - \gamma\nu| \leq \left|\frac{a_1\sigma_3(\mathbf{x})}{\sigma_1(\mathbf{x})}\right| + \left|\frac{a_2\sigma_4(\mathbf{x})}{\sigma_1(\mathbf{x})}\right|,$$

with

$$\gamma = -\frac{a_0\sigma_2(\mu)}{\sigma_1(\mu)}, \quad \nu = \frac{\sigma_2(u)}{\sigma_1(u)}.$$

We distinguish two cases.

Subsubcase $(\star)$. We assume that $1 - \gamma\nu = 0$, that is,

$$\sigma_1(\mathbf{x}) + a_0\sigma_2(\mathbf{x}) = 0.$$

Let η_1, η_2 be a fundamental system of units of $\mathbb{Q}(\xi)$. Then, there are a root of unity ζ in $\mathbb{Q}(\xi)$ and integers b_1, b_2 such that

$$\mathbf{x} = \mu u = \mu\zeta\eta_1^{b_1}\eta_2^{b_2},$$

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and we have

$$\frac{\sigma_1(\mu\zeta)}{-a_0\sigma_2(\mu\zeta)}\left(\frac{\sigma_1(\eta_1)}{\sigma_2(\eta_1)}\right)^{b_1}\left(\frac{\sigma_1(\eta_2)}{\sigma_2(\eta_2)}\right)^{b_2}=1. \tag{3.2}$$

By Theorem 2.2, this implies that there exist rational integers n_0, n_1, n_2, not all of which are zero, such that

$$\left(\frac{\sigma_1(\mu\zeta)}{-a_0\sigma_2(\mu\zeta)}\right)^{n_0}\left(\frac{\sigma_1(\eta_1)}{\sigma_2(\eta_1)}\right)^{n_1}\left(\frac{\sigma_1(\eta_2)}{\sigma_2(\eta_2)}\right)^{n_2}=1,$$

with

$$|n_0|\ll 1,\quad |n_1|,|n_2|\ll h_*(\mu).$$

If $\frac{\sigma_1(\eta_1)}{\sigma_2(\eta_1)}$ and $\frac{\sigma_1(\eta_2)}{\sigma_2(\eta_2)}$ are multiplicatively independent, then we derive from (3.2) that $b_1n_0=n_1$, $b_2n_0=n_2$, and

$$\max\{|b_1|,|b_2|\}\ll h_*(\mu),$$

giving $h(u)\ll h_*(\mu)$ and $h(\mathbf{x})\ll h_*(\mu)$, as wanted.

Otherwise, there exist nonzero integers c_1, c_2, with $|c_1|,|c_2|\ll 1$ such that

$$\left(\frac{\sigma_1(\eta_1)}{\sigma_2(\eta_1)}\right)^{c_1}\left(\frac{\sigma_1(\eta_2)}{\sigma_2(\eta_2)}\right)^{c_2}=1.$$

Combined with (3.2), this gives

$$\left(\frac{\sigma_1(\mu\zeta)}{-a_0\sigma_2(\mu\zeta)}\right)^{c_1}\left(\frac{\sigma_1(\eta_2)}{\sigma_2(\eta_2)}\right)^{-c_2b_1+c_1b_2}=1.$$

We deduce that

$$\max\{|b_1|,|b_2|\}\ll h\Big(\frac{\sigma_1(\mu\zeta)}{-a_0\sigma_2(\mu\zeta)}\Big)\ll h_*(\mu),$$

and we conclude as above that $h(\mathbf{x})\ll h_*(\mu)$.

Subsubcase $(\star\star)$. We assume that $|1-\gamma\nu|>0$. We treat this case below, simultaneously with the case $d\geq 6$.

Case $d\geq 6$. It follows from [1, Lemma 5.1] that there are nonzero $a_0, a_1, a_2, b_0, b_1, b_2$ in the Galois closure of K such that

$$\sigma_1(\xi^j)+\sum_{i=0}^{2}a_i\sigma_{d-i}(\xi^j)=0\quad\text{and}\quad\sigma_2(\xi^j)+\sum_{i=0}^{2}b_i\sigma_{d-i}(\xi^j)=0,\quad\text{for } j=0,1,2.$$

This implies

$$b_0\sigma_1(\xi^j)-a_0\sigma_2(\xi^j)+\sum_{i=1}^{2}(a_ib_0-a_0b_i)\sigma_{d-i}(\xi^j)=0,\quad\text{for } j=0,1,2.$$

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Consequently, we have $a_1b_0 \neq a_0b_1$. Thus, $a_1\sigma_{d-1}(\mathbf{x}) + a_0\sigma_d(\mathbf{x})$ and $b_1\sigma_{d-1}(\mathbf{x}) + b_0\sigma_d(\mathbf{x})$ cannot be both 0. By permuting σ_1 and σ_2 if necessary, we assume that $a_1\sigma_{d-1}(\mathbf{x}) + a_0\sigma_d(\mathbf{x})$ is nonzero. By linearity, we have

$$\sigma_1(\mathbf{x}) + \sum_{i=0}^{2} a_i\sigma_{d-i}(\mathbf{x}) = 0,$$

thus $\sigma_1(\mathbf{x}) + a_2\sigma_{d-2}(\mathbf{x})$ is nonzero. Consequently, by dividing by $\sigma_1(\mathbf{x})$ and recalling that $\mathbf{x} = \mu u$, we get

$$0 < |1 - \gamma\nu| \leq \sum_{i=0}^{1} \left| \frac{a_i\sigma_{d-i}(\mathbf{x})}{\sigma_1(\mathbf{x})} \right|, \quad \text{with} \quad \gamma = -\frac{a_2\sigma_{d-2}(\mu)}{\sigma_1(\mu)}, \nu = \frac{\sigma_{d-2}(u)}{\sigma_1(u)}.$$

This corresponds exactly to the inequalities obtained in the case $d = 4$. So, we treat both cases simultaneously.

Since $|\sigma_{d-1}(\mathbf{x})| = |\sigma_d(\mathbf{x})| < 1$, this gives $|1 - \gamma\nu| \ll |\sigma_1(\mathbf{x})|^{-1}$, thus

$$\log|1 - \gamma\nu| \leq -h(\mathbf{x})/2, \quad \text{if } h(\mathbf{x}) \text{ is large enough.}$$

As, by Dirichlet's unit theorem, the unit ν can be expressed as a product of a root of unity in K times integral powers of elements of a fundamental system of units in K, it follows from Theorem 2.1 that

$$\frac{h(\mathbf{x})}{2} \leq -\log|1 - \gamma\nu| \ll h_*(\gamma)\log_* \frac{h_*(\nu)}{h_*(\gamma)}. \tag{3.3}$$

Furthermore, by (3.1), we get

$$h(\nu) \ll h(u) \ll (h(\mathbf{x}) + h_*(\mu)) \ll h(\mathbf{x}),$$

and we derive from (3.3) that

$$h(\mathbf{x}) \ll h_*(\gamma) \ll h_*(\mu).$$

It then follows from Lemma 3.1 that there exist effectively computable, positive real numbers c, κ, depending only on ξ, such that

$$|\xi - \alpha| > c\,H(\alpha)^{-\frac{d}{2}+\kappa},$$

for every algebraic number α which is a root of $x_0 + x_1X + x_2X^2$, as soon as we are in Case 1.

Case 2. Assume now that the embeddings cannot be numbered in such a way that $\sigma_d(\mathbf{x}) = \mathbf{x} = P_\alpha(\xi)$. Then, there is a Galois conjugate ξ_0 of ξ such that

$$|P_\alpha(\xi_0)| < |P_\alpha(\xi)| = |P_\alpha(\overline{\xi})|.$$

Consequently, we have

$$1 \leq m = |\mathrm{Norm}_{K/\mathbb{Q}}(P_\alpha(\xi))| \ll |P_\alpha(\xi)|^3 \cdot H(\alpha)^{d-3} \ll |\xi - \alpha|^3 \cdot H(\alpha)^d,$$

giving that $|\xi - \alpha| \gg H(\alpha)^{-d/3} \gg H(\alpha)^{-d/2+2/3}$, since $d \geq 4$.

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Conclusion. We have established in both cases that

$$|\xi - \alpha| \gg H(\alpha)^{-\frac{d}{2}+\min\{\kappa,2/3\}},$$

for every complex nonreal quadratic algebraic number α. This completes the proof of Theorem 1.1.

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Department of Mathematics, University of British Columbia, Vancouver, B.C., V6T 1Z2 Canada
e-mail: pbajpai@cs.ubc.ca

I.R.M.A., UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg, France

Institut universitaire de France
e-mail: bugeaud@math.unistra.fr

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