NORMAL TRANSVERSALITY AND UNIFORM BOUNDS

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1. *Introduction*

Let A be a commutative ring. A graded A-algebra $U = \bigoplus_{n\geqslant 0} U_n$ is a *standard* A-algebra if $U_0 = A$ and $U = A[U_1]$ is generated as an A-algebra by the elements of U_1. A graded U-module $F = \bigoplus_{n\geqslant 0} F_n$ is a *standard* U-module if F is generated as a U-module by the elements of F_0, that is, $F_n = U_n F_0$ for all $n \geqslant 0$. In particular, $F_n = U_1 F_{n-1}$ for all $n \geqslant 1$. Given I, J, two ideals of A, we consider the following standard algebras: the *Rees algebra of* I, $\mathscr{R}(I) = \bigoplus_{n\geqslant 0} I^n t^n = A[It] \subset A[t]$, and the *multi-Rees algebra of* I *and* J, $\mathscr{R}(I,J) = \bigoplus_{n\geqslant 0}(\bigoplus_{p+q=n} I^p J^q u^p v^q) = A[Iu, Jv] \subset A[u,v]$. Consider the *associated graded ring of* I, $\mathscr{G}(I) = \mathscr{R}(I) \otimes A/I = \bigoplus_{n\geqslant 0} I^n/I^{n+1}$, and the *multi-associated graded ring of* I *and* J, $\mathscr{G}(I,J) = \mathscr{R}(I,J) \otimes A/(I+J) = \bigoplus_{n\geqslant 0}(\bigoplus_{p+q=n} I^p J^q/(I+J) I^p J^q)$. We can always consider the tensor product of two standard A-algebras $U = \bigoplus_{p\geqslant 0} U_p$ and $V = \bigoplus_{q\geqslant 0} V_q$ as a standard A-algebra with the natural grading $U \otimes V = \bigoplus_{n\geqslant 0}(\bigoplus_{p+q=n} U_p \otimes V_q)$. If M is an A-module, we have the standard modules: the *Rees module of* I *with respect to* M, $\mathscr{R}(I;M) = \bigoplus_{n\geqslant 0} I^n M t^n = M[It] \subset M[t]$ (a standard $\mathscr{R}(I)$-module), and the *multi-Rees module of* I *and* J *with respect to* M, $\mathscr{R}(I,J;M) = \bigoplus_{n\geqslant 0}(\bigoplus_{p+q=n} I^p J^q M u^p v^q) = M[Iu, Jv] \subset M[u,v]$ (a standard $\mathscr{R}(I,J)$-module). Consider the *associated graded module of* M *with respect to* I, $\mathscr{G}(I;M) = \mathscr{R}(I;M) \otimes A/I = \bigoplus_{n\geqslant 0} I^n M/I^{n+1} M$ (a standard $\mathscr{G}(I)$-module), and the *multi-associated graded module of* M *with respect to* I *and* J, $\mathscr{G}(I,J;M) = \mathscr{R}(I,J;M) \otimes A/(I+J) = \bigoplus_{n\geqslant 0}(\bigoplus_{p+q=n} I^p J^q M/(I+J) I^p J^q M)$ (a standard $\mathscr{G}(I,J)$-module). If U, V are two standard A-algebras, F is a standard U-module and G is a standard V-module, then $F \otimes G = \bigoplus_{n\geqslant 0}(\bigoplus_{p+q=n} F_p \otimes G_q)$ is a standard $U \otimes V$-module.

Denote by $\pi : \mathscr{R}(I) \otimes \mathscr{R}(J;M) \longrightarrow \mathscr{R}(I,J;M)$ and $\sigma : \mathscr{R}(I,J;M) \longrightarrow \mathscr{R}(I+J;M)$ the natural surjective graded morphisms of standard $\mathscr{R}(I) \otimes \mathscr{R}(J)$-modules. Let $\varphi : \mathscr{R}(I) \otimes \mathscr{R}(J;M) \longrightarrow \mathscr{R}(I+J;M)$ be $\sigma \circ \pi$. Denote by $\bar{\pi} : \mathscr{G}(I) \otimes \mathscr{G}(J;M) \longrightarrow \mathscr{G}(I,J;M)$ and $\bar{\sigma} : \mathscr{G}(I,J;M) \longrightarrow \mathscr{G}(I+J;M)$ the tensor product of π and σ by $A/(I+J)$; these are two natural surjective graded morphisms of standard $\mathscr{G}(I) \otimes \mathscr{G}(J)$-modules. Let $\bar{\varphi} : \mathscr{G}(I) \otimes \mathscr{G}(J;M) \longrightarrow \mathscr{G}(I+J;M)$ be $\bar{\sigma} \circ \bar{\pi}$. The first purpose of this paper is to prove the following theorem.

THEOREM 1.1. *Let A be a noetherian ring, I, J be two ideals of A and M be a finitely generated A-module. The following two conditions are equivalent:*

(i) $\bar{\varphi} : \mathscr{G}(I) \otimes \mathscr{G}(J;M) \longrightarrow \mathscr{G}(I+J;M)$ *is an isomorphism.*

(ii) $\operatorname{Tor}_1(A/I^p, \mathscr{R}(J;M)) = 0$ *and* $\operatorname{Tor}_1(A/I^p, \mathscr{G}(J;M)) = 0$ *for all integers* $p \geqslant 1$.

In particular, $\mathscr{G}(I) \otimes \mathscr{G}(J) \simeq \mathscr{G}(I+J)$ *if and only if* $\operatorname{Tor}_1(A/I^p, A/J^q) = 0$ *and* $\operatorname{Tor}_2(A/I^p, A/J^q) = 0$ *for all integers* $p, q \geqslant 1$.

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The morphism $\bar{\varphi}$ has been studied by Hironaka [**5**], Grothendieck [**3**] and Hermann, Ikeda and Orbanz [**4**], among others, but assuming always that A is normally flat along I (see [**4**, 21.11]). We will see how Theorem 1.1 generalizes all this previous work.

Let us now recall some definitions in order to state the second purpose of this paper. If U is a standard A-algebra and F is a graded U-module, put $s(F) = \min\{r \geqslant 1 \mid F_n = 0 \text{ for all } n \geqslant r+1\}$, where $s(F)$ may possibly be infinite. If $U_+ = \bigoplus_{n>0} U_n$ and $r \geqslant 1$, the following three conditions are equivalent: F can be generated by elements of degree at most r; $s(F/U_+F) \leqslant r$; and $F_n = U_1 F_{n-1}$ for all $n \geqslant r+1$. If $\varphi: G \longrightarrow F$ is a surjective graded morphism of graded U-modules, we denote by $E(\varphi)$ the graded A-module $E(\varphi) = \ker\varphi / U_+ \ker\varphi = \ker\varphi_0 \oplus (\bigoplus_{n\geqslant 1} \ker\varphi_n / U_1 \ker\varphi_{n-1}) = \bigoplus_{n\geqslant 0} E(\varphi)_n$. If F is a standard U-module, take $\mathbf{S}(U_1)$ the symmetric algebra of U_1, $\alpha: \mathbf{S}(U_1) \longrightarrow U$ the surjective graded morphism of standard A-algebras induced by the identity on U_1 and

$$\gamma: \mathbf{S}(U_1) \otimes F_0 \xrightarrow{\alpha\otimes 1} U \otimes F_0 \longrightarrow F$$

the composition of $\alpha \otimes 1$ with the structural morphism. Since F is a standard U-module, γ is a surjective graded morphism of graded $\mathbf{S}(U_1)$-modules. The *module of effective n-relations* of F is defined to be $E(F)_n = E(\gamma)_n = \ker\gamma_n / U_1 \ker\gamma_{n-1}$ (for $n = 0$, $E(F)_n = 0$). Put $E(F) = \bigoplus_{n\geqslant 1} E(F)_n = \bigoplus_{n\geqslant 1} E(\gamma)_n = E(\gamma) = \ker\gamma / \mathbf{S}_+(U_1)\ker\gamma$. The *relation type* of F is defined to be $\mathrm{rt}(F) = s(E(F))$, that is, $\mathrm{rt}(F)$ is the minimum positive integer $r \geqslant 1$ such that the effective n-relations are zero for all $n \geqslant r+1$. A *symmetric presentation* of a standard U-module F is a surjective graded morphism of standard V-modules $\varphi: G \longrightarrow F$, with

$$\varphi: G = V \otimes M \xrightarrow{f\otimes h} U \otimes F_0 \longrightarrow F,$$

where V is a symmetric A-algebra, $f: V \longrightarrow U$ is a surjective graded morphism of standard A-algebras, $h: M \longrightarrow F_0$ is an epimorphism of A-modules and $U \otimes F_0 \longrightarrow F$ is the structural morphism. One can show (see [**8**]) that $E(F)_n = E(\varphi)_n$ for all $n \geqslant 2$ and $s(E(F)) = s(E(\varphi))$. Thus the module of effective n-relations and the relation type of a standard U-module are independent of the chosen symmetric presentation. Roughly speaking, the relation type of F is the largest degree of any minimal homogeneous system of generators of the submodule defining F as a quotient of a polynomial ring with coefficients in F_0. For an ideal I of A and an A-module M, the module of effective n-relations and the relation type of I with respect to M are defined to be $E(I; M)_n = E(\mathscr{R}(I; M))_n$ and $\mathrm{rt}(I; M) = \mathrm{rt}(\mathscr{R}(I; M))$, respectively.

THEOREM 1.2. *Let A be a commutative ring, U and V be two standard A-algebras, F be a standard U-module and G be a standard V-module. Then $U \otimes V$ is a standard A-algebra, $F \otimes G$ is a standard $U \otimes V$-module and* $\mathrm{rt}(F \otimes G) \leqslant \max(\mathrm{rt}(F), \mathrm{rt}(G))$.

As a consequence of Theorems 1.1 and 1.2, one deduces the existence of a uniform bound for the relation type of all maximal ideals of an excellent ring.

THEOREM 1.3. *Let A be an excellent (or $J-2$) ring and let M be a finitely generated A-module. Then there exists an integer $s \geqslant 1$ such that, for all maximal ideals $\mathfrak{m}$ of A, the relation type of $\mathfrak{m}$ with respect to M satisfies* $\mathrm{rt}(\mathfrak{m}; M) \leqslant s$.

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In fact, Theorem 1.3 could also be deduced from the proof of [**9**, Theorem 4]. Finally, and using [**8**, Theorem 2], one can recover the following result of Duncan and O'Carroll.

COROLLARY 1.4 [**2**]. *Let A be an excellent (or $J-2$) ring and let $N \subseteq M$ be two finitely generated A-modules. Then there exists an integer $s \geqslant 1$ such that, for all integers $n \geqslant s$ and for all maximal ideals $\mathfrak{m}$ of A, $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-s}(\mathfrak{m}^s M \cap N)$.*

2. *Normal transversality*

LEMMA 2.1. *Let A be a commutative ring, I be an ideal of A, U be a standard A-algebra, F and G be two standard U-modules and $\varphi: G \longrightarrow F$ be a surjective graded morphism of standard A-algebras. If $\bar{A} = A/I$, then $\bar{U} = U \otimes \bar{A}$ is a standard $\bar{A}$-algebra, $\bar{F} = F \otimes \bar{A}$ and $\bar{G} = G \otimes \bar{A}$ are two standard $\bar{U}$-modules and $\bar{\varphi} = \varphi \otimes 1_{\bar{A}}: \bar{G} \longrightarrow \bar{F}$ is a surjective graded morphism of standard $\bar{U}$-modules. Moreover, $s(E(\bar{\varphi})) \leqslant s(E(\varphi))$.*

Proof. If we tensor

$$0 \longrightarrow \ker \varphi_i \longrightarrow G_i \xrightarrow{\varphi_i} F_i \longrightarrow 0$$

by $\bar{A}$ we get

$$\ker \varphi_i \otimes \bar{A} \longrightarrow \bar{G}_i \xrightarrow{\bar{\varphi}_i} \bar{F}_i \longrightarrow 0,$$

exact sequences which induce epimorphisms $\ker \varphi_i \otimes \bar{A} \longrightarrow \ker \bar{\varphi}_i$. On the other hand, if we tensor $U_1 \otimes \ker \varphi_{n-1} \longrightarrow \ker \varphi_n \longrightarrow E(\varphi)_n \longrightarrow 0$ by $\bar{A}$ we get the exact sequence

$$\bar{U}_1 \otimes \ker \varphi_{n-1} \otimes \bar{A} \longrightarrow \ker \varphi_n \otimes \bar{A} \longrightarrow E(\varphi)_n \otimes \bar{A} \longrightarrow 0.$$

We thus have the commutative diagram of exact rows shown in Figure 1, from which we deduce an epimorphism $E(\varphi)_n \otimes \bar{A} \longrightarrow E(\bar{\varphi})_n$. In particular, $s(E(\bar{\varphi})) \leqslant s(E(\varphi))$. □

$$\begin{array}{ccccccc}
\bar{U}_1 \otimes \ker \varphi_{n-1} \otimes \bar{A} & \longrightarrow & \ker \varphi_n \otimes \bar{A} & \longrightarrow & E(\varphi)_n \otimes \bar{A} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\bar{U}_1 \otimes \ker \bar{\varphi}_{n-1} & \longrightarrow & \ker \bar{\varphi}_n & \longrightarrow & E(\bar{\varphi})_n & \longrightarrow & 0
\end{array}$$

FIGURE 1.

LEMMA 2.2. *Let A be a commutative ring, I, J be two ideals of A and M be an A-module. Consider $\sigma: \mathscr{R}(I, J; M) \longrightarrow \mathscr{R}(I+J; M)$ and $\bar{\sigma}: \mathscr{G}(I, J; M) \longrightarrow \mathscr{G}(I+J; M)$. Then the following hold:*

(a) $\ker(\sigma_1) \simeq IM \cap JM$.

(b) $\ker(\bar{\sigma}_1) = 0$ *if and only if* $IM \cap JM \subset I(I+J)M \cap (I+J)JM$.

(c) *If $I^p M \cap J^q M = I^p J^q M$ for all integers $p, q \geqslant 1$, then $s(E(\sigma)) = 1$ and $\bar{\sigma}$ is an isomorphism.*

Proof. Consider

$$0 \longrightarrow IM \cap JM \xrightarrow{\rho} IM \oplus JM \xrightarrow{\sigma_1} (I+J)M \longrightarrow 0$$

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where $\rho(a)=(a,-a)$ and $\sigma_1(a,b)=a+b$. Clearly it is an exact sequence of A-modules. Thus $\ker(\sigma_1)=\rho(IM\cap JM)\simeq IM\cap JM$. If we tensor this exact sequence by $A/(I+J)$ we get

$$(IM\cap JM)\otimes A/(I+J)\xrightarrow{\bar{\rho}}(IM\oplus JM)\otimes A/(I+J)\xrightarrow{\bar{\sigma}_1}(I+J)\,M/(I+J)^2\,M\longrightarrow 0.$$

Then

$$\ker(\bar{\sigma}_1)=\operatorname{im}\bar{\rho}=\{(\bar{a},-\bar{a})\in IM/I(I+J)\,M\oplus JM/(I+J)\,JM\mid a\in IM\cap JM\}.$$

Hence $\ker(\bar{\sigma}_1)=0$ if and only if $IM\cap JM\subset I(I+J)\,M\cap(I+J)\,JM$. Now, let us prove (c). Let $z\in\ker\sigma_n\subset\mathscr{R}(I,J;M)_n=\bigoplus_{p+q=n}I^pJ^qMu^pv^q\subset M[u,v]$. Thus, $z=a_0u^n+a_1u^{n-1}v+\ldots+a_{n-1}uv^{n-1}+a_nv^n$, $a_i\in I^{n-i}J^iM$, and $0=\sigma_n(z)=(a_0+a_1+\ldots+a_{n-1}+a_n)\,t^n\in\mathscr{R}(I+J;M)_n=(I+J)^n\,Mt^n$, so $a_0+a_1+\ldots+a_{n-1}+a_n=0$. Let us denote (see [**10**, p. 134]):

$$\begin{cases}b_0=a_0\in I^nM\cap JM=I^nJM\\ b_1=a_0+a_1\in I^{n-1}M\cap J^2M=I^{n-1}J^2M \text{ and } a_1=b_1-b_0\\ b_2=a_0+a_1+a_2\in I^{n-2}M\cap J^3M=I^{n-2}J^3M \text{ and } a_2=b_2-b_1\\ \ldots\\ b_{n-2}=a_0+\ldots+a_{n-2}\in I^2M\cap J^{n-1}M=I^2J^{n-1}M \text{ and } a_{n-2}=b_{n-2}-b_{n-3}\\ b_{n-1}=a_0+\ldots+a_{n-1}\in IM\cap J^nM=IJ^nM \text{ and } a_{n-1}=b_{n-1}-b_{n-2}\\ a_n=-b_{n-1}\in IJ^nM.\end{cases}$$

We can rewrite z in $M[u,v]$ in the following manner:

$$\begin{aligned}z&=a_0u^n+a_1u^{n-1}v+\ldots+a_{n-1}uv^{n-1}+a_nv^n\\&=b_0u^n+(b_1-b_0)\,u^{n-1}v+(b_2-b_1)\,u^{n-2}v^2+\ldots\\&\quad+(b_{n-2}-b_{n-3})\,u^2v^{n-2}+(b_{n-1}-b_{n-2})\,uv^{n-1}+(-b_{n-1})\,v^n\\&=\underbrace{(b_0u^{n-1}+b_1u^{n-2}v+b_2u^{n-3}v^2+\ldots+b_{n-2}uv^{n-2}+b_{n-1}v^{n-1})}_{p(u,v)}(u-v):=p(u,v)\,(u-v),\end{aligned}$$

where $p(u,v)\in A[Iu,Jv]_{n-1}\cdot(IJM)=\mathscr{R}(I,J)_{n-1}\cdot(IJM)$. Since by hypothesis $IM\cap JM=IJM$, then $\ker(\sigma_1)=(IJM)\,(u-v)$, $\ker(\bar{\sigma}_1)=0$ and $z=p(u,v)\,(u-v)\in\mathscr{R}(I,J)_{n-1}\cdot(IJM)\,(u-v)=\mathscr{R}(I,J)_{n-1}\cdot\ker\sigma_1$. Thus $\ker\sigma_n=\mathscr{R}(I,J)_{n-1}\cdot\ker\sigma_1$ for all $n\geqslant 2$ and $s(E(\sigma))=1$. By Lemma 2.1, $s(E(\bar{\sigma}))\leqslant s(E(\sigma))=1$. Therefore $\ker(\bar{\sigma}_n)=\mathscr{G}(I,J)_{n-1}\cdot\ker(\bar{\sigma}_1)=0$ for all $n\geqslant 2$ and $\bar{\sigma}$ is an isomorphism. □

PROPOSITION 2.3. *Let A be a noetherian ring, I, J be two ideals of A and M be a finitely generated A-module. The following two conditions are equivalent:*

(i) $\bar{\sigma}:\mathscr{G}(I,J;M)\longrightarrow\mathscr{G}(I+J;M)$ *is an isomorphism.*

(ii) $I^pM\cap J^qM=I^pJ^qM$ *for all integers $p,q\geqslant 1$.*

Proof. Remark that we can suppose that A is local. By Lemma 2.2, (ii) ⇒ (i). Let us see (i) ⇒ (ii), proving by double induction in $p,q\geqslant 1$ that

$$I^pM\cap J^qM\subset I^p(I+J)\,J^{q-1}M\cap(I+J)^p\,J^qM.$$

Remark that if $I^pM\cap J^qM\subset I^p(I+J)\,J^{q-1}M$ for all $p,q\geqslant 1$, then $I^pM\cap J^qM\subset I^{p+1}M+I^pJ^qM$ and $I^pM\cap J^qM\subset I^{p+1}M\cap J^qM+I^pJ^qM$. Recursively, and using A

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is noetherian local and M is finitely generated, $I^pM \cap J^qM \subset \bigcap_{r\geqslant 1}(I^{p+r}M \cap J^qM + I^pJ^qM) \subset \bigcap_{n\geqslant 1}(I^nM + I^pJ^qM) = I^pJ^qM$, concluding (ii). Take $q = 1$. Let us prove by induction in $p \geqslant 1$ that

$$I^pM \cap JM \subset I^p(I+J)\,M \cap (I+J)^p\,JM.$$

For $p = 1$, we apply Lemma 2.2(b), using the hypothesis $\bar{\sigma}_1$ is an isomorphism. Suppose that

$$I^pM \cap JM \subset I^p(I+J)\,M \cap (I+J)^p\,JM$$

is true and let us prove that

$$I^{p+1}M \cap JM \subset I^{p+1}(I+J)\,M \cap (I+J)^{p+1}\,JM.$$

Then $I^{p+1}M \cap JM \subset I^pM \cap JM \subset (I+J)^pJM$. Consider the short complex of A-modules

$$I^{p+1}M \cap JM \xrightarrow{\alpha} I^{p+1}M \oplus (I+J)^pJM \xrightarrow{\beta} (I+J)^{p+1}M,$$

where $\alpha(a) = (a, -a)$ and $\beta(a, b) = a + b$. Remark that $\beta \circ \alpha = 0$, β is surjective and that there exists a natural epimorphism γ of A-modules such that $\beta \circ \gamma = \sigma_{p+1}$. If we tensor this short complex by $A/(I+J)$ we obtain

$$(I^{p+1}M \cap JM) \otimes A/(I+J) \xrightarrow{\bar{\alpha}} I^{p+1}M/I^{p+1}(I+J)\,M \oplus (I+J)^pJM/(I+J)^{p+1}JM$$

$$I^{p+1}M/I^{p+1}(I+J)\,M \oplus (I+J)^pJM/(I+J)^{p+1}JM \xrightarrow{\bar{\beta}} (I+J)^{p+1}M/(I+J)^{p+2}M,$$

with $\bar{\beta} \circ \bar{\alpha} = 0$. Since $\bar{\sigma}_{p+1} = \bar{\beta} \circ \bar{\gamma}$ is an isomorphism and $\bar{\gamma}$ is an epimorphism, then $\bar{\gamma}$ is an isomorphism, $\bar{\beta}$ is an isomorphism, $\bar{\alpha} = 0$ and

$$I^{p+1}M \cap JM \subset I^{p+1}(I+J)\,M \cap (I+J)^{p+1}\,JM.$$

In particular, $I^pM \cap JM \subset I^p(I+J)\,M$ for all $p \geqslant 1$, so $I^pM \cap JM \subset I^{p+1}M + I^pJM$ and therefore $I^pM \cap JM \subset I^{p+1}M \cap JM + I^pJM$. Recursively, and using A is noetherian local and M is finitely generated, $I^pM \cap JM \subset \bigcap_{r\geqslant 1}(I^{p+r}M \cap JM + I^pJM) \subset \bigcap_{n\geqslant 1}(I^nM + I^pJM) = I^pJM$ concluding that $I^pM \cap JM = I^pJM$ for all $p \geqslant 1$. Remark that, by the symmetry of the problem, $IM \cap J^qM = IJ^qM$ for all $q \geqslant 1$. Now, fix $q \geqslant 1$ and suppose that

$$I^pM \cap J^qM \subset I^p(I+J)\,J^{q-1}M \cap (I+J)^pJ^qM$$

holds for all $p \geqslant 1$ (in particular, $I^pM \cap J^qM = I^pJ^qM$ for all $p \geqslant 1$). Let us prove, by induction in $p \geqslant 1$, that

$$I^pM \cap J^{q+1}M \subset I^p(I+J)\,J^qM \cap (I+J)^pJ^{q+1}M.$$

For $p = 1$, we have $IM \cap J^{q+1}M = IJ^{q+1}M \subset I(I+J)\,J^qM \cap (I+J)\,J^{q+1}M$. Suppose that

$$I^pM \cap J^{q+1}M \subset I^p(I+J)\,J^qM \cap (I+J)^pJ^{q+1}M$$

and let us prove that

$$I^{p+1}M \cap J^{q+1}M \subset I^{p+1}(I+J)\,J^qM \cap (I+J)^{p+1}\,J^{q+1}M.$$

Then $I^{p+1}M \cap J^{q+1}M \subset I^pM \cap J^{q+1}M \subset (I+J)^pJ^{q+1}M$ and $I^{p+1}M \cap J^{q+1}M \subset I^{p+1}M \cap J^qM = I^{p+1}J^qM$. Consider the short complex of A-modules:

$$I^{p+1}M \cap J^{q+1}M \xrightarrow{\alpha} I^{p+q+1}M \oplus \ldots \oplus I^{p+1}J^qM \oplus (I+J)^pJ^{q+1}M \xrightarrow{\beta} (I+J)^{p+q+1}M,$$

where $\alpha(a) = (0, \ldots, 0, a, -a)$ and $\beta(a_1, \ldots, a_{q+2}) = a_1 + \ldots + a_{q+2}$. Remark that $\beta \circ \alpha = 0$, β is surjective and that there exists a natural epimorphism γ of A-modules such

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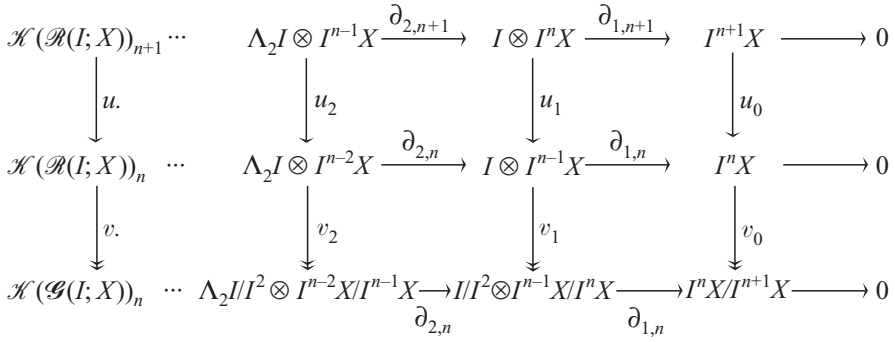


FIGURE 2.

that $\beta \circ \gamma = \sigma_{p+q+1}$. If we tensor this complex by $A/(I+J)$ we obtain $\bar{\beta} \circ \bar{\alpha} = 0$. Since $\bar{\sigma}_{p+q+1} = \bar{\beta} \circ \bar{\gamma}$ is an isomorphism and $\bar{\gamma}$ is an epimorphism, then $\bar{\gamma}$ is an isomorphism, $\bar{\beta}$ is an isomorphism, $\bar{\alpha} = 0$ and

$$I^{p+1}M \cap J^{q+1}M \subset I^{p+1}(I+J)\,J^q M \cap (I+J)^{p+1}\,J^{q+1}M. \qquad \square$$

PROPOSITION 2.4. *Let A be a commutative ring, I be an ideal of A and $\lambda : M \otimes N \longrightarrow P$ be an epimorphism of A-modules. Consider $f : \mathscr{R}(I;M) \otimes N \longrightarrow \mathscr{R}(I;P)$ and $\bar{f} = f \otimes 1_{A/I} : \mathscr{G}(I;M) \otimes N \longrightarrow \mathscr{G}(I;P)$ the natural surjective graded morphisms of standard modules. Then, for each integer $n \geqslant 2$, there exists an exact sequence of A-modules $E(f)_{n+1} \longrightarrow E(f)_n \longrightarrow E(\bar{f})_n \longrightarrow 0$. In particular, if A is noetherian, M, N, P are finitely generated and $\bar{f}$ is an isomorphism, then f is an isomorphism.*

Proof. For each integer $n \geqslant 1$, tensor $0 \longrightarrow I^n M \longrightarrow M \longrightarrow M/I^n M \longrightarrow 0$ by N and get the exact sequence $I^n M \otimes N \longrightarrow M \otimes N \longrightarrow M \otimes N/I^n(M \otimes N) \longrightarrow 0$. In particular, there exist epimorphisms $I^n M \otimes N \longrightarrow I^n(M \otimes N)$ which induce a surjective graded morphism of standard $\mathscr{R}(I)$-modules $g : \mathscr{R}(I;M) \otimes N \longrightarrow \mathscr{R}(I;M \otimes N)$. Clearly λ induces a surjective graded morphism of standard $\mathscr{R}(I)$-modules $h : \mathscr{R}(I;M \otimes N) \longrightarrow \mathscr{R}(I;P)$. The composition defines the surjective graded morphism of standard $\mathscr{R}(I)$-modules $f = h \circ g : \mathscr{R}(I;M) \otimes N \longrightarrow \mathscr{R}(I;P)$. If we tensor f by A/I, we get $\bar{f} : \mathscr{G}(I;M) \otimes N \longrightarrow \mathscr{G}(I;P)$ a surjective graded morphism of standard $\mathscr{G}(I)$-modules.

Let X be an A-module. Figure 2 is a commutative diagram of exact columns with rows the last three non-zero terms of the complexes $\mathscr{K}(\mathscr{R}(I;X))_{n+1}$, $\mathscr{K}(\mathscr{R}(I;X))_n$ and $\mathscr{K}(\mathscr{G}(I;X))_n$ (see [**8**, Proposition 2.6] for more details). In other words,

$$\mathscr{K}(\mathscr{R}(I;X))_{n+1} \xrightarrow{u_.} \mathscr{K}(\mathscr{R}(I;X))_n \xrightarrow{v_.} \mathscr{K}(\mathscr{G}(I;X))_n \longrightarrow 0$$

is an exact sequence of complexes. It induces the morphisms in homology:

$$H_1(\mathscr{K}(\mathscr{R}(I;X))_{n+1}) \xrightarrow{u} H_1(\mathscr{K}(\mathscr{R}(I;X))_n)$$

and

$$H_1(\mathscr{K}(\mathscr{R}(I;X))_n) \xrightarrow{v} H_1(\mathscr{K}(\mathscr{G}(I;X))_n).$$

By [**8**, Proposition 2.6], $H_1(\mathscr{K}(\mathscr{R}(I;X))_n) = E(I;X)_n$ and $H_1(\mathscr{K}(\mathscr{G}(I;X))_n) = E(\mathscr{G}(I;X))_n$. Thus we have

$$E(I;X)_{n+1} \xrightarrow{u} E(I;X)_n \xrightarrow{v} E(\mathscr{G}(I;X))_n.$$

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Since $v_{\cdot} \circ u_{\cdot} = 0$, then $v \circ u = 0$. Since u_0 is injective, then $\ker v \subset \operatorname{im} u$. Since $H_0(\mathscr{K}(\mathscr{R}(I;X))_{n+1}) = 0$, then v is surjective. Thus

$$E(I;X)_{n+1} \xrightarrow{u} E(I;X)_n \xrightarrow{v} E(\mathscr{G}(I;X))_n \longrightarrow 0$$

is an exact sequence of A-modules. For $X = P$ we get the exact sequence of A-modules

$$E(I;P)_{n+1} \xrightarrow{u} E(I;P)_n \xrightarrow{v} E(\mathscr{G}(I;P))_n \longrightarrow 0.$$

Take $X = M$ in

$$\mathscr{K}(\mathscr{R}(I;X))_{n+1} \xrightarrow{u_{\cdot}} \mathscr{K}(\mathscr{R}(I;X))_n \xrightarrow{v_{\cdot}} \mathscr{K}(\mathscr{G}(I;X))_n \longrightarrow 0$$

and tensor it by N. Then we get the exact sequence of complexes

$$\mathscr{K}(\mathscr{R}(I;M))_{n+1} \otimes N \xrightarrow{\alpha_{\cdot} = u_{\cdot} \otimes 1} \mathscr{K}(\mathscr{R}(I;M))_n \otimes N \xrightarrow{\beta_{\cdot} = v_{\cdot} \otimes 1} \mathscr{K}(\mathscr{G}(I;M))_n \otimes N \longrightarrow 0.$$

That is, we obtain the exact sequence

$$\mathscr{K}(\mathscr{R}(I;M) \otimes N)_{n+1} \xrightarrow{\alpha_{\cdot}} \mathscr{K}(\mathscr{R}(I;M) \otimes N)_n \xrightarrow{\beta_{\cdot}} \mathscr{K}(\mathscr{G}(I;M) \otimes N)_n \longrightarrow 0,$$

which induces the morphisms in homology

$$H_1(\mathscr{K}(\mathscr{R}(I;M) \otimes N)_{n+1}) \xrightarrow{\alpha} H_1(\mathscr{K}(\mathscr{R}(I;M) \otimes N)_n) \xrightarrow{\beta} H_1(\mathscr{K}(\mathscr{G}(I;M) \otimes N)_n).$$

Again, by [**8**, Proposition 2.6], $H_1(\mathscr{K}(\mathscr{R}(I;M) \otimes N)_n) = E(\mathscr{R}(I;M) \otimes N)_n$ and $H_1(\mathscr{K}(\mathscr{G}(I;M) \otimes N)_n) = E(\mathscr{G}(I;M) \otimes N)_n$. Moreover, since $\beta_{\cdot} \circ \alpha_{\cdot} = 0$, then $\beta \circ \alpha = 0$, and since $H_0(\mathscr{K}(\mathscr{R}(I;M) \otimes N)_{n+1}) = 0$, then β is an epimorphism. Thus we have

$$E(\mathscr{R}(I;M) \otimes N)_{n+1} \xrightarrow{\alpha} E(\mathscr{R}(I;M) \otimes N)_n \xrightarrow{\beta} E(\mathscr{G}(I;M) \otimes N)_n \longrightarrow 0$$

with $\beta \circ \alpha = 0$ and β surjective. Remark that since we do not know if $\alpha_0 = u_0 \otimes 1$ is injective, we cannot deduce that $\ker \beta \subset \operatorname{im} \alpha$. On the other hand, consider $g : \mathbf{S}(I) \otimes M \otimes N \longrightarrow \mathscr{R}(I;M) \otimes N$ and $\bar{g} : \mathbf{S}(I/I^2) \otimes M \otimes N \longrightarrow \mathscr{G}(I;M) \otimes N$ the natural surjective graded morphisms of standard modules, where $\mathbf{S}(I), \mathbf{S}(I/I^2)$ stand for the symmetric algebras of I and I/I^2, respectively. By [**8**, Lemma 2.3], for each $n \geqslant 2$, there exist exact sequences of A-modules $E(g)_n \longrightarrow E(f \circ g)_n \longrightarrow E(f)_n \longrightarrow 0$ and $E(\bar{g})_n \longrightarrow E(\bar{f} \circ \bar{g})_n \longrightarrow E(\bar{f})_n \longrightarrow 0$. In other words, we have exact sequences

$$E(\mathscr{R}(I;M) \otimes N)_n \longrightarrow E(\mathscr{R}(I;P))_n \longrightarrow E(f)_n \longrightarrow 0$$

and

$$E(\mathscr{G}(I;M) \otimes N)_n \longrightarrow E(\mathscr{G}(I;P))_n \longrightarrow E(\bar{f})_n \longrightarrow 0.$$

Consider the commutative diagram of exact columns shown in Figure 3. The commutativity induces two morphisms $\xi : E(f)_{n+1} \longrightarrow E(f)_n$ and $\mu : E(f)_n \longrightarrow E(\bar{f})_n$. Since $v \circ u = 0$, then $\mu \circ \xi = 0$. Since v is surjective, then μ is surjective too. Since β is surjective and the middle row is exact, then $\ker \mu \subset \operatorname{im} \xi$. Therefore,

$$E(f)_{n+1} \xrightarrow{\xi} E(f)_n \xrightarrow{\mu} E(\bar{f})_n \longrightarrow 0$$

is an exact sequence of A-modules. Finally, if A is noetherian and M, N and P are finitely generated, then $E(f)_n = 0$ for $n \gg 0$ big enough. □

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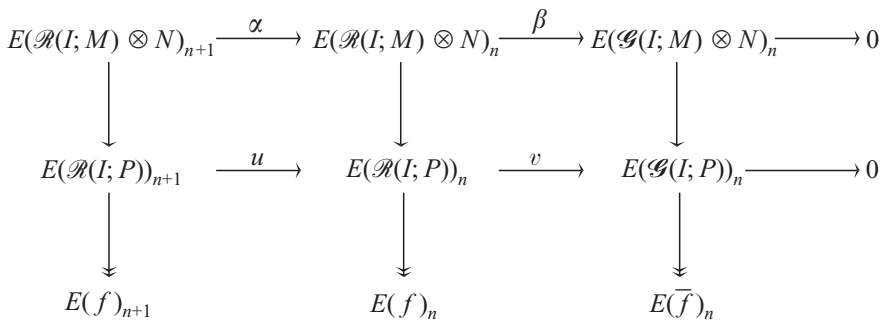


FIGURE 3.

THEOREM 2.5. *Let A be a noetherian ring, I, J be two ideals of A and M be a finitely generated A-module. The following two conditions are equivalent:*

(i) $\bar{\varphi}:\mathscr{G}(I)\otimes\mathscr{G}(J;M)\longrightarrow\mathscr{G}(I+J;M)$ *is an isomorphism.*

(ii) $\mathrm{Tor}_1(A/I^p,\mathscr{R}(J;M))=0$ *and* $\mathrm{Tor}_1(A/I^p,\mathscr{G}(J;M))=0$ *for all integers* $p\geqslant 1$.

In particular, $\mathscr{G}(I)\otimes\mathscr{G}(J)\simeq\mathscr{G}(I+J)$ *if and only if* $\mathrm{Tor}_1(A/I^p,A/J^q)=0$ *and* $\mathrm{Tor}_2(A/I^p,A/J^q)=0$ *for all integers* $p,q\geqslant 1$.

Proof. Remark that $\mathrm{Tor}_1(A/I^p,J^qM)=\ker(\pi_{p,q}:I^p\otimes J^qM\longrightarrow I^pJ^qM)$. Moreover, under the hypothesis $\mathrm{Tor}_1(A/I^p,\mathscr{R}(J;M))=0$ for all $p\geqslant 1$, then the following two conditions are equivalent:

(1) $\mathrm{Tor}_1(A/I^p,\mathscr{G}(J;M))=0$ for all $p\geqslant 1$.

(2) $I^pM\cap J^qM=I^pJ^qM$ for all $p,q\geqslant 1$.

Suppose that (ii) holds, that is, $\mathrm{Tor}_1(A/I^p,J^qM)=0$ and $I^pM\cap J^qM=I^pJ^qM$ for all $p,q\geqslant 1$. Then, $\pi:\mathscr{R}(I)\otimes\mathscr{R}(J;M)\longrightarrow\mathscr{R}(I,J;M)$ is an isomorphism and, by Lemma 2.2, $\bar{\sigma}:\mathscr{G}(I,J;M)\longrightarrow\mathscr{G}(I+J;M)$ is an isomorphism. Thus $\bar{\varphi}=\bar{\sigma}\circ\bar{\pi}$ is an isomorphism and (i) holds. Let us now prove (i) $\Rightarrow$ (ii). If $\bar{\varphi}=\bar{\sigma}\circ\bar{\pi}$ is an isomorphism, then $\bar{\sigma}$ and $\bar{\pi}$ are two isomorphisms. By Proposition 2.3, $\bar{\sigma}$ an isomorphism implies that $I^pM\cap J^qM=I^pJ^qM$ for all $p,q\geqslant 1$. In particular,

$$\mathscr{R}(I;J^qM/J^{q+1}M)_p=\frac{I^pJ^qM+J^{q+1}M}{J^{q+1}M}=\frac{I^pJ^qM}{I^pJ^qM\cap J^{q+1}M}=\frac{I^pJ^qM}{I^pJ^{q+1}M}=\mathscr{G}(J;I^pM)_q$$

and

$$\mathscr{G}(I;J^qM/J^{q+1}M)_p=\frac{I^pJ^qM+J^{q+1}M}{I^{p+1}J^qM+J^{q+1}M}=\frac{I^pJ^qM}{(I+J)\,I^pJ^qM}=\mathscr{G}(I,J;M)_{p,q}.$$

Fix $q\geqslant 1$. Since $\bar{\pi}_{p,q}:\mathscr{G}(I)_p\otimes\mathscr{G}(J;M)_q\longrightarrow\mathscr{G}(I,J;M)_{p,q}$ is an isomorphism for all $p\geqslant 1$ and $\mathscr{G}(I,J;M)_{p,q}=\mathscr{G}(I;J^qM/J^{q+1}M)_p$, then $\bar{\pi}_{*,q}:\mathscr{G}(I)\otimes J^qM/J^{q+1}M\longrightarrow\mathscr{G}(I;J^qM/J^{q+1}M)$ is an isomorphism for all $q\geqslant 1$. By Proposition 2.4, $\mathscr{R}(I)\otimes J^qM/J^{q+1}M\longrightarrow\mathscr{R}(I;J^qM/J^{q+1}M)$ is an isomorphism for all $q\geqslant 1$. Since $I^p\otimes J^qM/J^{q+1}M\longrightarrow\mathscr{R}(I;J^qM/J^{q+1}M)_p$ is an isomorphism for all $p,q\geqslant 1$ and $\mathscr{R}(I;J^qM/J^{q+1}M)_p=\mathscr{G}(J;I^pM)_q$, then $I^p\otimes\mathscr{G}(J;M)\longrightarrow\mathscr{G}(J;I^pM)$ is an isomorphism for all $p\geqslant 1$. By Proposition 2.4 again, $I^p\otimes\mathscr{R}(J;M)\longrightarrow\mathscr{R}(J;I^pM)$ is an isomorphism for all $p\geqslant 1$, so $\pi:\mathscr{R}(I)\otimes\mathscr{R}(J;M)\longrightarrow\mathscr{R}(I,J;M)$ is an isomorphism and $\mathrm{Tor}_1(A/I^p,\mathscr{R}(J,M))=0$ for all $p\geqslant 1$. Since $\mathrm{Tor}_1(A/I^p,\mathscr{R}(J;M))=0$ for all $p\geqslant 1$ and $I^pM\cap J^qM=I^pJ^qM$ for all $p,q\geqslant 1$, then $\mathrm{Tor}_1(A/I^p,\mathscr{G}(J;M))=0$ for all $p\geqslant 1$. □

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3. *Some examples*

EXAMPLE 3.1. Let A be a noetherian local ring, I, J be two ideals of A and M be a finitely generated A-module. If $I = (x)$ is principal and x is A-regular, then $\bar{\varphi}: \mathscr{G}(I) \otimes \mathscr{G}(J; M) \longrightarrow \mathscr{G}(I+J; M)$ is an isomorphism if and only if x is a non-zero divisor in $\mathscr{R}(J; M)$ and in $\mathscr{G}(J; M)$. Indeed, let $\mathscr{K}(y; N)$ denote the Koszul complex of a sequence of elements $y = y_1, \ldots, y_m$ of A with respect to an A-module N and let $H_i(y; N)$ denote its ith Koszul homology group. Then $\operatorname{Tor}_1(A/I, N) = H_1(\mathscr{K}(x; A) \otimes N) = H_1(x; N) = 0$ if and only if x is a non-zero divisor in N.

EXAMPLE 3.2. Let A be a noetherian local ring and let $I = (x)$ and $J = (y)$ be two principal ideals of A. If $(0:x) \subset (y)$ and $(0:y) \subset (x)$, then $\bar{\varphi}: \mathscr{G}(I) \otimes \mathscr{G}(J) \longrightarrow \mathscr{G}(I+J)$ is an isomorphism if and only if x, y is an A-regular sequence.

EXAMPLE 3.3. Let R be a noetherian local ring and let z, t be an R-regular sequence. Let $A = R/(zt)$, $x = z+(zt)$, $y = t+(zt)$, $I = (x)$ and $J = (y)$. Then $\bar{\sigma}: \mathscr{G}(I, J) \longrightarrow \mathscr{G}(I+J)$ is an isomorphism, but $\bar{\pi}: \mathscr{G}(I) \otimes \mathscr{G}(J) \longrightarrow \mathscr{G}(I, J)$ is not an isomorphism.

An example of a pair of ideals I, J with the property $\operatorname{Tor}_1(A/I^p, A/J^q) = 0$ for all integers $p, q \geqslant 1$ arises from a product of affine varieties (see [**10**, pp. 130–136, especially Proposition 5.5.7]). The next result is well known (see, for instance, [**4**, 21.9, 21.11]). We give a proof here for the sake of completeness.

PROPOSITION 3.4. *Let A be a noetherian local ring, I and J be two ideals of A and M be a finitely generated A-module. Let $x = x_1, \ldots, x_r$ be a system of generators of I and $y = y_1, \ldots, y_r, y_i = \bar{x}_i = x_i + J$, be a system of generators of the ideal $\bar{I} = I+J/J$ of the quotient ring $\bar{A} = A/J$. If $\mathscr{G}(J)$ and $\mathscr{G}(J; M)$ are free $\bar{A}$-modules and y is an $\bar{A}$-regular sequence in $\bar{I}$, then x is an A-regular sequence in I and $\bar{\varphi}: \mathscr{G}(I) \otimes \mathscr{G}(J; M) \longrightarrow \mathscr{G}(I+J; M)$ is an isomorphism.*

Proof. Since $J^q M/J^{q+1}M$ is $\bar{A}$-free for all $q \geqslant 1$ and y is an $\bar{A}$-regular sequence, then y is a $J^q M/J^{q+1}M$-regular sequence in $\bar{I}$ for all $q \geqslant 1$. In particular, x is a $J^q M/J^{q+1}M$-regular sequence in I for all $q \geqslant 1$. Thus x is an $M/J^q M$-regular sequence in I for all $q \geqslant 1$ and x is an M-regular sequence in I. Analogously, but using the hypothesis $\mathscr{G}(J)$ is $\bar{A}$-free, we deduce that x is an A-regular sequence in I. Therefore $\operatorname{Tor}_i(A/I, M) = 0$ and $\operatorname{Tor}_i(A/I, M/J^q M) = 0$ for all $i, q \geqslant 1$. Using the long exact sequences in homology associated to the short exact sequences

$$0 \longrightarrow J^q M \longrightarrow M \longrightarrow M/J^q M \longrightarrow 0$$

and

$$0 \longrightarrow J^q M/J^{q+1}M \longrightarrow M/J^{q+1}M \longrightarrow M/J^q M \longrightarrow 0,$$

we deduce that $\operatorname{Tor}_1(A/I, \mathscr{R}(J; M)) = 0$ and $\operatorname{Tor}_1(A/I, \mathscr{G}(J; M)) = 0$. Since I^p/I^{p+1} is A/I-free, then $\operatorname{Tor}_1(I^p/I^{p+1}, \mathscr{R}(J; M)) = \operatorname{Tor}_1(A/I, \mathscr{R}(J; M)) \otimes I^p/I^{p+1} = 0$ and $\operatorname{Tor}_1(I^p/I^{p+1}, \mathscr{G}(J; M)) = \operatorname{Tor}_1(A/I, \mathscr{G}(J; M)) \otimes I^p/I^{p+1} = 0$. Applying the long exact sequences in homology to the short exact sequences $0 \longrightarrow I^p/I^{p+1} \longrightarrow A/I^{p+1} \longrightarrow A/I^p \longrightarrow 0$, we deduce that $\operatorname{Tor}_1(A/I^p, \mathscr{R}(J; M)) = 0$ and $\operatorname{Tor}_1(A/I^p, \mathscr{G}(J; M)) = 0$ for all $p \geqslant 1$. □

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4. *Relation type of tensor products*

LEMMA 4.1. *Let U be a standard A-algebra and F be a standard U-module. If M is an A-module, then $F \otimes M$ is a standard U-module and* $\mathrm{rt}(F \otimes M) \leqslant \mathrm{rt}(F)$. *If $\lambda : M \longrightarrow N$ is an epimorphism of A-modules, then $1 \otimes \lambda : F \otimes M \longrightarrow F \otimes N$ is a surjective graded morphism of standard U-modules. Moreover, for each integer $n \geqslant 1$,* $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$. *In particular, for each $n \geqslant 2$, there exists an epimorphism of A-modules $E(F \otimes M)_n \longrightarrow E(F \otimes N)_n$ and* $\mathrm{rt}(F \otimes N) \leqslant \mathrm{rt}(F \otimes M)$.

Proof. Clearly $F \otimes M$ is a standard U-module and $1 \otimes \lambda : F \otimes M \longrightarrow F \otimes N$ is a surjective graded morphism of standard U-modules. Consider the symmetric presentation $\gamma : \mathbf{S}(U_1) \otimes F_0 \longrightarrow F$ of F. If we tensor

$$0 \longrightarrow \ker \gamma_i \longrightarrow \mathbf{S}_i(U_1) \otimes F_0 \xrightarrow{\gamma_i} F_i \longrightarrow 0$$

by M we get exact sequences

$$\ker \gamma_i \otimes M \longrightarrow \mathbf{S}_i(U_1) \otimes F_0 \otimes M \xrightarrow{\gamma_i \otimes 1} F_i \otimes M \longrightarrow 0.$$

They induce natural epimorphisms $\ker \gamma_i \otimes M \longrightarrow \ker(\gamma_i \otimes 1_M)$. On the other hand, if we tensor $U_1 \otimes \ker \gamma_{n-1} \longrightarrow \ker \gamma_n \longrightarrow E(\gamma)_n \longrightarrow 0$ by M we get the exact sequence

$$U_1 \otimes \ker \gamma_{n-1} \otimes M \longrightarrow \ker \gamma_n \otimes M \longrightarrow E(\gamma)_n \otimes M \longrightarrow 0.$$

We thus have the commutative diagram of exact rows shown in Figure 4 from which we deduce epimorphisms $E(\gamma)_n \otimes M \longrightarrow E(\gamma \otimes 1_M)_n$ for all $n \geqslant 1$. In particular, $\mathrm{rt}(F \otimes M) = s(E(\gamma \otimes 1_M)) \leqslant s(E(\gamma)) = \mathrm{rt}(F)$. Consider now, for each $n \geqslant 1$, the commutative diagram of exact columns and rows shown in Figure 5. Using a diagram-chasing argument, one deduces that $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$ for all $n \geqslant 1$. Consider the symmetric presentation $\gamma \otimes 1 : \mathbf{S}(U_1) \otimes F_0 \otimes M \longrightarrow F \otimes M$ of $F \otimes M$. By [**8**, Lemma 2.3], there exists an exact sequence of A-modules $E(\gamma \otimes 1)_n \longrightarrow E((1 \otimes \lambda) \circ (\gamma \otimes 1))_n \longrightarrow E(1 \otimes \lambda)_n \longrightarrow 0$ for all $n \geqslant 1$. However $E(\gamma \otimes 1)_n = E(F \otimes M)_n$, $E((1 \otimes \lambda) \circ (\gamma \otimes 1))_n = E(F \otimes N)_n$ and $E(1 \otimes \lambda)_n = 0$ for all $n \geqslant 2$. Thus $E(F \otimes M)_n \longrightarrow E(F \otimes N)_n$ is surjective for all $n \geqslant 2$ and $\mathrm{rt}(F \otimes N) \leqslant \mathrm{rt}(F \otimes M)$. □

THEOREM 4.2. *Let A be a commutative ring, U and V be two standard A-algebras, and F be a standard U-module and G be a standard V-module. Then $U \otimes V$ is a standard A-algebra, $F \otimes G$ is a standard $U \otimes V$-module and* $\mathrm{rt}(F \otimes G) \leqslant \max(\mathrm{rt}(F), \mathrm{rt}(G))$.

Proof. Clearly $U \otimes V$ is a standard A-algebra and $F \otimes G$ is a standard $U \otimes V$-module. Take $\varphi : X \longrightarrow F$ and $\psi : Y \longrightarrow G$ as two symmetric presentations of F and G, respectively. Then $\varphi \otimes \psi : X \otimes Y \longrightarrow F \otimes G$ is a symmetric presentation of $F \otimes G$. Since $\varphi \otimes \psi = (\varphi \otimes 1_G) \circ (1_X \otimes \psi)$, then, for each integer $n \geqslant 2$, there exists an exact sequence of A-modules

$$E(1_X \otimes \psi)_n \longrightarrow E(\varphi \otimes \psi)_n \longrightarrow E(\varphi \otimes 1_G)_n \longrightarrow 0.$$

Since $\psi : Y \longrightarrow G$ is a symmetric presentation of G, then $1_{X_0} \otimes \psi : X_0 \otimes Y \longrightarrow X_0 \otimes G$ is a symmetric presentation of $X_0 \otimes G$ and $E(X_0 \otimes G)_n = E(1_{X_0} \otimes \psi)_n$ for

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$$\begin{array}{ccccccc}
U_1 \otimes \ker \gamma_{n-1} \otimes M & \longrightarrow & \ker \gamma_n \otimes M & \longrightarrow & E(\gamma)_n \otimes M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
U_1 \otimes \ker(\gamma_{n-1} \otimes 1_M) & \longrightarrow & \ker(\gamma_n \otimes 1_M) & \longrightarrow & E(\gamma \otimes 1_M)_n & \longrightarrow & 0
\end{array}$$

FIGURE 4.

$$\begin{array}{ccccccc}
 & & (\ker\partial_n) \otimes M & \xrightarrow{1 \otimes \lambda} & (\ker\partial_n) \otimes N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
U_1 \otimes \ker(1_{F_{n-1}} \otimes \lambda) & \longrightarrow & U_1 \otimes F_{n-1} \otimes M & \xrightarrow{1 \otimes 1 \otimes \lambda} & U_1 \otimes F_{n-1} \otimes N & \longrightarrow & 0 \\
\downarrow & & \downarrow \partial_n \otimes 1_M & & \downarrow \partial_n \otimes 1_N & & \\
0 \longrightarrow \ker(1_{F_n} \otimes \lambda) & \longrightarrow & F_n \otimes M & \xrightarrow[1 \otimes \lambda]{} & F_n \otimes N & \longrightarrow & 0
\end{array}$$

FIGURE 5.

all $n \geqslant 2$. Using Lemma 4.1, $\ker(1_{X_i} \otimes \psi_{n-i}) = U_1 \cdot \ker(1_{X_{i-1}} \otimes \psi_{n-i})$ for all $i \geqslant 1$. Then

$$\begin{aligned}
E(1_X \otimes \psi)_n &= \frac{\ker(1_X \otimes \psi)_n}{(U \otimes V)_1 \cdot \ker(1_X \otimes \psi)_{n-1}} \\
&= \frac{\bigoplus_{i=0}^{n} \ker(1_{X_i} \otimes \psi_{n-i})}{\left(\bigoplus_{i=0}^{n-1} U_1 \cdot \ker(1_{X_i} \otimes \psi_{n-i})\right) + \left(\bigoplus_{i=0}^{n-1} V_1 \cdot \ker(1_{X_i} \otimes \psi_{n-i})\right)} \\
&= \frac{\ker(1_{X_0} \otimes \psi_n)}{V_1 \cdot \ker(1_{X_0} \otimes \psi_{n-1})} \oplus \frac{\ker(1_{X_1} \otimes \psi_{n-1})}{U_1 \cdot \ker(1_{X_0} \otimes \psi_{n-1}) + V_1 \cdot \ker(1_{X_1} \otimes \psi_{n-2})} \oplus \cdots \\
&\quad \oplus \frac{\ker(1_{X_{n-1}} \otimes \psi_1)}{U_1 \cdot \ker(1_{X_{n-2}} \otimes \psi_1) + V_1 \cdot \ker(1_{X_{n-1}} \otimes \psi_0)} \oplus \frac{\ker(1_{X_n} \otimes \psi_0)}{U_1 \cdot \ker(1_{X_{n-1}} \otimes \psi_0)} \\
&= E(1_{X_0} \otimes \psi)_n.
\end{aligned}$$

Therefore $E(1_X \otimes \psi)_n = E(1_{X_0} \otimes \psi)_n = E(X_0 \otimes G)_n$ for all $n \geqslant 2$. Analogously, $E(\varphi \otimes 1_G)_n = E(\varphi \otimes 1_{G_0})_n = E(F \otimes G_0)_n$ for all $n \geqslant 2$. Hence there exists an exact sequence of A-modules

$$E(X_0 \otimes G)_n \longrightarrow E(F \otimes G)_n \longrightarrow E(F \otimes G_0)_n \longrightarrow 0$$

for all $n \geqslant 2$ and, by Lemma 4.1, $\mathrm{rt}(F \otimes G) \leqslant \max(\mathrm{rt}(F \otimes G_0), \mathrm{rt}(X_0 \otimes G)) \leqslant \max(\mathrm{rt}(F), \mathrm{rt}(G))$. □

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REMARK 4.3. Let A be a commutative ring and let U and V be two standard A-algebras. If $\mathrm{Tor}_1^A(U,V)=0$, then $E(U\otimes V)=E(U)\oplus E(V)$. This follows from the characterization $E(U)=H_1(A,U,A)$ (see [**7**, Remark 2.3; **1**, Proposition 19.3]).

5. *Uniform bounds*

LEMMA 5.1. *Let $(A,\mathfrak{m})$ be a noetherian local ring and M be a finitely generated A-module. Let $\mathfrak{p}$ be a prime ideal of A such that $A/\mathfrak{p}$ is regular local and $\mathcal{G}(\mathfrak{p})$ and $\mathcal{G}(\mathfrak{p};M)$ are free $A/\mathfrak{p}$-modules. Then* $\mathrm{rt}(\mathfrak{m};M)\leqslant \mathrm{rt}(\mathfrak{p};M)$.

Proof. Since $A/\mathfrak{p}$ is regular local, there exists a sequence of elements $x=x_1,\dots,x_r$ in A such that $y=y_1,\dots,y_r$, defined by $y_i=x_i+\mathfrak{p}$, is a system of generators of $\mathfrak{m}/\mathfrak{p}$ and an $\bar{A}$-regular sequence. Let I be the ideal of A generated by x. In particular, $I+\mathfrak{p}/\mathfrak{p}=\mathfrak{m}/\mathfrak{p}$ and $I+\mathfrak{p}=\mathfrak{m}$. By Proposition 3.4, x is an A-regular sequence and $\mathrm{Tor}_1(A/I^p,\mathcal{R}(\mathfrak{p};M))=0$ and $\mathrm{Tor}_1(A/I^p,\mathcal{G}(\mathfrak{p};M))=0$ for all $p\geqslant 1$. By Theorem 2.5, $\bar{\varphi}:\mathcal{G}(I)\otimes\mathcal{G}(\mathfrak{p};M)\longrightarrow\mathcal{G}(\mathfrak{m};M)$ is an isomorphism. By Theorem 4.2, $\mathrm{rt}(\mathcal{G}(\mathfrak{m};M))\leqslant \max(\mathrm{rt}(\mathcal{G}(I)),\mathrm{rt}(\mathcal{G}(\mathfrak{p};M)))$. By [**8**, Remark 2.7], $\mathrm{rt}(\mathcal{G}(J;M))=\mathrm{rt}(J;M)$ for any ideal J of A. Since I is generated by a regular sequence, then $\mathrm{rt}(I)=1$ (see, for instance, [**10**, p. 30]). Thus $\mathrm{rt}(\mathfrak{m};M)\leqslant\mathrm{rt}(\mathfrak{p};M)$. □

The next result is a slight generalization of a well-known theorem of Duncan and O'Carroll [**2**]. In fact the proof of our theorem is directly inspired by theirs. We sketch it here for the sake of completeness.

THEOREM 5.2. *Let A be an excellent (or $J-2$) ring and let M be a finitely generated A-module. Then there exists an integer $s\geqslant 1$ such that, for all maximal ideals $\mathfrak{m}$ of A, the relation type of $\mathfrak{m}$ with respect to M satisfies* $\mathrm{rt}(\mathfrak{m};M)\leqslant s$.

Proof. For every $\mathfrak{p}\in\mathrm{Spec}(A)$, let us construct a non-empty open subset $U(\mathfrak{p})$ of $V(\mathfrak{p})=\{\mathfrak{q}\in\mathrm{Spec}(A)\mid\mathfrak{q}\supseteq\mathfrak{p}\}\simeq\mathrm{Spec}(A/\mathfrak{p})$. Remark that $A/\mathfrak{p}$ is a noetherian domain, $\mathcal{G}(\mathfrak{p})$ is a finitely generated $A/\mathfrak{p}$-algebra and $\mathcal{G}(\mathfrak{p};M)$ is a finitely generated $\mathcal{G}(\mathfrak{p})$-module. By generic flatness [**6**, Theorem 22.A], there exist $f,g\in A-\mathfrak{p}$ such that $\mathcal{G}(\mathfrak{p})_f$ is an $(A/\mathfrak{p})_f$-free module and $\mathcal{G}(\mathfrak{p};M)_g$ is an $(A/\mathfrak{p})_g$-free module. Since A is $J-2$, the set $\mathrm{Reg}(A/\mathfrak{p})=\{\mathfrak{q}\in V(\mathfrak{p})\mid(A/\mathfrak{p})_\mathfrak{q}\text{ is regular local}\}$ is a non-empty open subset of $V(\mathfrak{p})$. Define $U(\mathfrak{p})$ as the intersection $D(f)\cap D(g)\cap\mathrm{Reg}(A/\mathfrak{p})=\{\mathfrak{q}\in V(\mathfrak{p})\mid\mathfrak{q}\not\supseteq f,\ \mathfrak{q}\not\supseteq g,\ (A/\mathfrak{p})_\mathfrak{q}\text{ is regular local}\}$, which is a non-empty open subset of $V(\mathfrak{p})$. Remark that for all $\mathfrak{q}\in U(\mathfrak{p})$, $(A/\mathfrak{p})_\mathfrak{q}$ is regular local and $\mathcal{G}(\mathfrak{p})_\mathfrak{q}$ and $\mathcal{G}(\mathfrak{p};M)_\mathfrak{q}$ are free $\mathcal{G}(\mathfrak{p})_\mathfrak{q}$-modules. By Lemma 5.1, $\mathrm{rt}(\mathfrak{q}A_\mathfrak{q};M_\mathfrak{q})\leqslant\mathrm{rt}(\mathfrak{p}A_\mathfrak{q};M_\mathfrak{q})\leqslant\mathrm{rt}(\mathfrak{p};M)$ for all $\mathfrak{q}\in U(\mathfrak{p})$. In particular, $\mathrm{rt}(\mathfrak{m};M)\leqslant\mathrm{rt}(\mathfrak{p};M)$ for all maximal ideals $\mathfrak{m}\in U(\mathfrak{p})$. For each minimal prime $\mathfrak{p}_i$ of A, let $V(\mathfrak{p}_i)-U(\mathfrak{p}_i)=V(\mathfrak{p}_{i,1})\cup\dots\cup V(\mathfrak{p}_{i,r_i})$ be the decomposition into irreducible closed subsets of the proper closed subset $V(\mathfrak{p}_i)-U(\mathfrak{p}_i)$, $\mathfrak{p}_{i,j}\in\mathrm{Spec}(A)$, $\mathfrak{p}_{i,j}\supsetneq\mathfrak{p}_i$. Then $\mathrm{Spec}(A)=\bigcup_i V(\mathfrak{p}_i)=\bigcup_i U(\mathfrak{p}_i)\cup(V(\mathfrak{p}_i)-U(\mathfrak{p}_i))=\bigcup_{i,j}U(\mathfrak{p}_i)\cup V(\mathfrak{p}_{i,j})=\bigcup_{i,j}U(\mathfrak{p}_i)\cup U(\mathfrak{p}_{i,j})\cup(V(\mathfrak{p}_{i,j})-U(\mathfrak{p}_{i,j}))$. Since A is noetherian, one deduces that $\mathrm{Spec}(A)$ can be covered by finitely many locally closed sets of type $U(\mathfrak{p})$, that is, there exist a finite number of prime ideals $\mathfrak{q}_1,\dots,\mathfrak{q}_m$, such that $\mathrm{Spec}(A)=\bigcup_{i=1}^m U(\mathfrak{q}_i)$. Hence, $\mathrm{rt}(\mathfrak{m};M)\leqslant\max\{\mathrm{rt}(\mathfrak{q}_i;M)\mid i=1,\dots,m\}$ for any maximal ideal $\mathfrak{m}$ of A. □

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Using [**8**, Theorem 2] we deduce the result of Duncan and O'Carroll in [**2**].

COROLLARY 5.3 [**2**]. *Let A be an excellent (or $J-2$) ring and let $N \subseteq M$ be two finitely generated A-modules. Then there exists an integer $s \geqslant 1$ such that, for all integers $n \geqslant s$ and for all maximal ideals $\mathfrak{m}$ of A, $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-s}(\mathfrak{m}^s M \cap N)$.*

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