NOTE ON A THEOREM ENUNCIATED BY TODD

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J. A. Todd* has enunciated the following theorem:

"If the canonical adjoints Σ of an irreducible curve Γ contain fixed parts, then, if Γ is not elliptic, all the fixed curves are simple, rational, determined uniquely by the base points, which present independent conditions for them, and have no free intersections with each other or with the variable part of the system Σ."

This is as much as to say that the fixed curves are exceptional curves (of the first kind), and can be transformed simultaneously into the neighbourhoods of a set of distinct simple points of the new surface.

Unfortunately the theorem, in this form, is certainly untrue, as can be seen from a consideration of the following simple example:

Let O_1, O_2, O_3 be on a line l, and O_4, O_5 on another line l', and let Γ be a general 12-ic, having 4-ple points at O_1, O_2, O_3, and 6-ple points at O_4, O_5. The system $|\Sigma|$ consists of 9-ics, with triple points at O_1, O_2, O_3, and 5-ple points at O_4, O_5; this system has negative intersection number with l', which is accordingly a fixed part; but $|\Sigma - l'|$ has negative intersection number with l, so that l is also a fixed part; $|\Sigma - l' - l|$ has again negative intersection number with l', so that altogether $l + 2l'$ is the fixed part of $|\Sigma|$. Here the fixed part l' is not simple, the fixed part l is redundantly determined by the base points on it, and the two have a free intersection. Thus the theorem is incorrect.

It is obvious, however, that the two curves l, l', are together exceptional, in the sense that they can be transformed (e.g. by the homaloidal net of cubics, having a double base point at O_5, and simple base points at the other four points) into the neighbourhoods of two points, one of which (that corresponding to l') is in the neighbourhood of the other. It may thus be hoped that the theorem can be salvaged in the following form:

"If $|\Sigma|$ has fixed parts, these are an exceptional set of curves";

i.e. the plane can be transformed into a surface on which these curves, say r in number, appear as the neighbourhoods of r points, simple for the surface, of which however some may be in the neighbourhoods of others.

* J. A. Todd, "On the determination of plane curves by means of assigned singularities", *Proc. Camb. Phil. Soc.* 27 (1931), 291–305 (298).

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Of this a proof can be constructed without difficulty if we assume that the actual postulation (namely 0) of the total fixed part of $|\Sigma|$ for $|\Sigma|$ is equal to its virtual postulation. But this assumption is quite unjustified; it is in fact an assumption equivalent to this (for the single fixed curve) which seems to be the flaw in Todd's proof (namely the equation (3·4) on p. 297). And that the assumption is no more legitimate when we consider the total fixed part can again be seen by an example: Let $(O_1, \ldots, O_9)$ be nine associated points, and let l be the line O_1O_2. If now Γ is a general octavic curve, with 4-ple points at O_1, O_2 and double points at $(O_3, \ldots, O_9)$, Σ consists of quintics with triple base-points at O_1, O_2 and simple base-points at $(O_3, \ldots, O_9)$; this evidently contains l as part; but since $|\Sigma|$ is a pencil—Γ being of genus 2—and the curves consisting of $2l$ and a cubic through $(O_1, \ldots, O_9)$ are already a pencil, they are the whole system, and $|\Sigma|$ has $2l$ as a fixed part, though the virtual postulation of this is easily seen to be 1, not 0.

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