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Integral inequalities for increasing functions

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Abstract. For numbers of increasing real functions $f(x)$ with $\int_{-1}^{+1} f(x)\,dx \geqslant 0$ we give new integral inequalities. They generalize classical results. The proofs are short and simple being based on sequences.

1. *Introduction.* Let E be the set of all real functions $f(x)$ defined and increasing for $-1 \leqslant x \leqslant 1$. Let F be the set of members of E with

$$0 \leqslant \int_{-1}^{1} f = \int_{-1}^{1} f(x)\,dx. \tag{1}$$

Also let G be the subset of F with equality in (1). Our main results are:

THEOREM 1. *If* $f_1, \ldots, f_r \in E$ *and*

$$0 \leqslant f_i(0) + \int_0^1 f_i(x)\,dx \quad \text{for} \quad 1 \leqslant i \leqslant r \tag{2}$$

then

$$0 \leqslant \left(\int_0^1 f_1\right) \ldots \left(\int_0^1 f_r\right) \leqslant \int_0^1 f_1 \ldots f_r.$$

Notice that for f_i defined and increasing for $0 \leqslant x \leqslant 1$ the condition (2) simply requires that f_i can be extended to lie in F.

THEOREM 2. *If* r *is even and* $f_1, \ldots, f_r \in G$ *then*

$$0 \leqslant \int_{-1}^{1} f_1 \ldots f_r.$$

THEOREM 3. *If* $f_1, \ldots, f_r, g_1, \ldots, g_s \in F$ *and* $0 \leqslant \theta \leqslant 1$ *then*

$$\left(\int_\theta^1 f_1 \ldots f_r\right)\left(\int_\theta^1 g_1 \ldots g_s\right) \leqslant (1-\theta)\int_\theta^1 f_1 \ldots f_r g_1 \ldots g_s.$$

THEOREM 4. *If* r, s *are odd and* $f_1, \ldots, f_r, g_1, \ldots, g_s \in G$ *then*

$$\left(\int_{-1}^1 f_1 \ldots f_r\right)\left(\int_{-1}^1 g_1 \ldots g_s\right) \leqslant \int_{-1}^1 f_1 \ldots f_r g_1 \ldots g_s.$$

These results will follow immediately from their analogues for sequences which we proceed to prove and discuss.

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2. *Finite increasing sequences.* Let n be a fixed positive integer. Abusing our notation we now let E be the set of all real sequences $f(1) \leqslant \ldots \leqslant f(n)$. We let F be the members of E with

$$0 \leqslant \Sigma f = f(1) + \ldots + f(n). \tag{3}$$

Also we let G be the subset of F with equality in (3).

THEOREM 1′. *If* $f_1, \ldots, f_r \in E$ *and*

$$0 \leqslant (n-1)f_i(1) + f_i(2) + f_i(3) + \ldots + f_i(n) \quad \textit{for} \quad 1 \leqslant i \leqslant r$$

then

$$0 \leqslant (n^{-1}\Sigma f_1) \ldots (n^{-1}\Sigma f_r) \leqslant n^{-1}\Sigma f_1 \ldots f_r.$$

The familiar Chebychev type inequality ((3), 2·17) says that if $f_1, \ldots, f_r \in E$ and are *non-negative* then for positive integers s

$$(n^{-1}\Sigma f_1^s)^{1/s} \ldots (n^{-1}\Sigma f_r^s)^{1/s} \leqslant (n^{-1}\Sigma (f_1 \ldots f_r)^s)^{1/s}.$$

Clearly this inequality follows immediately from Theorem 1′. It is more convenient to prove a slightly different form of Theorem 1′ namely

THEOREM 1″. *If* $f_1, \ldots, f_r \in F$ *and* t *is an integer in* $\frac{1}{2}n \leqslant t \leqslant n$, *then*

$$\mu_1 \ldots \mu_r \leqslant m^{-1}\Sigma f_1(x) \ldots f_r(x), \tag{4}$$

where

$$0 \leqslant \mu_i = m^{-1}\Sigma f_i(x) \quad \textit{for} \quad 1 \leqslant i \leqslant r,$$

and $m = n - t + 1$ *while summation is over* $t \leqslant x \leqslant n$.

Proof. We may assume that there is a smallest integer p in $t - 1 \leqslant p \leqslant n$ such that for each i in $1 \leqslant i \leqslant r$ we have $0 < f_i(p+1) = \ldots = f_i(n) = g_i$ say. If $t - 1 = p$ then (4) holds with equality. So assume $t - 1 < p$ and put $q = n - p$ and $h_i = f_i(p)$ and

$$k_i = (qg_i + h_i)/(q+1).$$

Then for each i because $f_i \in F$ we have $k_i \leqslant g_i$ and $0 \leqslant ph_i + qg_i$ so $|h_i| \leqslant g_i$ so $h_i \leqslant k_i$ and $0 \leqslant k_i$. We change f_i to a new function f_i^* by changing $f_i(x)$ to k_i for $p \leqslant x \leqslant n$. Then $f_i^* \in F$ and has the same μ_i as f_i. Further $\Sigma f_1^* \ldots f_r^* \leqslant \Sigma f_1 \ldots f_r$ with summation over $t \leqslant x \leqslant n$, because

$$(q+1)\,\Pi k_i \leqslant (q\Pi g_i) + \Pi h_i \tag{5}$$

with products over $1 \leqslant i \leqslant r$. The result (4) follows by repetition of this process. It is easy to prove (5) by induction on r.

If r is to be allowed to get large the condition $\frac{1}{2}n \leqslant t$ of Theorem 1″ is necessary. To see this let all $f_i \in G$ and be 1 for $\frac{1}{2}n \leqslant x \leqslant n$ and constant elsewhere.

THEOREM 2′. *If* r *is even and* $f_1, \ldots, f_r \in G$ *then* $0 \leqslant \Sigma f_1 \ldots f_r$ *with summation over* $1 \leqslant x \leqslant n$.

Proof. Split the sum at $\frac{1}{2}n$ and apply Theorem 1″ to each half.

Inversion and change of sign for each f_i shows there is no such result for r odd.

THEOREM 3′. *If* $f_1, \ldots, f_r, g_1, \ldots, g_s \in F$ *and* $\frac{1}{2}n \leqslant t \leqslant n$ *put*

$$A = \Sigma f_1 \ldots f_r, \quad B = \Sigma g_1 \ldots g_s, \quad C = \Sigma f_1 \ldots f_r g_1 \ldots g_s \tag{6}$$

with summation over $t \leqslant x \leqslant n$ *then* $AB \leqslant (n - t + 1)C$.

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Proof. We may assume $f_i(n) = g_j(n) = 1$ for all i, j. Then there will be a smallest integer p in $t-1 \leqslant p \leqslant n$ such that $f_i(x) = g_j(x) = 1$ for $p < x \leqslant n$ and all i, j. If $t-1 = p$ the result holds with equality, so assume $t-1 < p$.

Now $-1 \leqslant f_i(p), g_j(p) \leqslant 1$ for all i, j. If say, $f_1(p), f_2(p) < 0$ then we change f_1, f_2 into two new functions f_1^*, f_2^* by changing $f_1(p)$, $f_2(p)$ into $-f_1(p)$, $-f_2(p)$ respectively. Clearly $f_1^*, f_2^* \in F$ and A, B, C do not change. So we may assume $0 \leqslant f_2(p), \ldots, f_r(p)$ and that $c = f(p) < 1$ where f now denotes f_1.

Put $q = n-p$ and $d = f_2(p) \ldots f_r(p)$ and $e = g_1(p) \ldots g_s(p)$ and $b = (q+cd)/(q+d)$. Notice that $0 \leqslant d \leqslant 1$ so $-1 \leqslant c \leqslant b$ and $0 \leqslant b$, and trivially $-1 \leqslant e \leqslant 1$. We change f into a new function f^* by changing $f(x)$ to b for $p \leqslant x \leqslant n$. Let A^*, B^*, C^* denote the corresponding new values of A, B, C. Now f^* is increasing and the inequality $\Sigma f \leqslant \Sigma f^*$ is equivalent to $0 \leqslant q(1-c)(1-d)$ so $f^* \in F$. Observe that $A^* = A$ by definition of b, and trivially $B^* = B$. Finally the inequality $C^* \leqslant C$ holds because it is equivalent to $qb + bde \leqslant q + cde$ which is $0 \leqslant qd(1-c)(1-e)$.

If $b = 0$ then $f^* = 0$ and the result holds. If $0 < b$ we divide f^* by b and go back to the beginning of the proof. The theorem follows by repetition of this process.

THEOREM 4′. *If r, s are odd and $f_1, \ldots, f_r, g_1, \ldots, g_s \in G$ and A, B, C are defined by* (6) *with summation over $1 \leqslant x \leqslant n$ then $AB \leqslant \frac{1}{2}nC$.*

Proof. Suppose first that n is even. We use (6) to define A_1, B_1, C_1 with summation over $1 \leqslant x \leqslant \frac{1}{2}n$ and A_2, B_2, C_2 with summation over $\frac{1}{2}n < x \leqslant n$. Thus $A = A_1 + A_2$ and similarly for B, C. Now Theorem 3′ says that $A_2 B_2 \leqslant \frac{1}{2}nC_2$. If we multiply all f_i, g_j by -1 it also says that $A_1 B_1 \leqslant \frac{1}{2}nC_1$. Similarly from Theorem 1″ we find that $A_1, B_1 \leqslant 0 \leqslant A_2, B_2$. It is now clear that $AB \leqslant \frac{1}{2}nC$. This case n even of this theorem yields Theorem 4 which in turn contains the case n odd of this theorem.

We now give an example to show that the constant $\frac{1}{2}n$ in Theorem 4′ is best possible. We let all f_i be $-1, \ldots, -1, 0, \ldots, 0, p$ and all g_j be $a, \ldots, a, 1, \ldots, 1$ with

$$a = -(\tfrac{1}{2}n - 1)/(\tfrac{1}{2}n + 1)$$

then $A \sim p^r$ and $B \sim \frac{1}{2}n - 1$ while $C \sim p^r$. Examples of the form $-1, \ldots, -1, n-1$ and $-n+1, 1, \ldots, 1$ indicate that there are no other inequalities between AB or $|A|\,|B|$ and C or $|C|$ with summation over $1 \leqslant x \leqslant n$.

DEFINITION. We say non-negative real numbers $w(t), \ldots, w(n)$ are *good weights* if $\frac{1}{2}n \leqslant t$ and for all $f_1, \ldots, f_r \in F$ we have

$$0 \leqslant \Sigma w f_1 \ldots f_r \tag{7}$$

with summation over $t \leqslant x \leqslant n$.

Thus good weights are related to Theorems 1, 1′, 1″. We could not find weights for the other theorems.

Let H be the set of all $f \in G$ of the form $-p/q, \ldots, -p/q, 0, \ldots, 0, 1, \ldots, 1$ where $q, n-p-q, p$ terms have the value $-p/q$, 0, 1 respectively and the positive integers p, q have $p+q \leqslant n$. It is easy to see that H is a basis for G. If we adjoin the function 1, ..., 1 to H we get a basis for F.

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THEOREM 5. *The non-negative reals* $w(t), \ldots, w(n)$ *with* $\frac{1}{2}n \leqslant t$ *are good weights if and only if* (7) *holds whenever* $r = 1$ *and* $f_1 \in H$.

Proof. Necessity is obvious, so to show sufficiency let $f_1, \ldots, f_r \in F$. By linearity we may assume $f_1, \ldots, f_r \in H$. There is a least p in $t-1 \leqslant p \leqslant n$ such that $f_i(x) = 1$ for all i and $p < x \leqslant n$. If $t-1 = p$ then (7) clearly holds, so assume $t-1 < p$ and $f_1(p) < 1$. Then by inspection of the functions in H we see that $w(x)f_1(x) \leqslant w(x)f_1(x) \ldots f_r(x)$ for $t \leqslant x \leqslant m$ and the theorem is proved.

3. *Remarks on lattices.* The FKG and GKS inequalities of physics have many applications (see (1, 2, 4, 5)). It was trying to generalize them that led to this paper. Let L be the lattice of subsets of a finite set. Examples show that our above results do not generalize to L. For $\alpha, \lambda \in L$ let $\sigma_\alpha(\lambda)$ be 1 if $\alpha \subset \lambda$ but -1 otherwise. The case $|\alpha| = 1$ of these functions σ_α is used in physics. We do not allow $|\alpha| = 0$. Then it is easy to see that $\Sigma\sigma_\alpha \leqslant 0 \leqslant \Sigma\sigma_\alpha\sigma_\beta$ where summation is over $\lambda \in L$. We have proved that $\Sigma\sigma_\alpha\sigma_\beta\sigma_\gamma$ is > 0 if $1 = |\alpha| < |\beta|$ and $\alpha \not\subset \beta$ and $\alpha \cup \beta \subset \gamma$, is $= 0$ if $\alpha = \{1, 2\}, \beta = \{2, 3\}, \gamma = \{1, 3\}$, but is < 0 otherwise. Elementary arguments show that $0 \leqslant (-1)^r\Sigma\sigma_{\alpha_1} \ldots \sigma_{\alpha_r}$ if $r = 4$ and $|\alpha_i| = 2$ or if $r \leqslant 2^{s-1}$ and $s \leqslant |\alpha_i|$. We omit the proofs.

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