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On intrinsic eigenstates in plasticity with generalized variables

By R. HILL

Department of Applied Mathematics and Theoretical Physics, University of Cambridge

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Abstract. The classical specification of elastic/plastic or rigid/plastic response in metals is reformulated in generalized variables. The allowed measures of strain are sets of any geometric magnitudes that jointly determine the shape of a material element; the allowed measures of stress are generated by work-conjugacy. The choice of variables affects the parameters and qualitative features of the constitutive framework; these dependences are made explicit by concise formulae of transformation.

Eigenstates intrinsic to the material are considered wherein the strain can change while the conjugate stress is either stationary or coupled differentially with the strain. Such configurations are associated with incipient branching of the strain response to a prescribed variation of the conjugate stress. Sensitivity to the loading environment is evaluated for both elastic/plastic and rigid/plastic materials. This synoptic approach to eigenproblems is illustrated in the context of materials testing.

1. *Objectives.* It is proposed to study singularities of the classical specification of elastic/plastic or rigid/plastic response in polycrystalline metals. The singularities are associated with incipient branching of the constitutive relation between increments of objective stress and strain. In particular, the strain can change while the stress is stationary; hence the material configurations will be called *intrinsic eigenstates*, as in crystal elasticity (**9**).

Inevitably, their locations are in part dependent on the choice of measures for stress and strain. In order to examine this relativity in proper context, the classical law will be reformulated in generalized variables. The allowed measures of strain will be any sets of geometric magnitudes that jointly determine the shape of a material element; the allowed measures of stress will be generated from these by work-conjugacy. The choice of variables affects most parameters in the constitutive framework, as well as certain qualitative features; all measure dependences will be made explicit by formulae of transformation. These go beyond previous studies (**4**, **5**, **8**, **10**), more especially in regard to the critical parameter whose vanishing gives rise to eigenstates. Additionally, the formulae set in perspective some controversial notions that have become attached to the classical framework: for instance, that there is a natural measure of the rate of hardening; that this rate is uniform over each yield surface; and that the accumulated hardening is solely a function of plastic work.

Different choices of variables present themselves unavoidably in practice, and hence induce mutual transformations. Typically, the material response is given relative to a conventional measure of strain (commonly the tensor logarithm), whereas

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an eigenstate is generally sought with respect to some unrelated measure of stress (work-conjugate for instance to the tensor stretch). All such eigenproblems will be reduced to a universal formula for the critical rate of hardening in the preferred variables. A similar formula is derived for a still wider class of eigenproblems generated by differential coupling between conjugate measures (not by simple stationarity). This synoptic approach will be applied to respresentative examples from materials testing.

Constitutive branching at intrinsic eigenstates has recently been analyzed in detail for finitely-strained elastic materials (13). Nothing comparable will be undertaken here for elastic/plastic materials. A thorough study of the incidence of eigenstates, and of their measure dependence, is regarded as a necessary preliminary in this context.

2. *Constitutive framework.* For the present purpose we allow a homogeneous strain to be specified by any six parameters, tensorial or not, that together determine the eventual shape of a reference cube of material. In the spirit of Lagrangian mechanics such strain parameters will be regarded as *generalized coordinates* and, following tradition, denoted by q_r $(r = 1, \ldots, 6)$ or collectively by the vector $\mathbf{q}$. The differential work of deformation per unit reference volume is expressible as a Pfaffian $\mathbf{p}\,d\mathbf{q}$, where the components p_r have the character of generalized stresses and are here called *work-conjugate variables*. In principle they are calculable functions of the chosen coordinates and the material loading; for a systematic exposition and various applications see Hill (10, 12, 13).

Consider a representative macro-element of some metal polycrystal, and any stage Q of a path of plastic deformation in $\mathbf{q}$-space. It is classically presumed that there exists an associated *elastic domain* of states $\mathbf{q}$ attainable from Q by purely elastic paths. The domain is bounded by a *yield surface* passing through Q and consisting of points where plastic deformation can be resumed. The surface may have any shape; it need not be convex, but is assumed smooth. Still adhering to the classical view, the response to incremental deformation at any yieldpoint is taken to be piecewise linear:

$$\left.\begin{aligned} d\mathbf{p} &= \mathscr{L}\,d\mathbf{q}, \quad \boldsymbol{\lambda}\,d\mathbf{q} \leqslant 0 \quad \text{(elastic)};\\ d\mathbf{p} &= L\,d\mathbf{q}, \quad \boldsymbol{\lambda}\,d\mathbf{q} > 0 \quad \text{(plastic)}. \end{aligned}\right\} \tag{2·1}$$

Here vector $\boldsymbol{\lambda}$ is outwardly normal to the yield surface at Q, and the (6×6) matrices of generalized moduli, $\mathscr{L}$ and L, do not depend on $d\mathbf{q}$. Additionally, we stipulate the symmetries

$$\mathscr{L}^T = \mathscr{L}, \quad L^T = L, \tag{2·2}$$

provided the moduli are associated with work-conjugate variables. As a specific way of extending classical concepts to finite strain, this postulate can be traced back in embryo to Hill (2, 3).

The two regimes are compatible provided that

$$(\mathscr{L} - L)\,d\mathbf{q} = \mathbf{0} \quad \text{whenever} \quad \boldsymbol{\lambda}\,d\mathbf{q} = 0.$$

This implies that

$$\mathscr{L} - L = \boldsymbol{\lambda} \otimes \boldsymbol{\lambda}/g, \tag{2·3}$$

which in spirit is the so-called normality flow-rule, but here specifically relates to work-conjugate variables. Scalar g is required to be positive,

$$g > 0, \tag{2·4}$$

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in order that $$d\mathbf{q}(\mathscr{L}-L)\,d\mathbf{q} = (\boldsymbol{\lambda}\,d\mathbf{q})^2/g \geqslant 0,$$

to reflect the softer response of yielding metals. The plastic regime can now be stated as

$$d\mathbf{p} = \mathscr{L}\,d\mathbf{q} - \boldsymbol{\lambda}\,d\gamma, \quad g\,d\gamma = \boldsymbol{\lambda}\,d\mathbf{q}, \quad d\gamma > 0. \tag{2·5}$$

Scalar $d\gamma$ governs the magnitude of the non-elastic component of material response; its role will appear more precisely later.

Let $\mathscr{M}$ be the matrix of elastic compliances, inverse to $\mathscr{L}$; for metals this inverse exists since the determinant $|\mathscr{L}|$ is always positive with any reasonable choice of generalized coordinates. Define $\boldsymbol{\mu}$ such that

$$\boldsymbol{\lambda} = \mathscr{L}\boldsymbol{\mu}, \quad \boldsymbol{\mu} = \mathscr{M}\boldsymbol{\lambda}. \tag{2·6}$$

Then $\boldsymbol{\mu}$ is outwardly normal to the trace yield surface in $\mathbf{p}$-space, since

$$\boldsymbol{\lambda}\,d\mathbf{q} = \boldsymbol{\mu}\,d\mathbf{p} \leqslant 0,$$

for elastic increments at a yieldpoint. Resolve (2·5) along the normal, producing

$$\boldsymbol{\lambda}\,d\mathbf{q} - \boldsymbol{\mu}\,d\mathbf{p} = \boldsymbol{\lambda}\boldsymbol{\mu}\,d\gamma,$$

and correspondingly introduce h such that

$$g - h = \boldsymbol{\lambda}\boldsymbol{\mu}.$$

Then (2·5) is equivalent to

$$d\mathbf{q} = \mathscr{M}\,d\mathbf{p} + \boldsymbol{\mu}\,d\gamma, \quad h\,d\gamma = \boldsymbol{\mu}\,d\mathbf{p}, \quad d\gamma > 0. \tag{2·8}$$

On evaluating the determinant $|L|$ from (2·3) with (2·7), we find

$$g|L| = h|\mathscr{L}|. \tag{2·9}$$

Therefore, provided $h \neq 0$, L has an inverse M which can be read from (2·8):

$$M - \mathscr{M} = \boldsymbol{\mu} \otimes \boldsymbol{\mu}/h. \tag{2·10}$$

Deployment of generalized variables in the classical approach originates with Hill (**10**), as does the explicit algebraic duality embodied in (2·3), (2·6), (2·7), and (2·10), which is patterned on (**5**, **6**). Textural anisotropy can be accommodated by the framework, as well as progressive changes in $\mathscr{L}$, $\mathscr{M}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, g, h during deformation; some possible hypotheses are reviewed later.

Three categories of plastic regime are indicated in relation to generalized variables (**5**). They are distinguished according to how $d\mathbf{p}$ is directed with respect to the yield surface.

(i) Strain hardening ($h > 0$): $\boldsymbol{\mu}\,d\mathbf{p} > 0$, $d\mathbf{p}$ outwards.
(ii) Ideally plastic ($h = 0$): $\boldsymbol{\mu}\,d\mathbf{p} = 0$, $d\mathbf{p}$ tangential.
(iii) Strain softening ($h < 0$): $\boldsymbol{\mu}\,d\mathbf{p} < 0$, $d\mathbf{p}$ inwards. (2·11)

In (ii) and (iii) $\boldsymbol{\mu}\,d\mathbf{p} > 0$ is inadmissible since it conflicts with $h\,d\gamma \leqslant 0$. In all cases $\boldsymbol{\mu}\,d\mathbf{p} \leqslant 0$ permits elastic response, so in (iii) both $M\,d\mathbf{p}$ and $\mathscr{M}\,d\mathbf{p}$ are possible deformations under admissible $d\mathbf{p}$. Another implication of the framework is worth noting at this point; it seems not to be in the literature, except for a formal analogue in thermoelasticity (**12**). From (2·3) and (2·7) there follows

$$(\mathscr{L}-L)\boldsymbol{\mu} = (g-h)\boldsymbol{\lambda}/g,$$

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and by (2·6) this may be re-arranged dually as

$$L\boldsymbol{\mu} = \frac{h}{g}\mathscr{L}\boldsymbol{\mu}, \quad M\boldsymbol{\lambda} = \frac{g}{h}\mathscr{M}\boldsymbol{\lambda}. \tag{2·12}$$

The constitutive interpretation is that the L and $\mathscr{L}$ responses to a deformation co-directional with $\boldsymbol{\mu}$ are both co-directional with $\boldsymbol{\lambda}$, but in the ratio $h:g$. Likewise the M and $\mathscr{M}$ responses to a loading co-directional with $\boldsymbol{\lambda}$ are both co-directional with $\boldsymbol{\mu}$, but in the ratio $g:h$. Parts $(L-\mathscr{L})\,d\mathbf{q}$ and $(M-\mathscr{M})\,d\mathbf{p}$ of the L and M responses are of course *always* co-directional with the normals $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, regardless of the imposed $d\mathbf{q}$ and $d\mathbf{p}$, respectively.

It remains to choose a conventional magnitude for $\boldsymbol{\mu}$ at each yieldpoint. Let the work expended in the plastic part of $d\mathbf{q}$ be expressed as

$$\mathbf{p}(d\mathbf{q} - \mathscr{M}\,d\mathbf{p}) = \tau d\gamma, \tag{2·13}$$

per unit reference volume, where

$$\tau = \boldsymbol{\mu}\mathbf{p}, \tag{2·14}$$

by (2·8). It is presumed that the yield surface in $\mathbf{p}$-space is star-shaped with respect to the origin, so that $\boldsymbol{\mu}\mathbf{p}$ is positive when $\boldsymbol{\mu}$ is the outward normal. Then $\boldsymbol{\mu}$ can be arbitrarily scaled over the surface so that $\boldsymbol{\mu}\mathbf{p}$ is *uniform*:

$$\tau = \sigma \quad \text{(constant)}, \tag{2·15}$$

where a natural choice for σ is the current Kirchhoff stress in uniaxial tension (along a designated fibre when the material is anisotropic). The magnitude of $\boldsymbol{\lambda}$ at each yield-point is then determined by (2·6); in principle $\mathscr{L}$ and $\mathscr{M}$ are known functions of $\mathbf{q}$ over a domain, via an elastic potential which in general depends on the path of plastic deformation leading to that domain. Over each yield surface $\mathbf{p}$ may be regarded as determined by $d\mathbf{q} - \mathscr{M}\,d\mathbf{p}$, since this decides the local normal and thereby the yield-point. Consequently, $d\gamma$ given by (2·13) subject to (2·15) is a homogeneous function of degree one in $d\mathbf{q} - \mathscr{M}\,d\mathbf{p}$. Moreover, in relation to a given yield surface, $d\gamma$ is a *work-equivalent measure of incremental plastic strain* (11), in that the same work would be expended in every deformation $d\mathbf{q} - \mathscr{M}\,d\mathbf{p}$ generating the same $d\gamma$.

The constitutive prescription is completed by assigning appropriate path-dependences to the parameters g and h. These are necessarily linked by (2·7), so we concentrate on h and the associated hardening rule. In the absence of clear-cut observational data, it is usually presumed that the *same* neighbouring yield surface in $\mathbf{p}$-space can be generated from *any* point P of a current yield surface by an appropriate increment of plastic deformation at P. Then, if $d\gamma$ between neighbouring surfaces is uniform in value, so is $\tau d\gamma$; and conversely. In other words, under this uniformity hypothesis, the integrated measure $\int d\gamma$ and the plastic work $\int \tau d\gamma$ are corresponding determinants of hardening along a path of deformation. This simple theorem is almost tautologous in content by virtue of the work-equivalent property of $d\gamma$; different definitions of strain equivalence have been proposed in the past, and for those the correspondence is generally not valid.

It has often been assumed that successive yield surfaces form a one-parameter family with the origin as a perpetual centre of similarity. Supposing this to be an acceptable approximation in certain variables $\mathbf{p}$, the family is parametrizable as

$$f(\mathbf{p}) = \sigma, \tag{2·16}$$

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where function f is homogeneous of degree one and the history dependence is felt through σ alone. Then

$$\mathbf{p}\frac{\partial f}{\partial \mathbf{p}} \equiv f(\mathbf{p}), \quad \frac{\partial f}{\partial \mathbf{p}} d\mathbf{p} = df,$$

so the stipulated scaling of $\boldsymbol{\mu}(\mathbf{p})$ is achieved with

$$\boldsymbol{\mu} = \frac{\partial f}{\partial \mathbf{p}}, \quad \boldsymbol{\mu} d\mathbf{p} = d\sigma. \tag{2·17}$$

It follows that $\boldsymbol{\mu}$ is constant along a radius and that

$$d\sigma = h d\gamma, \tag{2·18}$$

by $(2·8)_2$. Therefore, if h is uniform over each surface, so is $d\gamma$ between neighbouring surfaces:

$$d\gamma = d\epsilon \quad \text{(constant)}. \tag{2·19}$$

In the uniaxial test that determines σ, as agreed previously, the work expended in the plastic part of the deformation is $\sigma d\epsilon$ per unit reference volume (the uniform value of $\tau d\gamma$). So $d\epsilon$ is the plastic part of the increment in *logarithmic* strain of the designated fibre. Then

$$h = \frac{d\sigma}{d\epsilon}, \tag{2·20}$$

the gradient of the standard representation of material behaviour in tension. Equivalently, if σ in (2·16) is regarded as a function of the accumulated plastic work w, where $dw = \sigma d\epsilon$, then $h = \sigma d\sigma/dw$.

3. *Relativity and invariance.* When assessing any constitutive framework its dependence on the choice of variables should be examined, so that what is significant can be distinguished from what is adventitious. For elastic/plastic materials the examination was at first confined to various classes of tensor variables (**5**, **7**, **8**), then later extended to generalized coordinates without restriction (**10**, **12**). Not all matters were treated, however, because of manipulative obstacles. These are now removed with the help of a concise symbolism. For completeness we begin by re-deriving known formulae in the new notation.

Consider any two ways of specifying a deformation by generalized coordinates, $\mathbf{q}$ and $\tilde{\mathbf{q}}$ respectively. A tilde will similarly distinguish all associated elements in the constitutive specification. We regard each $\tilde{q}_r$ as a known function of all $q_s (r, s = 1, \ldots, 6)$, and write the transformation in Jacobian form as

$$d\tilde{\mathbf{q}} = \mathscr{A}\, d\mathbf{q}, \quad \mathscr{A}_{rs} = \frac{\partial \tilde{q}_r}{\partial q_s}. \tag{3·1}$$

Now $\mathbf{p}\, d\mathbf{q}/\rho$ is coordinate-invariant by definition of $\mathbf{p}$, where ρ is the mass density in the reference configuration. Therefore, for a given deformation and loading, the associated conjugate variables are related by

$$\frac{\tilde{\rho}}{\rho}\mathbf{p} = \mathscr{A}^T \tilde{\mathbf{p}}. \tag{3·2}$$

Consequently, in a differential increment of deformation,

$$\frac{\tilde{\rho}}{\rho}(d\mathbf{p} - \mathscr{C}\, d\mathbf{q}) = \mathscr{A}^T d\tilde{\mathbf{p}} \tag{3·3}$$

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where
$$\frac{\tilde{\rho}}{\rho}\mathscr{C}_{rs} = \tilde{p}_t \frac{\partial^2 \tilde{q}_t}{\partial q_r \partial q_s}, \quad \mathscr{C}^T = \mathscr{C}. \tag{3·4}$$

It follows that the generalized moduli $\mathscr{L}$ and L transform according to

$$\frac{\tilde{\rho}}{\rho}(\mathscr{L} - \mathscr{C}) = \mathscr{A}^T \tilde{\mathscr{L}} \mathscr{A}, \quad \frac{\tilde{\rho}}{\rho}(L - \mathscr{C}) = \mathscr{A}^T \tilde{L} \mathscr{A}. \tag{3·5}$$

An immediate corollary is that the postulated symmetry of the moduli is indeed coordinate-invariant. Also, by eliminating $\mathscr{C}$,

$$\frac{\tilde{\rho}}{\rho}(\mathscr{L} - L) = \mathscr{A}^T(\tilde{\mathscr{L}} - \tilde{L})\mathscr{A}. \tag{3·6}$$

Equivalently, the bilinear form

$$(\delta \mathbf{p}\, d\mathbf{q} - d\mathbf{p}\, \delta \mathbf{q})/\rho = d\mathbf{q}(\mathscr{L} - L)\, \delta \mathbf{q}/\rho,$$

is coordinate-invariant, where δ and d are independent differentials and $\delta \mathbf{p} = \mathscr{L}\, \delta \mathbf{q}$, $d\mathbf{p} = L\, d\mathbf{q}$ (8, 12).

Let $\tilde{\boldsymbol{\lambda}}$ be locally normal to the yield surface in $\tilde{\mathbf{q}}$-space. Then $\tilde{\boldsymbol{\lambda}}\, \delta \tilde{\mathbf{q}} = 0$ whenever $\boldsymbol{\lambda}\, \delta \mathbf{q} = 0$, implying $\mathscr{A}^T \tilde{\boldsymbol{\lambda}} \propto \boldsymbol{\lambda}$ by (3·1). Suppose the factor of proportionality chosen so as to keep $\boldsymbol{\lambda}\, d\mathbf{q}/\rho$ coordinate-invariant for arbitrary $d\mathbf{q}$. Then $\tilde{\boldsymbol{\lambda}}$ is always outward from the elastic domain and transforms like $\tilde{\mathbf{p}}$:

$$\frac{\tilde{\rho}}{\rho}\boldsymbol{\lambda} = \mathscr{A}^T \tilde{\boldsymbol{\lambda}}. \tag{3·7}$$

Now the equi-dyadic structure of $\mathscr{L} - L$ in (2·3) is coordinate-invariant, as is the consequent normality flow-rule, since the argument from symmetry and branch compatibility is universal. Therefore, by (3·6) and (3·7), the parameter g transforms according to

$$\frac{\tilde{\rho}}{\rho} g = \tilde{g}. \tag{3·8}$$

In other words g/ρ is coordinate-invariant, along with $\boldsymbol{\lambda}\, d\mathbf{q}/\rho$ and hence so is $d\gamma$ defined by $(2·5)_2$. On the other hand, it is seen from (2·13) that $\tau\, d\gamma/\rho$ is not coordinate-invariant when $\mathscr{M} \neq 0$, and therefore neither is τ/ρ. The invariance of τ/ρ could have been secured by normalizing $\boldsymbol{\lambda}$ differently, but at the cost of non-invariant g/ρ and $d\gamma$; the present course is preferred on balance.

To calculate $\tilde{\boldsymbol{\mu}}$ we re-write (3·7) by $(2·6)_1$, and then use $(3·5)_1$:

$$\frac{\tilde{\rho}}{\rho}\mathscr{L}\boldsymbol{\mu} = \mathscr{A}^T \tilde{\mathscr{L}} \tilde{\boldsymbol{\mu}} = \frac{\tilde{\rho}}{\rho}(\mathscr{L} - \mathscr{C})\mathscr{A}^{-1} \tilde{\boldsymbol{\mu}}.$$

Whence
$$\boldsymbol{\mu} = (I - \mathscr{M}\mathscr{C})\mathscr{A}^{-1} \tilde{\boldsymbol{\mu}}, \tag{3·9}$$

which is an explicit new formula. Hitherto the transformation of $\boldsymbol{\mu}$ has been stated only implicitly via the invariance of $\boldsymbol{\mu}\, d\mathbf{p}/\rho$, where $d\mathbf{p}$ is any *elastic* increment (10). We recall the proposal to scale $\boldsymbol{\mu}$ over a yield surface so that $\tau = \boldsymbol{\mu}\mathbf{p}$ is uniform. It is evident from (3·2) and (3·9) that $\tilde{\boldsymbol{\tau}} = \tilde{\boldsymbol{\mu}}\tilde{\mathbf{p}}$ can not also be uniform when $\mathscr{M} \neq 0$, since $\mathscr{A}$ and $\mathscr{C}$ vary with $\mathbf{q}$ within an elastic domain. In general, therefore, the proposed scaling is possible for only *one* choice of coordinates, $\mathbf{q}$ say, if $\tilde{\boldsymbol{\lambda}}$ is normalized in advance by (3·7). Likewise, when $\tilde{\boldsymbol{\tau}}$ varies over a yield surface, $d\tilde{\gamma} = \tilde{\mathbf{p}}(d\tilde{\mathbf{q}} - \tilde{\mathscr{M}}\, d\tilde{\mathbf{p}})/\tilde{\boldsymbol{\tau}}$ no longer

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acts as a work-equivalent measure of incremental plastic strain, despite being coordinate-invariant at each yieldpoint. The explanation of this seeming paradox is simply that the partitioning of incremental strain into elastic and plastic parts is itself not an invariant concept. On the contrary, it is relative to whichever conjugate variable $\tilde{\mathbf{p}}$ is cycled, by adding and then removing an increment $d\tilde{\mathbf{p}}$ (5).

It remains to determine the hardening parameter $\tilde{h}$. From (2·7) with its counterpart, we construct

$$h - \frac{\rho}{\tilde{\rho}}\tilde{h} = \frac{\rho}{\tilde{\rho}}\tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{\mu}} - \boldsymbol{\lambda}\boldsymbol{\mu},$$

taking note of (3·8). Then, by (3·7) and (3·9) successively, there follows

$$h - \frac{\rho}{\tilde{\rho}}\tilde{h} = \boldsymbol{\lambda}(\mathscr{A}^{-1}\tilde{\boldsymbol{\mu}} - \boldsymbol{\mu}) = \boldsymbol{\mu}\mathscr{C}\mathscr{A}^{-1}\tilde{\boldsymbol{\mu}}, \tag{3·10}$$

along with the variants

$$h - \frac{\rho}{\tilde{\rho}}\tilde{h} = \boldsymbol{\mu}\mathscr{C}(I - \mathscr{M}\mathscr{C})^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}(I - \mathscr{C}\mathscr{M})^{-1}\mathscr{C}\boldsymbol{\mu}, \tag{3·11}$$

since $\mathscr{C}$ is symmetric. These transformations are new, though prototypes have been given for a particular family of tensor variables based on the current configuration (5).

The situation is much simplified when the material is rigid/plastic ($\mathscr{M} = 0$), with the constitutive specification

$$d\mathbf{q} = \boldsymbol{\mu}\, d\gamma, \quad h\, d\gamma = \boldsymbol{\mu}\, d\mathbf{p}, \quad d\gamma > 0. \tag{3·12}$$

Assuming no plastic change in volume, we take $\tilde{\rho} = \rho$ since nothing would be gained by referring the coordinates to a configuration inaccessible to the material. Then the basic connexions are

$$d\tilde{\mathbf{q}} = \mathscr{A}\, d\mathbf{q}, \quad \mathbf{p} = \mathscr{A}^T\tilde{\mathbf{p}}, \quad d\mathbf{p} - \mathscr{C}\, d\mathbf{q} = \mathscr{A}^T d\tilde{\mathbf{p}}. \tag{3·13}$$

Since the deformation is invariable within each yield surface, the values of $\mathbf{q}$, $\mathscr{A}$, and $\mathscr{C}$ are *constant* there, and a surface can only be represented in $\mathbf{p}$-space. Its convexity, if any, is invariant since relation $(3·13)_2$ is linear at constant $\mathscr{A}$. Its local normal transforms like $d\mathbf{q}$, namely

$$\tilde{\boldsymbol{\mu}} = \mathscr{A}\boldsymbol{\mu}, \tag{3·14}$$

when $d\gamma$ is arbitrarily made coordinate-invariant; this is consistent with the limit of (3·9) as $\mathscr{M} \to 0$. Then $\tau = \boldsymbol{\mu}\mathbf{p}$ is automatically coordinate-invariant, and likewise its surface uniformity. Moreover, $d\gamma = \mathbf{p}\, d\mathbf{q}/\tau$ acts as an invariant work-equivalent measure of incremental strain. On the other hand, even if yield surfaces relative to particular variables, $\mathbf{p}$ say, are parametrizable as in (2·16), the geometric similarity does not extend to the family $f(\mathscr{A}^T\tilde{\mathbf{p}}) = \sigma$ in $\tilde{\mathbf{p}}$-space, since $\mathscr{A}$ changes from one surface to another. As to the transformation of the hardening parameter itself, we have

$$\tilde{h} = h - \boldsymbol{\mu}\mathscr{C}\boldsymbol{\mu}, \tag{3·15}$$

from (3·11) with $\mathscr{M} = 0$ and $\tilde{\rho} = \rho$. A prototype of this was originally derived in seeking the parameter most closely connected with the speeds of acceleration waves in rigid/plastic media (4); only the main types of stress then extant were contemplated.

4. *Intrinsic eigenstates.* In the constitutive specification the modulus matrix L for the plastic regime was seen to be invertible except when h vanishes. In that event, by

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$(2{\cdot}12)_1$, $d\mathbf{p}$ is zero for any $d\mathbf{q}$ co-directional with the normal $\boldsymbol{\mu}$ to the yield surface in $\mathbf{p}$-space. Such differential deformations will be called *eigenmodes* (being possible under stationary $\mathbf{p}$), while the material configurations will be called *eigenstates*. Since associated with follower loading via work-conjugacy, they reflect properties intrinsic to the material (9), as distinct from rotation-affected modes under general dead loading (6).

Intrinsic eigenstates manifest themselves not only by eigenmodes, but more generally by *constitutive branching*. If a loading path in $\mathbf{p}$-space is prescribed, the differential response $d\mathbf{q}$ first becomes non-unique at an eigenstate. With $\boldsymbol{\mu}\,d\mathbf{p} = 0$ there, the elastic branch is $d\mathbf{q} = \mathscr{M}d\mathbf{p}$, but characteristically the plastic branch depends on the path value of $d^2\mathbf{p}$ as well as on $d\mathbf{p}$. In this respect the non-uniqueness when $h = 0$ differs from that when $h < 0$, where $d\mathbf{q}$ is either $\mathscr{M}d\mathbf{p}$ or $M\,d\mathbf{p}$ simply. Branching at eigenstates where $|\mathscr{L}| = 0$ has been analysed for finitely deformable elastic materials (13); a cognate analysis for elastic/plastic materials will not be undertaken here. Attention is confined to the simple incidence of eigenstates and to their coordinate dependence.

To be definite without losing generality, $\boldsymbol{\mu}$ is normalized by (2·14) and (2·15), but no particular hardening rule is pre-supposed. During progressive plastic deformation the value of h steadily decreases, so an eigenstate relative to the variables $(\mathbf{p}, \mathbf{q})$ is eventually encountered unless h remains positive indefinitely. During the same deformation an eigenstate relative to different variables $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ is encountered when $\tilde{h}$ vanishes (if at all); this requires that

$$h = \boldsymbol{\mu}\mathscr{C}(I - \mathscr{M}\mathscr{C})^{-1}\boldsymbol{\mu}, \quad (|\mathscr{L} - \mathscr{C}| \neq 0), \tag{4·1}$$

by (3·11). In appreciating this criterion it should be kept in mind that $\boldsymbol{\mu}$ and $\mathscr{M}$, as well as $\mathscr{C}$, may vary during the deformation. The associated eigenmode, when expressed in the original coordinates $\mathbf{q}$, is co-directional with

$$\mathscr{A}^{-1}\tilde{\boldsymbol{\mu}} = (I - \mathscr{M}\mathscr{C})^{-1}\boldsymbol{\mu}, \tag{4·2}$$

by (3·9). Such formulae are needed when material properties are stated in one kind of coordinates (say $\mathbf{q}$) but criticality is sought in a different kind (say $\tilde{\mathbf{q}}$). From that standpoint, freely chosen, any path of plastic deformation is dense with latent eigenstates, each tied to some particular generalized variables. This same material deformation induces a unique material loading, which is represented by different paths in the various $\mathbf{p}$, $\tilde{\mathbf{p}}$, ... spaces. A particular eigenstate, say $\tilde{h} = 0$, can be activated by an environment that enforces the path in the corresponding $\tilde{\mathbf{p}}$-space (regardless of any branching and the resulting change in material loading).

An eigenmode can also be viewed ambivalently, namely as a deformation possible under $d\tilde{\mathbf{p}} = \mathbf{0}$, or under

$$d\mathbf{p} = \mathscr{C}\,d\mathbf{q}, \tag{4·3}$$

by (3·3). The last relation can be regarded as a notional coupling between load and deformation. In conjunction with the plastic response, $d\mathbf{p} = L\,d\mathbf{q}$, it permits non-zero $d\mathbf{q}$ provided $|L - \mathscr{C}| = 0$. This condition is reducible by (2·3) to

$$g = \boldsymbol{\lambda}(\mathscr{L} - \mathscr{C})^{-1}\boldsymbol{\lambda}, \quad (|\mathscr{L} - \mathscr{C}| \neq 0) \tag{4·4}$$

or by (2·7) to

$$h = \boldsymbol{\mu}\mathscr{L}(\mathscr{L} - \mathscr{C})^{-1}\mathscr{C}\boldsymbol{\mu}, \tag{4·5}$$

which is a variant of (4·1).

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This viewpoint prompts the investigation of eigenstates in an environment whose *passive mode* is

$$d\mathbf{p} = C\,d\mathbf{q}, \tag{4·6}$$

where matrix C may depend on the total deformation. The coupling envisaged here is not interpretable through a transformation of coordinates, as in (3·4), nor need it even be symmetric. Nevertheless, the environment still allows a purely intrinsic response, since its influence is prescribed in objective variables. Eigenstates now occur when

$$|L - C| = 0. \tag{4·7}$$

The critical values of g and h are given by equations similar to (4·1), (4·4), (4·5) as they stand, with C in place of $\mathscr{C}$. Under change of coordinates the coupling is expressed by

$$d\tilde{\mathbf{p}} = \tilde{C}\,d\tilde{\mathbf{q}}, \quad \text{where} \quad \mathscr{A}^T\tilde{C}\mathscr{A} = \frac{\tilde{\rho}}{\rho}(C - \mathscr{C}), \tag{4·8}$$

which is a transformation like (3·5) for a modulus matrix, even though C may be unsymmetric. The incidence of eigenstates in an actual environment of this kind is illustrated later.

In a rigid/plastic material the eigenstate $\tilde{h} = 0$ is located in the coordinates $\mathbf{q}$ where

$$h = \boldsymbol{\mu}\mathscr{C}\boldsymbol{\mu}, \tag{4·9}$$

and the eigenmode is always co-directional with $\boldsymbol{\mu}$ (as are deformations of any kind in this material). These conclusions emerge from (4·1) and (4·2) with $\mathscr{M} = 0$, but can be better appreciated in relation to the rigid/plastic law itself. According to (3·12) this is

$$h\,d\mathbf{q} = (\boldsymbol{\mu}\,d\mathbf{p})\,\boldsymbol{\mu}, \tag{4·10}$$

which is identically the same as

$$(h - \boldsymbol{\mu}C\boldsymbol{\mu})\,d\mathbf{q} = \{\boldsymbol{\mu}(d\mathbf{p} - C\,d\mathbf{q})\}\,\boldsymbol{\mu}, \tag{4·11}$$

for any C, symmetric or not, since $(\boldsymbol{\mu}C\boldsymbol{\mu})\,d\mathbf{q} \equiv (\boldsymbol{\mu}C\,d\mathbf{q})\,\boldsymbol{\mu}$ when $d\mathbf{q} \propto \boldsymbol{\mu}$ as in (4·10). It is apparent from (4·11) that an incremental deformation $d\mathbf{q}$ is possible under a passive constraint $d\mathbf{p} = C\,d\mathbf{q}$ provided that

$$h = \boldsymbol{\mu}C\boldsymbol{\mu}. \tag{4·12}$$

Note that the critical value of h is not affected by the skew part of C. When the environment is such that C is of type (3·4). the criterion returns to (4·9).

5. *Illustrations.* We consider incompressible rigid/plastic materials with a one-parameter family of yield surfaces. These are supposed isotropic and geometrically similar in the space of the Cauchy stress tensor, and can therefore be parametrized by

$$f(\sigma_1, \sigma_2, \sigma_3) = \sigma, \tag{5·1}$$

where function f is homogeneous of degree one and symmetric in the principal values $\sigma_1, \sigma_2, \sigma_3$. Following custom the stress-strain relation in uniaxial tension is taken to be

$$\sigma = c\epsilon^n, \quad \epsilon = \log a, \quad 0 < n < 1, \tag{5·2}$$

where c and n are material constants and a is the axial stretch (final length of a unit fibre). Paths of deformation are envisaged such that the same three fibres are perpetually orthogonal. Then, because of the material properties, all relevant tensors

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are coaxial with this triad, and their principal values can serve as a full set of generalized variables. Accordingly we make the identifications

$$\mathbf{p} \equiv \sigma_r, \quad \mathbf{q} \equiv \epsilon_r = \log a_r, \quad \boldsymbol{\mu} \equiv \mu_r = \partial f/\partial \sigma_r \quad (r = 1, 2, 3). \tag{5·3}$$

Matrices $\mathscr{A}$ and $\mathscr{C}$ defined by (3·1) and (3·4) are now diagonal, with leading elements

$$\mathscr{A}_r = \frac{d\tilde{q}_r}{dq_r}, \quad \mathscr{C}_r = \tilde{p}_r \frac{d^2\tilde{q}_r}{dq_r^2}, \quad \text{(no sum)} \tag{5·4}$$

where $\tilde{p}_r = p_r/\mathscr{A}_r$ and $\tilde{q}_r = \tilde{q}(a_r)$ is any other measure of strain, such that $\tilde{q}(1) = 0$, $\tilde{q}'(1) = 1$.

Suppose, for instance, that eigenstates are sought under stationary nominal stress (load per unit reference area). Then the appropriate transformation is

$$\tilde{q}(a) = a - 1, \quad \frac{d\tilde{q}}{dq} = a, \quad \frac{d^2\tilde{q}}{dq^2} = a;$$

$$\tilde{p}_r = \sigma_r/a_r, \quad \mathscr{A}_r = a_r, \quad \mathscr{C}_r = \sigma_r.$$

From (4·9) the eigenstates are located where

$$h = \sigma_1\mu_1^2 + \sigma_2\mu_2^2 + \sigma_3\mu_3^2, \tag{5·5}$$

in terms of quantities associated with the logarithmic measure of strain. This criterion is algebraically consistent with one for primary eigenstates under general *dead* loading ((6), p. 385), when that result is specialized for isotropy and account is taken of a different normalization ($|\boldsymbol{\mu}| = 1$). Under plane stress, $\sigma_3 = 0$, the criterion also coincides with one proposed for the termination of homogeneous deformation by diffuse necking ((1), p. 24 footnote); this states a critical value for h/σ in terms of a yield function $f(\sigma_1, \sigma_2)$ whose degree of homogeneity is left arbitrary.

As a specific illustration of (5·5) we take the Mises family of surfaces:

$$f(\sigma_1, \sigma_2, \sigma_3) \equiv \frac{1}{\sqrt{2}}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}^{\frac{1}{2}} = \sigma,$$

$$\tfrac{2}{3}\sigma\mu_r = \sigma_r - \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3), \quad |\boldsymbol{\mu}| = \sqrt{(\tfrac{3}{2})}.$$

The result can be arranged in various ways, for example

$$\frac{2h}{3} = \frac{\sigma_1(2\sigma_1 - \sigma_2 - \sigma_3)^2 + \ldots + \ldots}{(2\sigma_1 - \sigma_2 - \sigma_3)^2 + \ldots + \ldots}. \tag{5·6}$$

A variant can be seen in (6); note that the symbol h there signifies the present $2h/3$. In plane stress the expression reduces to

$$h = \frac{(\sigma_1 + \sigma_2)(4\sigma_1^2 - 7\sigma_1\sigma_2 + 4\sigma_2^2)}{4(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)}.$$

This tallies with a formula of Swift (15), who intended it as a criterion of incipient instability in a flat sheet under biaxial loading. In that regard it has often been cited in experimental studies of ductility in sheet metal, most recently by Mellor (14). Its relevance presumably depends on whether the constraints in an actual test permit the specimen to take advantage of this particular eigenstate, by deviating along a path that leads to inhomogeneity and eventual localized necking. As to Swift's view

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of instability, it might be paraphrased as follows in present terms: a rigid/plastic material is stable under homogeneous deformations provided the value of $\boldsymbol{\mu}\,d\mathbf{p}$ demanded constitutively exceeds the value of $\boldsymbol{\mu}\,d\mathbf{p}$ supplied environmentally, for all admissible $d\mathbf{q}$. This requires $h\boldsymbol{\mu}\,d\mathbf{q}/\boldsymbol{\mu}^2 > \boldsymbol{\mu}\mathscr{C}\,d\mathbf{q}$ when $d\mathbf{q} \propto \boldsymbol{\mu}$, which implies $h > \boldsymbol{\mu}\mathscr{C}\boldsymbol{\mu}$.

Reverting to the main development, let the hardening parameter be assumed uniform over each yield surface. Then, by (2·20) with $(5{\cdot}2)_1$,

$$h = n\sigma/\epsilon, \tag{5·7}$$

which remains positive. Consider a monotonic radial path in $\mathbf{p}$-space, such that

$$\sigma_r = \sigma\alpha_r \quad (\alpha_r \text{ constant}). \tag{5·8}$$

During this loading the values of (μ_1, μ_2, μ_3) do not change; their dependence on $(\alpha_1, \alpha_2, \alpha_3)$ is determined by the geometry of family (5·1), but is always such that

$$\alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3 = 1, \quad \mu_1 + \mu_2 + \mu_3 = 0. \tag{5·9}$$

The corresponding path in $\mathbf{q}$-space is likewise linear, since $d\mathbf{q} = \boldsymbol{\mu}\,d\gamma$ now integrates to

$$\epsilon_r = \epsilon\mu_r \quad (\epsilon > 0) \tag{5·10}$$

by (2·19) with $(5{\cdot}3)_2$. All other stress measures $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ vary by $(3{\cdot}13)_3$ in accordance with

$$\begin{aligned} \mathscr{A}_r\,d\tilde{p}_r &= d\sigma_r - \mathscr{C}_r\,d\epsilon_r \quad \text{(no sum)} \\ &= (h\alpha_r - \mathscr{C}_r\mu_r)\,d\epsilon, \end{aligned} \tag{5·11}$$

where $\mathscr{A}_r$ and $\mathscr{C}_r$ are given by (5·4). Multiplication by μ_r, followed by summation, confirms in passing that $\tilde{\boldsymbol{\mu}}\,d\tilde{\mathbf{p}} = 0$ by (3·14) and (5·9), when

$$h = \mathscr{C}_1\mu_1^2 + \mathscr{C}_2\mu_2^2 + \mathscr{C}_3\mu_3^2. \tag{5·12}$$

What is significant, however, is that all components of $d\tilde{\mathbf{p}}$ along the radial path are in general nonzero at the eigenstate $\tilde{h} = 0$, and any one $d\tilde{p}_r$ vanishes when and only when $h = \mathscr{C}_r\mu_r/\alpha_r$. This underscores the point that the eigenstate is *latent*; it could only manifest itself if the assigned $\mathbf{p}$-path were to be suspended and replaced by an assigned $\tilde{\mathbf{p}}$-path. The strain magnitude at the eigenstate is given by

$$\epsilon = n(\Gamma_1\mu_1^2 + \Gamma_2\mu_2^2 + \Gamma_3\mu_3^2)^{-1}, \tag{5·13}$$

from (5·7) and (5·12), provided the denominator is positive, where

$$\Gamma_r = \mathscr{C}_r/\sigma = \alpha_r \frac{d^2\tilde{q}_r}{dq_r^2} \Big/ \frac{d\tilde{q}_r}{dq_r}, \tag{5·14}$$

from (5·4). These dimensionless quantities are to be evaluated at $q_r = \epsilon\mu_r$, and are hence usually dependent on ϵ itself as well as on $(\alpha_1, \alpha_2, \alpha_3)$. However, $\Gamma_r = m\alpha_r$ when $\tilde{q}(a) = (a^m - 1)/m$ with arbitrary m, which is a widely used family of measures. As to the strain magnitude when component $d\tilde{p}_r = 0$, this is $\epsilon = n\alpha_r/\mu_r\Gamma_r$ by (5·7) and (5·14); therefore $\epsilon_r = n\alpha_r/\Gamma_r = n/m$ for the family mentioned, and in particular $\epsilon_r = n$ when the rth component of load is stationary on the stress path (5·8).

We turn to the eigencondition (4·12) and show how constraints of type $d\mathbf{p} = C\,d\mathbf{q}$ arise in practice. Consider a thin-walled circular cylinder with closed ends, loaded by internal fluid pressure P and an external tensile load L. The axial, circumferential, and through-thickness components of Cauchy stress are

$$\sigma_1 = \frac{Pr}{2t} + \frac{L}{2\pi rt}, \quad \sigma_2 = \frac{Pr}{t}, \quad \sigma_3 = 0,$$

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when the deformation is treated as homogeneous, with r and t as the current mean radius and wall thickness. The components of incremental logarithmic strain are

$$d\epsilon_1 = -\left(\frac{dr}{r}+\frac{dt}{t}\right), \quad d\epsilon_2 = \frac{dr}{r}, \quad d\epsilon_3 = \frac{dt}{t}.$$

With these variables any passive mode of the loading environment is expressible by

$$d\sigma_1 = C_{11}d\epsilon_1 + C_{12}d\epsilon_2, \quad d\sigma_2 = C_{21}d\epsilon_1 + C_{22}d\epsilon_2.$$

Particular couplings of interest are

(i) $dP = 0 = dL$; (ii) $dP = 0 = d(L+\pi r^2 P)$;
(iii) $dP = 0 = d(L/\pi r^2 P)$; (iv) $dL = 0 = d(L/\pi r^2 P)$.

The last is identical with $dL = 0 = d(\pi r^2 P)$, of course. The respective matrices C are readily found to be

$$\left.\begin{aligned} &\text{(i)} \begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_2 & 2\sigma_2\end{pmatrix}; \quad \text{(ii)} \begin{pmatrix}\sigma_1 & 0\\ \sigma_2 & 2\sigma_2\end{pmatrix};\\ &\text{(iii)} \begin{pmatrix}\sigma_1 & 2\sigma_1\\ \sigma_2 & 2\sigma_2\end{pmatrix}; \quad \text{(iv)} \begin{pmatrix}\sigma_1 & 0\\ \sigma_2 & 0\end{pmatrix}. \end{aligned}\right\} \tag{5·15}$$

Note that only the first of these has diagonal symmetry. The corresponding eigenconditions are

$$\left.\begin{aligned} &\text{(i)} \ h = \sigma_1\mu_1^2 + 2\sigma_2\mu_1\mu_2 + 2\sigma_2\mu_2^2;\\ &\text{(ii)} \ h = \sigma_1\mu_1^2 + \sigma_2\mu_1\mu_2 + 2\sigma_2\mu_2^2;\\ &\text{(iii)} \ h = (\mu_1+2\mu_2)(\mu_1\sigma_1+\mu_2\sigma_2) \equiv (\mu_1+2\mu_2)\sigma;\\ &\text{(iv)} \ h = \mu_1(\mu_1\sigma_1+\mu_2\sigma_2) \equiv \mu_1\sigma. \end{aligned}\right\} \tag{5·16}$$

The degree of dependence on the environment is well illustrated by the spread of these values. For present purposes the analysis need be pursued no farther.

As a final example we make contact with the theory of localized necking in a flat sheet under biaxial loading (1). Consider the unsymmetric coupling

$$d\sigma_1 = \sigma_1(d\epsilon_1 + d\epsilon_2), \quad d\sigma_2 = \sigma_2(d\epsilon_1 + d\epsilon_2). \tag{5·17}$$

The corresponding eigencondition is

$$h = (\mu_1+\mu_2)(\mu_1\sigma_1+\mu_2\sigma_2) \equiv (\mu_1+\mu_2)\sigma,$$

which leads by (5·7) and (5·10) to

$$\epsilon = n/(\mu_1+\mu_2) \quad (\epsilon > 0). \tag{5·18}$$

According to (5·17), and subject only to $\mu_1+\mu_2 > 0$, an eigenmode is possible under stationary $\sigma_1 t$ and $\sigma_2 t$, where t is the current thickness. However, when this coupling is not provided directly by external agency, the sheet may still be able to take advantage of the latent eigenstate by deforming inhomogeneously. In particular, when $\mu_1\mu_2 < 0$ as well as $\mu_1+\mu_2 > 0$, parts of the sheet can become rigid as so to concentrate the incremental deformation in a narrow linear zone. This is necessarily inclined to the axis of μ_1 at an angle whose tangent is $\sqrt{(-\mu_1/\mu_2)}$, which maintains the lengths of fibres lying along the zone. Then the nominal tractions acting on the rigid parts are

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indeed stationary when the principal stresses in the zone vary as in (5·17) and their axes remain aligned with the external loads. Although the same fibres do not remain principal, contrary to the premiss in (5·3), the value of $\boldsymbol{\mu}\, d\mathbf{p}$ is unaffected because of the isotropy. We can readily enlarge the class of eigenmodes to admit rotations of principal axes within the material, by a correspondingly wider prescription of the coupling $d\mathbf{p} = C\, d\mathbf{q}$.

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