Math. Proc. Camb. Phil. Soc. (2003), **134**, 33 © 2003 *Cambridge Philosophical Society*
DOI: 10.1017/S0305004102006242 *Printed in the United Kingdom*

A theorem on cardinal numbers associated with $\mathscr{L}_\infty$ Abelian groups

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(*Received* 16 *March* 2001)

Abstract

The topology of a topological group G is called an $\mathscr{L}_\infty$-topology if it can be represented as the intersection of a decreasing sequence of locally compact Hausdorff group topolgies on G. If $\mathscr{L}_1 < \mathscr{L}_2$ are two distinct $\mathscr{L}_\infty$-topologies on an Abelian group G, it is shown that the quotient of the corresponding character groups has cardinality $\geqslant 2^{\mathfrak{c}}$. A conjecture in this sense announced by J. B. Reade in his paper [**6**] is thereby proved.

1. *Introduction*

Varopoulos introduced in [**10**] the class $\mathscr{L}_\infty$ of all topological groups whose topologies are the intersection of a decreasing sequence of locally compact Hausdorff group topologies. He set the basis for the development of the harmonic analysis on $\mathscr{L}_\infty$ groups and proved that much of the classical theory of locally compact Abelian groups applies for Abelian $\mathscr{L}_\infty$ groups. Subsequently these groups have also been investigated by others, [**3**, **6**, **9**, **13**]. As a consequence of that research, we now have quite good information about their topological and algebraic structure. Proposition 1 summarizes the properties which are relevant for this paper. Throughout, all groups are assumed to be Abelian.

Proposition 1. *Let $(G, \mathscr{L})$ be any Abelian group equipped with an $\mathscr{L}_\infty$ topology. Then the following assertions are satisfied*:

(i) *$(G, \mathscr{L})$ is maximally almost periodic*;
(ii) *closed subgroups of $(G, \mathscr{L})$ are dually closed and dually embedded*;
(iii) *$(G, \mathscr{L})$ satisfies Pontryagin–van Kampen duality*;
(iv) *if K is a subset of G, which is compact as a subset of $b(G, \mathscr{L})$, the Bohr compactification of $(G, \mathscr{L})$, then K is compact as a subset of $(G, \mathscr{L})$ as well*;
(v) *$(G, \mathscr{L})$ is a locally σ-compact topological k-space in which every compact subset is contained in a compactly generated locally compact subgroup of G; and*
(vi) *there is an ordinal α, with $0 \leqslant \alpha \leqslant \aleph_0$, and another $\mathscr{L}_\infty$ group $(G_0, \mathscr{L}_0)$, contain-*

† Research partially supported by Spanish DGES, grant BFM2000-0913, and Fundació Caixa Castelló, grant P1B98-24.

https://doi.org/10.1017/S0305004102006242 Published online by Cambridge University Press

ing an open subgroup which is the strict inductive limit of an increasing sequence of compact groups, such that $(G, \mathscr{L})$ *is topologically isomorphic to*

$$\left(\bigoplus_{n=1}^{\alpha} \mathbb{R}\right) \times (G_0, \mathscr{L}_0).$$

Proof. Assertions (i)–(iii) are due to Varopoulos [**10**], (iv) to Sulley [**9**] and (v) to Venkataraman [**13**]; (vi) is a synthesis of the structure results given by Sulley and Venkataraman.

Along this line, Reade (cf. [**6**]) proved the following result about the character groups of $\mathscr{L}_\infty$ groups.

THEOREM 1 (Reade). *If* $\mathscr{L}_1 < \mathscr{L}_2$ *are two distinct* $\mathscr{L}_\infty$*-topologies on the same Abelian group* G *with corresponding character groups* $G(\mathscr{L}_1)\hat{}$, $G(\mathscr{L}_2)\hat{}$ *then*

$$\left|\frac{G(\mathscr{L}_2)\hat{}}{G(\mathscr{L}_1)\hat{}}\right| \geqslant 2^{\aleph_1}. \tag{$*$}$$

Reade's theorem extends a previous similar result (found separately and independently by [**5**, **8**, **12**], that holds for locally compact Abelian groups. However, in that case, they obtain a stronger result since, for those groups: the inequality $(*)$ is fulfilled with $\mathfrak{c}$ (the cardinal of the continuum) in place of $\aleph_1$. Thus, in the absence of the continuum hypothesis, Reade's statement does not yield for $\mathscr{L}_\infty$ groups the property proved to hold for locally compact groups by [**5**, **8**, **12**]. As a matter of fact, Reade himself [**6**] conjectures that Theorem 1 should also hold for the cardinal number $2^{\mathfrak{c}}$.

Our motivation for this research has been the verification of Reade's conjecture. We shall begin by introducing some definitions and notation which will be needed in the rest of the paper.

Let (G, τ) be an arbitrary topological abelian group. A character on (G, τ) is a continuous function χ from G to the complex numbers of modulus one $\mathbb{T}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all x and y in G. The pointwise product of two characters is again a character, and the set $\widehat{G}$ of all characters is a group with pointwise multiplication as the composition law. If $\widehat{G}$ is equipped with the compact open topology, it becomes a topological group $(\widehat{G}, \hat{\tau})$ which is called the *dual group* of (G, τ). There is a natural evaluation homomorphism (not necessarily continuous) $e_G: G \to \widehat{\widehat{G}}$ of G to its bidual group. A topological abelian group (G, τ) satisfies *Pontryagin duality* or, equivalently, is *Pontryagin reflexive* if the evaluation map e_G is a topological isomorphism onto.

Definition 1 ([**10**]). Let G and G' be two abelian groups then we say that they are in duality if and only if there is a function

$$\begin{aligned} \langle ., . \rangle : G \times G' &\longrightarrow \mathbb{T} \quad \text{such that} \\ \langle g_1 g_2, g' \rangle &= \langle g_1, g' \rangle \cdot \langle g_2, g' \rangle, \\ \langle g, g'_1 g'_2 \rangle &= \langle g, g'_1 \rangle \cdot \langle g, g'_2 \rangle \end{aligned}$$

for all $g_1, g_2, g \in G$ and $g', g'_1 g'_2 \in G'$ and it holds: (i) if $g \neq 0_G$, the neutral element of G, then there exists $g' \in G'$ such that $\langle g, g' \rangle \neq 1$; and (ii) if $g' \neq 0_{G'}$, the neutral element of G', there exists $g \in G$ such that $\langle g, g' \rangle \neq 1$.

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Definition 2 ([**10**]). If we have a duality $\langle G, G'\rangle$ we say that a topology τ on G is compatible with the duality when $(G,\tau)\hat{} = G'$.

Thus we can say that a topological abelian group (G,τ) is *maximally almost periodic* (MAP) when the topology τ is compatible with the duality $\langle G, \widehat{G}\rangle$ which is defined by $\langle g, \chi\rangle = \chi(g)$ for all $g \in G$ and $\chi \in \widehat{G}$.

For a given duality $\alpha = \langle G, G'\rangle$ we may associate two canonical topologies on G and G' respectively. The topology $w(G, G')$ on G is the weak topology generated by all the elements in G' considered as continuous homomorphisms into $\mathbb{T}$. The topology $w(G', G)$ is defined similarly. Both topologies are totally bounded and compatible with the given duality, therefore, their completions are compact groups denoted by $b(G, G')$ and $b(G', G)$ respectively (cf. [**10**]).

If Γ is a group of characters of a group G, or conversely, and S is a non-empty subset of G, we will denote the *annihilator*, $\{\kappa \in \Gamma\colon \kappa(S) = 1\}$ of S in Γ by $\mathrm{An}(S;\Gamma)$.

For the proof of our main result, we shall use the following lemma whose proof is implicit in Rosenthal's well-known theorem characterizing the Banach spaces which contain an isomorphic copy of l^1 (see [**2**, **7**]). Henceforth, $\mathbb{C}$ denotes the set of all complex numbers and $C_p(X, \mathbb{C})$ is the space of all complex-valued continuous functions on X provided with the topology of pointwise convergence.

Lemma 1 ([**7**]). *Let K be a compact Hausdorff space and consider a sequence $\{f_n\}_{n<\omega}$ in $C_p(K, \mathbb{C})$ containing no subsequence pointwise convergent in $\mathbb{C}^K$. Then, there are an infinite set $M \subseteq \omega$ and two disjoint disks in $\mathbb{C}$, I_1 and I_2, such that for each subsequence $\{f_n\colon n \in J\}$ of $\{f_n\colon n \in M\}$ there is $x \in K$ with $f_n(x) \in I_1$, for all $n \in J$, and $f_n(x) \in I_2$ for all $n \in M\backslash J$.*

The above Lemma is related to the following definition (cf. [**3**]): given two subsets A and B in a topological group G we say they are *separated* by Γ, a subgroup of $\widehat{G}$, if there are two disjoint closed intervals in $\mathbb{T}$, say I_0 and I_1, and a character $\chi \in \Gamma$ such that $\chi(A) \subseteq I_0$ and $\chi(B) \subseteq I_1$.

2. *Main result*

Theorem 2. *If $\mathscr{L}_1 < \mathscr{L}_2$ are two distinct $\mathscr{L}_\infty$-topologies on the same Abelian group G with corresponding character groups $G(\mathscr{L}_1)\hat{}$, $G(\mathscr{L}_2)\hat{}$ then*

$$\left|\frac{G(\mathscr{L}_2)\hat{}}{G(\mathscr{L}_1)\hat{}}\right| \geqslant 2^{\mathfrak{c}}.$$

For the reader's sake, we have separated the proof of the main result in several auxiliary lemmas.

Lemma 2. *If $\mathscr{L}_1$ and $\mathscr{L}_2$ are two distinct $\mathscr{L}_\infty$-topologies on the same Abelian group G and the identity mapping $\phi\colon (G, \mathscr{L}_1) \to (G, \mathscr{L}_2)$ is not continuous then there is a separable and closed subgroup H of $(G, \mathscr{L}_1)$ such that $\phi_{|H}\colon (H, \mathscr{L}_{1|H}) \to (H, \mathscr{L}_{2|H})$ is not continuous.*

Proof. We proceed by reduction to the absurd. Suppose that ϕ is continuous on each separable closed subgroup of $(G, \mathscr{L}_1)$. Then, for every $\chi \in G(\mathscr{L}_2)\hat{}$, it holds that $\psi = \chi \circ \phi$ is $\mathscr{L}_1$-continuous on each countable subset of G. In particular, ψ is sequentially continuous on $(G, \mathscr{L}_1)$. Now, taking into account Proposition 1(v) and

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a well known result of [**11**] asserting the continuity of any sequentially continuous group homomorphism defined on a locally compact group, it follows that ψ is continuous on $(G, \mathscr{L}_1)$. Thus, we have just verified that each $\mathscr{L}_2$-continuous character on G is $\mathscr{L}_1$-continuous as well. That is, $G(\mathscr{L}_2)\hat{} \subset G(\mathscr{L}_1)\hat{}$ and, as a consequence, the topology $w(G, G(\mathscr{L}_1)\hat{})$ is finer than $w(G, G(\mathscr{L}_2)\hat{})$. For that reason, if K is a $\mathscr{L}_1$-compact subset of G, we have that K is $w(G, G(\mathscr{L}_1)\hat{})$-compact and, therefore, $w(G, G(\mathscr{L}_2)\hat{})$-compact. Applying Proposition 1(iv), we obtain finally that K also is $\mathscr{L}_2$-compact. Therefore, $\mathscr{L}_1$ and $\mathscr{L}_2$ must coincide on K. Then, Proposition 1(v) yields the continuity of ϕ from $(G, \mathscr{L}_1)$ onto $(G, \mathscr{L}_2)$. This is a contradiction which completes the proof.

LEMMA 3. *Let $\mathscr{L}_1 < \mathscr{L}_2$ be two distinct $\mathscr{L}_\infty$-topologies on the same Abelian group G. Assuming that $(G, \mathscr{L}_1)$ is separable and $(G, \mathscr{L}_2)$ is hemicompact, then*

$$\left| \frac{G(\mathscr{L}_2)\hat{}}{G(\mathscr{L}_1)\hat{}} \right| \geqslant 2^{\mathfrak{c}}.$$

Proof. Firstly observe that, since $(G, \mathscr{L}_1)$ is separable, there is a countable subgroup H of G that separates the points of $G(\mathscr{L}_1)\hat{}$. Thus, the group $(G(\mathscr{L}_1)\hat{}, w(G(\mathscr{L}_1)\hat{}, H))$ is metrizable and totally bounded and, as a consequence, $|G(\mathscr{L}_1)\hat{}| \leqslant \mathfrak{c}$.

Now, suppose that for each $\mathscr{L}_2\hat{}$-compact subset of $G(\mathscr{L}_2)\hat{}$, it holds that $K \bigcap G(\mathscr{L}_1)\hat{}$ also is a $\mathscr{L}_1\hat{}$-compact subset of $G(\mathscr{L}_1)\hat{}$. Since $\mathscr{L}_2\hat{}_{|G(\mathscr{L}_1)\hat{}}$ is coarser than $\mathscr{L}_1\hat{}$, it follows that both topologies coincide on $K \bigcap G(\mathscr{L}_1)\hat{}$. Hence, the identity mapping $\hat{\phi}$: $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_2\hat{}_{|G(\mathscr{L}_1)\hat{}}) \to (G(\mathscr{L}_1)\hat{}, \mathscr{L}_1\hat{})$ is sequentially continuous.

On the other hand, the group $(G, \mathscr{L}_2)$ is hemicompact by hypothesis and this fact clearly implies the metrizability of $(G(\mathscr{L}_2)\hat{}, \mathscr{L}_2\hat{})$. Thus, the group $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_2\hat{}_{|G(\mathscr{L}_1)\hat{}})$ must also be metrizable and, as a consequence, we deduce the continuity of $\hat{\phi}$. Therefore, we have obtained that the groups $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_2\hat{}_{|G(\mathscr{L}_1)\hat{}})$ and $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_1\hat{})$ are topologically isomorphic.

Now, observe that $G(\mathscr{L}_1)\hat{} \subset G(\mathscr{L}_2)\hat{} \subset b(G(\mathscr{L}_1)\hat{}, G)$ and all of these groups have the same group G *qua* algebraic dual. This fact implies that $G(\mathscr{L}_1)\hat{}$ is $w(G(\mathscr{L}_2)\hat{}, G)$-dense in $G(\mathscr{L}_2)\hat{}$ and, by [**9**, section 2, p. 631], it follows easily that $G(\mathscr{L}_1)\hat{}$ is $\mathscr{L}_2\hat{}$-dense in $G(\mathscr{L}_2)\hat{}$. Now, applying a result of ([**1**]) the (topological) dual groups of $(G(\mathscr{L}_2)\hat{}, \mathscr{L}_2\hat{})$ and $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_2\hat{}_{|G(\mathscr{L}_1)\hat{}})$ coincide. This yields the following chain of topological isomophisms:

$$(G, \mathscr{L}_1) \cong (G(\mathscr{L}_1)\hat{}, \mathscr{L}_1\hat{})\hat{} \cong (G(\mathscr{L}_1)\hat{}, \mathscr{L}_2\hat{}_{|G(\mathscr{L}_1)\hat{}})\hat{} \cong (G(\mathscr{L}_2)\hat{}, \mathscr{L}_2\hat{})\hat{} \cong (G, \mathscr{L}_2),$$

which contradicts the hypothesis of the lemma.

Accordingly, there is no loss of generality in assuming the existence of an $\mathscr{L}_2\hat{}$-compact subset K of $G(\mathscr{L}_2)\hat{}$ such that $K \bigcap G(\mathscr{L}_1)\hat{}$ fails to be $\mathscr{L}_1\hat{}$-compact. Since $K \bigcap G(\mathscr{L}_1)\hat{}$ is $\mathscr{L}_2\hat{}_{|G(\mathscr{L}_1)\hat{}}$-closed (therefore, $\mathscr{L}_1\hat{}$-closed) and $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_1\hat{})$ is complete (as a consequence of being $(G, \mathscr{L}_1)$ a topological k-space), it follows that $K \bigcap G(\mathscr{L}_1)\hat{}$ may not be totally bounded in $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_1\hat{})$. Hence, we can obtain a sequence $\{\chi_n\}_{n<\omega} \subset K \bigcap G(\mathscr{L}_1)\hat{}$ and an open neighbourhood of the identity U such that

$$(\chi_n + U) \bigcap (\chi_m + U) = \emptyset \quad \text{for all } n \neq m. \tag{**}$$

On the other hand, the group $(G, \mathscr{L}_1)$ is separable and this fact, plus the assertion

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in Proposition 1 (vi) about the structure of $\mathscr{L}_\infty$ groups, implies that $(G, \mathscr{L}_1)$ is σ-compact. Thus there is a sequence $\{K_n\}_{n<\omega}$ of $\mathscr{L}_1$-compact subsets of G such that $G = \bigcup\{K_n : n < \omega\}$. Next we establish the following.

Claim. There is an index p and a subsequence $\{\chi_{n(m)}\}_{m<\omega}$ of $\{\chi_n\}_{n<\omega}$ such that no subsequence of $\{\chi_{n(m)}\}_{m<\omega}$ converges pointwise on $\mathbb{T}^{(K_1 \cup \ldots \cup K_p)}$.

Otherwise, we can define inductively a family of subsequences $\{\chi_n^{(p)}\}_{n<\omega}$ of $\{\chi_n\}_{n<\omega}$ such that

(i) $\{\chi_n^{(p+1)}\}_{n<\omega}$ is a subsequence of $\{\chi_n^{(p)}\}_{n<\omega}$.
(ii) The sequence $\{\chi_n^{(p)}\}_{n<\omega}$ converges pointwise in $K_1 \cup \ldots \cup K_p$ to $f_p \in \mathbb{T}^{(K_1 \cup \ldots \cup K_p)}$.
(iii) $(f_{p+1})_{|_{K_1 \cup \ldots \cup K_p}} = f_p$ for all $p < \omega$.

Consider now the diagonal subsequence $\{\chi_n^{(n)}\}_{n<\omega}$. This sequence is convergent to f_p in the topology of pointwise convergence on $K_1 \cup \cdots \cup K_p$ for all $p < \omega$. Define then the sequence of sets

$$A_p = \{f \in b(G(\mathscr{L}_1)\hat{}, G) \colon (f_{|_{K_1 \cup \ldots \cup K_p}} = f_p\}.$$

Every A_p is a nonempty closed subset of the compact group $b(G(\mathscr{L}_1)\hat{}, G)$ and $A_{p+1} \subseteq A_p$ for all $p < \omega$. Hence, there is some $f \in b(G(\mathscr{L}_1)\hat{}, G)$ such that $f \in \bigcap_p A_p$. It is then easily checked that $\{\chi_n^{(n)}\}_{n<\omega}$ converges pointwise to f on G. Thus, the sequence $\{\chi_n^{(n)} - \chi_{2n}^{(2n)}\}_{n<\omega}$ is $w(G(\mathscr{L}_1)\hat{}, G)$-convergent to the identity in $G(\mathscr{L}_1)\hat{}$. We can now apply [**9**, theorem 11] about the preservation of compactness to obtain that $\{\chi_n^{(n)} - \chi_{2n}^{(2n)}\}_{n<\omega}$ converges to identity in $(G(\mathscr{L}_1)\hat{}, \mathscr{L}_1\hat{})$. But this is impossible according to (**) and, therefore, the claim has been proved.

Hence, we can get a subsequence $\{\chi_{n(m)}\}_{m<\omega}$ and index p such that no subsequence of the former converges in $\mathbb{T}^{(K_1 \cup \ldots \cup K_p)}$. Now we are in position to apply Lemma 1 to the sequence $\{\chi_{n(m)}\}_{m<\omega}$ (regarded as a set of continuous functions on $K_1 \cup \cdots \cup K_p$), to obtain an infinite subset B of $\{\chi_{n(m)}\}_{m<\omega}$, each subset of which is separated by G from its complement in B. We now prove that B is discrete in $(G(\mathscr{L}_1)\hat{}, w(G(\mathscr{L}_1)\hat{}, G))$ and C^*-embedded in $b(G(\mathscr{L}_1)\hat{}, G)$.

Indeed, take an arbitrary element $b_0 \in B$. The subsets $\{b_0\}$ and $B\backslash\{b_0\}$ are separated by G so that there exist two disjoint closed intervals in $\mathbb{T}$, say I_0 and I_1, and an element $g \in G$ with $\langle g, b_0\rangle \in I_0$ and $\langle g, b\rangle \subseteq I_1$ for all $b \in B\backslash\{b_0\}$. Then $\langle g, .\rangle^{-1}(\mathbb{T}\backslash I_1)$ is a neighbourhood of b_0 in $(G(\mathscr{L}_1)\hat{}, w(G(\mathscr{L}_1)\hat{}, G))$ that meets B exactly at the point b_0. That is, B is $w(G(\mathscr{L}_1)\hat{}, G)$ -discrete. To see, on the other hand, that B is C^*-embedded in $b(G(\mathscr{L}_1)\hat{}, G)$, it is enough to prove that for each pair, B_0 and B_1, of disjoint subsets of B, there is a continuous real-valued function $f \in C(b(G(\mathscr{L}_1)\hat{}, G))$ such that $0 \leqslant f \leqslant 1$, $f(B_0) = \{0\}$ and $f(B_1) = \{1\}$ (see [**4**, p. 18]). Since B_0 and B_1 are separated by G we can find again two disjoint closed intervals in $\mathbb{T}$, say I_0 and I_1, and an element $g \in G$ such that $\langle g, B_0\rangle \subseteq I_0$ and $\langle g, B_1\rangle \subseteq I_1$. Let now $\varphi\colon \mathbb{T} \to \mathbb{R}$ be defined by

$$\varphi(t) = \frac{d(t, I_0)}{d(t, I_0) + d(t, I_1)}.$$

Clearly, φ is continuous, $0 \leqslant \varphi \leqslant 1$, $\varphi(I_0) = \{0\}$ and $\varphi(I_1) = \{1\}$. On the other hand, since every continuous character of a topological group extends continuously to the Bohr compactification of the group, it can be assumed that $\langle g, .\rangle$ is defined on

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$b(G(\mathscr{L}_1)^\wedge, G)$. Then it is sufficient to take $f = \varphi \circ \langle g, . \rangle$ in order to verify that B_0 and B_1 are completely separated by f in $b(G(\mathscr{L}_1)^\wedge, G)$.

Thus B is discrete and C^*-embedded in $b(G(\mathscr{L}_1)^\wedge, G)$. This means that $cl_{b(G(\mathscr{L}_1)^\wedge, G)}B$ is homeomorphic to $\beta\omega$, the Stone–Cech compactification of ω (the set of all natural numbers equipped with the discrete topology). Hence, the cardinality of $cl_{b(G(\mathscr{L}_1)^\wedge, G)}B$ is $2^{\mathfrak{c}}$ and, since K is $w(G(\mathscr{L}_2)^\wedge, G)$-compact and $B \subset K$, we obtain that K must have cardinality greater than or equal to $2^{\mathfrak{c}}$. Now, we have verified above that $G(\mathscr{L}_1)^\wedge$ has cardinality less than or equal to $\mathfrak{c}$ and, as a consequence, it follows that $|(G(\mathscr{L}_2)^\wedge)/(G(\mathscr{L}_1)^\wedge)| \geqslant 2^{\mathfrak{c}}$.

LEMMA 4. *Let $\mathscr{L}_1 < \mathscr{L}_2$ be two distinct $\mathscr{L}_\infty$-topologies on the same Abelian group G such that $(G, \mathscr{L}_1)$ is separable and $(G, \mathscr{L}_2)$ is discrete. Then*

$$\left| \frac{G(\mathscr{L}_2)^\wedge}{G(\mathscr{L}_1)^\wedge} \right| \geqslant 2^{\mathfrak{c}}.$$

Proof. In this case, $(G(\mathscr{L}_2)^\wedge, \mathscr{L}_2^\wedge) = (G(\mathscr{L}_2)^\wedge, w(G(\mathscr{L}_2)^\wedge, G)) = b(G(\mathscr{L}_2)^\wedge, G)$ which is a compact group and $G(\mathscr{L}_1)^\wedge$ is a proper dense subgroup of it. On the other hand, the group $(G(\mathscr{L}_1)^\wedge, \mathscr{L}_1^\wedge)$ may not be totally bounded since $(G, \mathscr{L}_1)$ is not discrete. Hence, we can obtain a sequence $\{\chi_n\}_{n<\omega} \subset G(\mathscr{L}_1)^\wedge$ and an open neighbourhood of the identity U in $(G(\mathscr{L}_1)^\wedge, \mathscr{L}_1^\wedge)$ such that

$$(\chi_n + U) \bigcap (\chi_m + U) = \emptyset \quad \text{for all } n \neq m$$

the proof then follows *mutatis mutandis* as in the last part of Lemma 3.

LEMMA 5. *Let $\mathscr{L}_1 < \mathscr{L}_2$ be two distinct $\mathscr{L}_\infty$-topologies on the same Abelian group G such that $(G, \mathscr{L}_2)$ is hemicompact or discrete. Then*

$$\left| \frac{G(\mathscr{L}_2)^\wedge}{G(\mathscr{L}_1)^\wedge} \right| \geqslant 2^{\mathfrak{c}}.$$

Proof. By Lemma 2 there is a separable closed subgroup H of $(G, \mathscr{L}_1)$ such that $(\mathscr{L}_1)_{|H} < (\mathscr{L}_2)_{|H}$. By Lemma 3 we have that $\left|(H(\mathscr{L}_2)^\wedge)/(H(\mathscr{L}_1)^\wedge)\right| \geqslant 2^{\mathfrak{c}}$. Since $H(\mathscr{L}_2)^\wedge \cong G(\mathscr{L}_2)^\wedge/\mathrm{An}(H, G(\mathscr{L}_2)^\wedge)$ and $H(\mathscr{L}_1)^\wedge \cong G(\mathscr{L}_1)^\wedge/\mathrm{An}(H, G(\mathscr{L}_1)^\wedge)$ (see [**10**]), it is readily deduced that $\left|(G(\mathscr{L}_2)^\wedge)/(G(\mathscr{L}_1)^\wedge)\right| \geqslant 2^{\mathfrak{c}}$.

Proof of Theorem 2. Let ϕ: $(G, \mathscr{L}_1) \to (G, \mathscr{L}_2)$ the identity mapping and let K be an $\mathscr{L}_2$-open hemicompact subgroup of G, that we know to exist by Proposition 1(vi). If $(\mathscr{L}_2)_{|K}$ is strictly finer than $(\mathscr{L}_1)_{|K}$, we can apply Lemma 5 to obtain that $\left|(K(\mathscr{L}_2)^\wedge)/(K(\mathscr{L}_1)^\wedge)\right| \geqslant 2^{\mathfrak{c}}$ and, in the same manner as in Lemma 5, this implies that $\left|(G(\mathscr{L}_2)^\wedge)/(G(\mathscr{L}_1)^\wedge)\right| \geqslant 2^{\mathfrak{c}}$. Therefore, we may assume that $(\mathscr{L}_1)_{|K}$ and $(\mathscr{L}_2)_{|K}$ coincide. Hence, K is $\mathscr{L}_1$-complete and, as a consequence, a closed subgroup of $(G, \mathscr{L}_1)$. Let ϕ/K: $(G/K, \mathscr{L}_1/K) \to (G/K, \mathscr{L}_2/K)$ denote the identity mapping on G/K, with $\mathscr{L}_i/K$ denoting the quotient topology for $i = 1, 2$. Observe that $(G/K, \mathscr{L}_2/K)$ is discrete. Thus, if ϕ/K is continuous, it follows that $(G/K, \mathscr{L}_1/K)$ is discrete as well. Hence, K must be open in $(G, \mathscr{L}_1)$ and this yields $\mathscr{L}_1 = \mathscr{L}_2$ on G, which is impossible. Accordingly, we have that $\mathscr{L}_2/K$ is strictly finer than $\mathscr{L}_1/K$ on G/K. Then, again by [**10**] and Lemma 5, it follows that

$$\left| \frac{\mathrm{An}(K, G(\mathscr{L}_2)^\wedge)}{\mathrm{An}(K, G(\mathscr{L}_1)^\wedge)} \right| = \left| \frac{(G/K)(\mathscr{L}_2/K)^\wedge}{(G/K)(\mathscr{L}_1/K)^\wedge} \right| \geqslant 2^{\mathfrak{c}}. \qquad (***)$$

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Since $\mathrm{An}(K, G(\mathscr{L}_1)\hat{}) = \mathrm{An}(K, G(\mathscr{L}_2)\hat{}) \cap G(\mathscr{L}_1)\hat{}$, we obtain that $(***)$ yields the inequality $|(G(\mathscr{L}_2)\hat{})/(G(\mathscr{L}_1)\hat{})| \geqslant 2^{\mathfrak{c}}$. This completes the proof.

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https://doi.org/10.1017/S0305004102006242 Published online by Cambridge University Press