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NOTES ON LINKAGE OF MODULES

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Abstract Let R be a Cohen–Macaulay local ring. It is shown that under some mild conditions, the Cohen–Macaulay property is preserved under linkage. We also study the connection of the (S_n) locus of a horizontally linked module and the attached primes of certain local cohomology modules of its linked module.

Keywords: linkage of modules; attached prime; local cohomology

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1. Introduction

The theory of linkage for subschemes of projective space goes back more than a century in some sense, but the modern study was introduced by Peskine and Szpiro [**27**] in 1974. Recall that two ideals I and J in a Cohen–Macaulay local ring R are said to be linked if there is a regular sequence α in their intersection such that $I = (\alpha : J)$ and $J = (\alpha : I)$. The first main result in the theory of linkage, due to Peskine and Szpiro, indicates that the Cohen–Macaulay property is preserved under linkage over Gorenstein rings. They also give a counterexample to show that the above result is no longer true if the base ring is Cohen–Macaulay but not Gorenstein. On the other hand, there are interesting extensions of the Peskine–Szpiro theorem to the case where R is Cohen–Macaulay and one of the ideals I or J is strongly Cohen–Macaulay (i.e. Koszul homology modules are Cohen–Macaulay) [**15**] or satisfies the sliding depth condition (on Koszul homology) [**14**]. In a different direction, Schenzel generalized the Peskine–Szpiro theorem to the vanishing of certain local cohomology modules. More precisely, for linked ideals I and J over a Gorenstein local ring R with maximal ideal $\mathfrak{m}$, the Serre condition (S_n) for R/I is equivalent to the vanishing of the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(R/J) = 0$ for all i, $\dim(R/J) - n < i < \dim(R/J)$ [**33**, Theorem 4.1].

A few authors advanced the theory to the setting of linkage of modules in different ways, for instance Martin [**19**], Yoshino and Isogawa [**35**], Martsinkovsky and Strooker [**20**], and Nagel [**24**]. Based on these generalizations, several works have studied the linkage theory in the context of modules; see, for example, [**6**–**10**, **16**, **26**, **28**, **29**]. In this

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paper, we are interested in linkage of modules in the sense of [20]. Martsinkovsky and Strooker generalized the notion of linkage for modules by using the composition of two functors: transpose and syzygy. They showed that ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by zero ideal if and only if $R/\mathfrak{a} \cong \lambda(R/\mathfrak{b})$ and $R/\mathfrak{b} \cong \lambda(R/\mathfrak{a})$, where $\lambda := \Omega\mathsf{Tr}$.

Recall that the (S_k) locus of M, denoted by $S_k(M)$, is the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p}$ satisfies the Serre condition (S_k). The Gorenstein locus of R, denoted by $\operatorname{Gor}(R)$, is the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is a Gorenstein local ring. In the first part of this paper, we study the connection of the (S_n) locus of a horizontally linked module and the attached primes of certain local cohomology modules of its linked module. As a consequence, we obtain the following result (see also Theorem 3.3 for a more general case).

Theorem A. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and let X be an open subset of $\operatorname{Spec} R$ which is contained in $\operatorname{Gor}(R)$. Assume that M is a horizontally linked R-module and that n is a positive integer. Then $X \subseteq S_n(M)$ if and only if $\operatorname{Att}_R(\mathrm{H}^i_\mathfrak{m}(\lambda M)) \subseteq \operatorname{Spec} R \setminus X$ for $d - n < i < d$.*

Note that for an Artinian R-module N, $\operatorname{Att}_R(N) = \emptyset$ if and only if $N = 0$. Therefore, the above theorem can be viewed as a generalization of Schenzel's result. We should note that Theorem A is new even in the case of cyclic modules. The above theorem helps us to study the attached primes of local cohomology modules of a linked module. As an application of Theorem A, for a horizontally linked module M over a Gorenstein local ring R and a closed subset X of $\operatorname{Spec} R$, it is shown that $\operatorname{Att}_R(\mathrm{H}^i_\mathfrak{m}(M)) \subseteq X$ for $0 < i < \dim R$ if and only if $\operatorname{Att}_R(\mathrm{H}^i_\mathfrak{m}(\lambda M)) \subseteq X$ for $0 < i < \dim R$ (see Corollary 3.8).

There is another interesting generalization of the Peskine–Szpiro theorem, due to Martin [18]. Let R be a Gorenstein local ring and let I be a Cohen–Macaulay ideal linked to J by an ideal $\mathfrak{a}$, where $\mathfrak{a}$ is Cohen–Macaulay and generically Gorenstein. Then J is Cohen–Macaulay if and only if $I + \mathfrak{b}$ is a Cohen–Macaulay ideal with the same height as $\mathfrak{b}$, where $\mathfrak{b}$ is an ideal of R, containing $\mathfrak{a}$, with $\mathfrak{b}/\mathfrak{a}$ equal to a canonical module of $\omega_{R/\mathfrak{a}}$ [18, Corolary 2.11]. In the second part of this paper, we study the theory of linkage for Cohen–Macaulay modules. As a consequence, we generalize Martin's result to the module case. More precisely, we have the following result (see also Corollary 4.3 for a more general case).

Theorem B. *Let R be a Cohen–Macaulay local ring with canonical module ω_R and let $\mathfrak{a}$ be a generically Gorenstein, Cohen–Macaulay ideal of R. Write $\omega_{R/\mathfrak{a}} \cong \mathfrak{c}/\mathfrak{a}$. Assume that M is a Cohen–Macaulay R-module which is linked to an R-module N by the ideal $\mathfrak{a}$. Then N is Cohen–Macaulay if and only if $\mathsf{depth}_R(M/\mathfrak{c}M) \geq \dim(R/\mathfrak{a}) - 1$.*

This paper is organized as follows. In § 2 we gather together preliminary notions, definitions and some known results which will be used in this paper. In § 3 we study the connection of the (S_n) locus of a horizontally linked module and the attached primes of certain local cohomology modules of its linked module. As a consequence, we prove Theorem A in this section. In § 4 we study the theory of linkage for Cohen–Macaulay modules. We prove Theorem B in this section. In § 5 we establish a duality on local cohomology modules of a linked module.

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2. Definitions and preliminary results

Throughout the paper R is a commutative Noetherian local ring and all modules over R are assumed to be finitely generated. We start by recalling several definitions and terminology from [2, 5, 20].

Let M be an R-module. For a positive integer i, we denote by $\Omega^i M$ the ith syzygy of M, namely, the image of the ith differential map in a minimal free resolution of M.

The *transpose* $\mathsf{Tr}M$ of M is defined as the cokernel of the R-dual map $\partial_1^* = \mathsf{Hom}_R(\partial_1, R)$ of the first differential map ∂_1 in a minimal free resolution of M. Hence there is an exact sequence of the form $0 \to M^* \to P_0^* \to P_1^* \to \mathsf{Tr}M \to 0$. Note that the modules $\Omega^i M$ and $\mathsf{Tr}M$ are uniquely determined up to isomorphism, since so is a minimal free resolution of M, and there is a stable isomorphism $\Omega^2 \mathsf{Tr}M \cong M^*$.

A *stable* module is a module with no non-zero free direct summands. An R-module M is called a *syzygy module* if it is embedded in a free R-module.

2.1. Linkage of modules

Two R-modules M and N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$, where $\lambda M := \Omega \mathsf{Tr} M$. Equivalently, M is horizontally linked (to λM) if and only if $M \cong \lambda^2 M$. An R-module M is horizontally linked if and only if it is stable and $\mathsf{Ext}^1_R(\mathsf{Tr}M, R) = 0$, equivalently M is stable and is a syzygy module [20, Theorem 2 and Proposition 3]. An R-module M is said to be *linked* to an R-module N, by an ideal $\mathfrak{c}$ of R, if $\mathfrak{c} \subseteq \operatorname{ann}_R(M) \cap \operatorname{ann}_R(N)$ and M and N are horizontally linked as $R/\mathfrak{c}$-modules. In this situation we write $M \underset{\mathfrak{c}}{\sim} N$.

Let M be a stable R-module and let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of M. Then $P_0^* \to P_1^* \to \mathsf{Tr}M \to 0$ is a minimal projective presentation of $\mathsf{Tr}M$ [1, Theorem 32.13]. We quote the following induced exact sequences:

$$0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow \lambda M \longrightarrow 0, \tag{2.1.1}$$

$$0 \longrightarrow \lambda M \longrightarrow P_1^* \longrightarrow \mathsf{Tr}M \longrightarrow 0. \tag{2.1.2}$$

2.2. Gorenstein dimension with respect to a semidualizing module

An R-module C is called a semidualizing module, if the homothety morphism $R \to \mathsf{Hom}_R(C, C)$ is an isomorphism and $\mathsf{Ext}^i_R(C, C) = 0$ for all $i > 0$. Semidualizing modules are initially studied by Foxby [11] and Golod [13]. It is clear that R itself is a semidualizing R-module. Over a Cohen–Macaulay local ring R, a canonical module ω_R of R is a semidualizing module with finite injective dimension.

An R-module M is said to be *totally C-reflexive* if the natural map $M \to \mathsf{Hom}(\mathsf{Hom}(M, C), C)$ is bijective and $\mathsf{Ext}^i_R(M, C) = 0 = \mathsf{Ext}^i_R(\mathsf{Hom}(M, C), C)$ for all $i \geq 1$. The smallest non-negative integer n for which there exists an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0,$$

such that each X_i is totally C-reflexive, is called the G_C*-dimension* of M. If M has G_C-dimension n, we write $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = n$. Therefore M is totally C-reflexive if and only if $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = 0$.

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Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective presentation of M. The transpose of M with respect to C, $\mathsf{Tr}_C M$, is defined to be Coker f^∇, where $(-)^\nabla := \mathsf{Hom}_R(-, C)$, which satisfies the exact sequence

$$0 \to M^\nabla \to P_0^\nabla \to P_1^\nabla \to \mathsf{Tr}_C M \to 0. \tag{2.2.1}$$

We catalogue some properties of Gorenstein dimension with respect to a semidualizing module.

(i) If C is a dualizing R-module, then $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) < \infty$ for all R-modules M.

(ii) If $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) < \infty$, then $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = \mathsf{depth}(R) - \mathsf{depth}(M)$.

(iii) If $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) < \infty$, then $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = \sup\{i \mid \mathsf{Ext}^i_R(M, C) \neq 0\}$.

(iv) $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = 0$ if and only if $\mathsf{G}_C\text{-}\mathsf{dim}_R(\mathsf{Tr}_C M) = 0$.

An R-module M is said to satisfy the property $\widetilde{S}_k$ if $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{k, \mathsf{depth}\, R_\mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Recall that the $\widetilde{S}_k$ locus of M, denoted by $\widetilde{S}_k(M)$, is defined by

$$\widetilde{S}_k(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \text{ satisfies } \widetilde{S}_k\}.$$

For a horizontally linked module M over a Cohen–Macaulay local ring R, the properties $\widetilde{S}_k$ and (S_k) are identical. An R-module M is called n-torsionfree, when $\mathsf{Ext}^i_R(\mathsf{Tr} M, R) = 0$ for $1 \leq i \leq n$.

The following theorem is due to Auslander and Bridger.

Theorem 2.3 (Auslander and Bridger [2, Theorem 4.25]). *Let M be an R-module of finite Gorenstein dimension. Then M is n-torsionfree if and only if M satisfies $\widetilde{S}_n$.*

2.4. Auslander class

Let C be a semidualizing R-module. The Auslander class with respect to C, denoted by $\mathcal{A}_C$, consists of all R-modules M satisfying the following conditions.

(i) The natural map $\mu : M \to \mathsf{Hom}_R(C, M \otimes_R C)$ is an isomorphism.

(ii) $\mathsf{Tor}^R_i(M, C) = 0 = \mathsf{Ext}^i_R(C, M \otimes_R C)$ for all $i > 0$.

Every module of finite projective dimension is in the Auslander class $\mathcal{A}_C$. Over a Cohen–Macaulay local ring R with canonical module ω_R, we have $M \in \mathcal{A}_{\omega_R}$ if and only if $\mathsf{G}\text{-}\mathsf{dim}_R(M) < \infty$ [12, Theorem 1].

2.5. Secondary representation

Let M be a non-zero Artinian R-module. We say that M is secondary if the multiplication by x on M is surjective or nilpotent for every $x \in R$. In this case, $\mathfrak{p} := \sqrt{\operatorname{ann}_R(M)}$ is a prime ideal of R and we say that M is $\mathfrak{p}$-secondary. Note that every Artinian R-module M has a minimal secondary representation $M = M_1 + \cdots + M_n$, where M_i is $\mathfrak{p}_i$-secondary, each M_i is not redundant and $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all $i \neq j$. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$

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is independent of the choice of the minimal secondary representation of M. This set is called the set of attached primes of M and denoted by $\operatorname{Att}_R(M)$.

We denote by $\widehat{M}$ the completion of M in the $\mathfrak{m}$-adic topology. In what follows we gather together some basic properties of attached primes of a module which will be used throughout the paper (see [5, 17]).

Theorem 2.6. *Let M be an Artinian R-module. The following statements hold.*

(i) $M \neq 0$ *if and if* $\operatorname{Att}_R(M) \neq \emptyset$.

(ii) M *has finite length if and only if* $\operatorname{Att}_R(M) \subseteq \{\mathfrak{m}\}$.

(iii) $\operatorname{Att}_R(M) = \operatorname{Ass}_R(\mathsf{Hom}_R(M, \mathrm{E}_R(k)))$, *where* $\mathrm{E}_R(k)$ *is the injective envelope of* k.

(iv) $\operatorname{Att}_R(M) = \{P \cap R \mid P \in \operatorname{Att}_{\widehat{R}}(M)\}$.

The following result plays an important role in the proof of Theorem A.

Theorem 2.7 (Nhan and Quy [25, Theorem 1.1]). *Let R be a ring which is the homomorphic image of a Cohen–Macaulay local ring. Then*

$$\operatorname{Att}_{R_\mathfrak{p}}(\mathrm{H}^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Att}_R(\mathrm{H}^i_\mathfrak{m}(M)), \mathfrak{q} \subseteq \mathfrak{p}\},$$

for every R-module M, integer $i \geq 0$ and prime ideal $\mathfrak{p}$ of R.

3. Linkage and the attached primes of local cohomology modules

Recall that a subset X of $\operatorname{Spec} R$ is called *stable under generalization* provided that if $\mathfrak{p} \in X$ and $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q} \in X$. Note that every open subset of $\operatorname{Spec} R$ is stable under generalization. For an integer n, set $X^n(R) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathsf{depth}\, R_\mathfrak{p} \leq n\}$.

We start by recalling the following result.

Lemma 3.1. *Let C be a semidualizing R-module and let M be an R-module. Assume that $M \in \mathcal{A}_C$ and that n is a positive integer. Consider the following statements.*

(i) $\mathsf{Ext}^i_R(\mathsf{Tr}M, R) = 0$ *for* $1 \leq i \leq n$.

(ii) $\mathsf{Ext}^i_R(\mathsf{Tr}M, C) = 0$ *for* $1 \leq i \leq n$.

(iii) M *satisfies* $\widetilde{S}_n$.

Then we have the following consequences.

(a) *(i)⇔(ii)⇒(iii)*

(b) *If* $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ *for all* $\mathfrak{p} \in \mathrm{X}^{n-1}(R)$ *(e.g.* $\mathrm{id}_{R_\mathfrak{p}}(C_\mathfrak{p}) < \infty$ *for all* $\mathfrak{p} \in \mathrm{X}^{n-1}(R)$*), then all the statements (i)–(iii) are equivalent.*

Proof. See [31, Theorem 4.1] and [8, Theorem 2.12]. □

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The proof of Theorem A is based on the following lemma, which is of independent interest.

Lemma 3.2. *Let C be a semidualizing R-module and let M be an n-torsionfree R-module. Assume that $M \in \mathcal{A}_C$ and that $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in X^n(R)$ (e.g. $\mathrm{id}_{R_\mathfrak{p}}(C_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in X^n(R)$). Then $\mathrm{Ass}_R(\mathsf{Ext}^{n+1}_R(\mathsf{Tr}M, C)) = \mathrm{Ass}_R(\mathsf{Ext}^{n+1}_R(\mathsf{Tr}M, R))$.*

Proof. We argue by induction on n. Let $n = 0$ and let $\mathfrak{p} \in \mathrm{Ass}(\mathsf{Ext}^1_R(\mathsf{Tr}M, R))$. Consider the exact sequence (see [21, Proposition 5])

$$0 \to \mathsf{Ext}^1_R(\mathsf{Tr}M, R) \to M \to M^{**} \to \mathsf{Ext}^2_R(\mathsf{Tr}M, R) \to 0. \tag{3.2.1}$$

It follows from the exact sequence (3.2.1) that $\mathfrak{p} \in \mathrm{Ass}_R(M)$. As $M \in \mathcal{A}_C$, we have $M \cong \mathsf{Hom}_R(C, C \otimes_R M)$. Therefore, $\mathfrak{p} \in \mathrm{Ass}_R(M \otimes_R C)$. Let us now consider the exact sequence (see [2, Proposition 2.6])

$$0 \to \mathsf{Ext}^1_R(\mathsf{Tr}M, C) \to M \otimes_R C \to \mathsf{Hom}_R(M^*, C) \to \mathsf{Ext}^2_R(\mathsf{Tr}M, C) \to 0. \tag{3.2.2}$$

If $\mathsf{depth}\, R_\mathfrak{p} = 0$ then $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ and so $\mathsf{Ext}^1_R(\mathsf{Tr}M, R)_\mathfrak{p} = 0$, which is a contradiction. Therefore, $\mathsf{depth}\, R_\mathfrak{p} > 0$ and so $\mathsf{depth}_{R_\mathfrak{p}}(\mathsf{Hom}_R(M^*, C)_\mathfrak{p}) \geq \min\{2, \mathsf{depth}\, R_\mathfrak{p}\} > 0$. Hence, it follows from the exact sequence (3.2.2) that $\mathsf{depth}_{R_\mathfrak{p}}(\mathsf{Ext}^1_R(\mathsf{Tr}M, C)_\mathfrak{p}) = 0$. In other words, $\mathfrak{p} \in \mathrm{Ass}_R(\mathsf{Ext}^1_R(\mathsf{Tr}M, C))$. Similarly, one can prove that $\mathrm{Ass}_R(\mathsf{Ext}^1_R(\mathsf{Tr}M, C)) \subseteq \mathrm{Ass}_R(\mathsf{Ext}^1_R(\mathsf{Tr}M, R))$. Now let $n > 0$. As M is n-torsionfree, $\mathsf{Ext}^i_R(\mathsf{Tr}M, R) = 0 = \mathsf{Ext}^i_R(\mathsf{Tr}M, C)$ for $1 \leq i \leq n$ by Lemma 3.1. Consider the universal pushforward of M:

$$0 \longrightarrow M \longrightarrow F \longrightarrow M_1 \longrightarrow 0, \tag{3.2.3}$$

where F is free and $\mathsf{Ext}^1_R(M_1, R) = 0$ (see [21]). Hence, by [2, Lemma 3.9], the exact sequence (3.2.3) induces the exact sequence

$$0 \longrightarrow \mathsf{Tr}M_1 \longrightarrow P \longrightarrow \mathsf{Tr}M \longrightarrow 0, \tag{3.2.4}$$

where P is a free R-module. It follows from the exact sequence (3.2.4) that M_1 is $(n-1)$-torsionfree. Also, by the exact sequence (3.2.3), $M_1 \in \mathcal{A}_C$. Hence, by induction hypothesis,

$$\begin{aligned}\mathrm{Ass}_R(\mathsf{Ext}^{n+1}_R(\mathsf{Tr}M, R)) &= \mathrm{Ass}_R(\mathsf{Ext}^n_R(\mathsf{Tr}M_1, R)) \\ &= \mathrm{Ass}_R(\mathsf{Ext}^n_R(\mathsf{Tr}M_1, C)) = \mathrm{Ass}_R(\mathsf{Ext}^{n+1}_R(\mathsf{Tr}M, C)).\end{aligned}$$

□

Recall that a ring R is called *equidimensional* if $\dim R/\mathfrak{p} = \dim R$ for every minimal prime ideal $\mathfrak{p}$ of R. We also say that R is *formally equidimensional* if its completion $\widehat{R}$ is equidimensional. Now we can present our first main result.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring of dimension d which is the homomorphic image of a Cohen–Macaulay ring. Assume that M is a horizontally linked R-module and that n is a positive integer. Assume further that X is a*

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subset of $\operatorname{Spec} R$ *which is stable under generalization and that* $R_{\mathfrak{p}}$ *is Cohen–Macaulay and* $\mathsf{G}\text{-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ *for all* $\mathfrak{p} \in X$ *(e.g.* $X \subseteq \operatorname{Gor}(R)$*). Then the following are equivalent.*

(i) $X \subseteq \widetilde{S}_n(M)$.

(ii) $\operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(\lambda M)) \subseteq \operatorname{Spec} R \setminus X$ *for* $d - n < i < d$.

Proof. (i)⇒(ii). Assume to the contrary that $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(\lambda M)) \cap X$ for some $d - n < i < d$. Therefore, $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}(\mathrm{H}^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(\lambda_{R_{\mathfrak{p}}} M_{\mathfrak{p}}))$ by Theorem 2.7. As $\mathfrak{p} \in \widetilde{S}_n(M) \cap X$, we have $\mathrm{H}^j_{\mathfrak{p}R_{\mathfrak{p}}}(\lambda_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$ for $\dim R_{\mathfrak{p}} - n < j < \dim R_{\mathfrak{p}}$ by [**7**, Theorem 4.2]. Note that by [**23**, Theorem 31.5], R is equidimensional and catenary. Hence $d = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$ and so $\dim R_{\mathfrak{p}} - n < i - \dim R/\mathfrak{p} < \dim R_{\mathfrak{p}}$. In particular, $\mathrm{H}^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(\lambda_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$, which is a contradiction by Theorem 2.6(i).

(ii)⇒(i). Assume to the contrary that $X \nsubseteq \widetilde{S}_n(M)$. Set $Y = \{\mathfrak{p} \in X \mid M_{\mathfrak{p}} \text{ does not satisfy } \widetilde{S}_n\}$. Let $\mathfrak{p}_0$ be a minimal element of Y (with respect to the inclusion relation). As $\mathsf{G}\text{-}\dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}) < \infty$, $\mathsf{Ext}^i_R(\mathsf{Tr}M, R)_{\mathfrak{p}_0} \neq 0$ for some $1 \leq i \leq n$ by Theorem 2.3. Set

$$t = \min\{j \mid \mathsf{Ext}^j_R(\mathsf{Tr}M, R)_{\mathfrak{p}_0} \neq 0, 1 \leq j \leq n\}. \tag{3.3.1}$$

As M is horizontally linked, $\mathsf{Ext}^1_R(\mathsf{Tr}M, R) = 0$ and so $1 < t \leq n$. Note that $\mathfrak{q} \in \widetilde{S}_n(M)$ for all prime ideals $\mathfrak{q}$ with $\mathfrak{q} \subsetneq \mathfrak{p}_0$, because $\mathfrak{p}_0$ is a minimal element of Y and X is stable under generalization. In other words, by Theorem 2.3,

$$\mathsf{Ext}^t_R(\mathsf{Tr}M, R)_{\mathfrak{q}} = 0 \quad \text{for all prime ideals } \mathfrak{q} \text{ with } \mathfrak{q} \subsetneq \mathfrak{p}_0. \tag{3.3.2}$$

Therefore $\mathfrak{p}_0$ is a minimal element of $\operatorname{Supp}_R(\mathsf{Ext}^t_R(\mathsf{Tr}M, R))$ and so $\mathfrak{p}_0 \in \mathsf{Min}_R(\mathsf{Ext}^t_R(\mathsf{Tr}M, R))$. Note thats by [**22**, Proposition 9.A],

$$\operatorname{Ass}_R(\mathsf{Ext}^t_R(\mathsf{Tr}M, R)) = \{P \cap R \mid P \in \operatorname{Ass}_{\widehat{R}}(\mathsf{Ext}^t_{\widehat{R}}(\widehat{\mathsf{Tr}M}, \widehat{R}))\}.$$

Let $P_0 \in \mathsf{Min}_{\widehat{R}}(\mathsf{Ext}^t_{\widehat{R}}(\widehat{\mathsf{Tr}M}, \widehat{R}))$ such that $P_0 \cap R = \mathfrak{p}_0$. It follows easily from (3.3.1) that

$$\mathsf{Ext}^i_{\widehat{R}}(\widehat{\mathsf{Tr}M}, \widehat{R})_{P_0} = 0 \quad \text{for all } 0 < i < t. \tag{3.3.3}$$

As $\mathsf{G}\text{-}\dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}) < \infty$ and the local homomorphism $R_{\mathfrak{p}_0} \to \widehat{R}_{P_0}$ is flat, $\mathsf{G}\text{-}\dim_{\widehat{R}_{P_0}}(\widehat{M}_{P_0}) < \infty$. As R is the homomorphic image of a Cohen–Macaulay ring, all formal fibres of R are Cohen–Macaulay. Therefore, $\widehat{R}_{P_0}/\mathfrak{p}_0\widehat{R}_{P_0} \cong (R_{\mathfrak{p}_0}/\mathfrak{p}_0 R_{\mathfrak{p}_0} \otimes_R \widehat{R})_{P_0}$ is Cohen–Macaulay and so $\widehat{R}_{P_0}$ is Cohen–Macaulay (see [**23**, p. 181]). Therefore, by Lemma 3.2, (3.3.3) and § 2.4,

$$P_0\widehat{R}_{P_0} \in \operatorname{Ass}(\mathsf{Ext}^t_{\widehat{R}_{P_0}}(\mathsf{Tr}_{\widehat{R}_{P_0}} \widehat{M}_{P_0}, \omega_{\widehat{R}_{P_0}})). \tag{3.3.4}$$

As $t > 1$, $\mathsf{Ext}^t_{\widehat{R}_{P_0}}(\mathsf{Tr}_{\widehat{R}_{P_0}} \widehat{M}_{P_0}, \omega_{\widehat{R}_{P_0}}) \cong \mathsf{Ext}^{t-1}_{\widehat{R}_{P_0}}(\widehat{\lambda M}_{P_0}, \omega_{\widehat{R}_{P_0}})$. Therefore, by (3.3.4), Theorem 2.6(iii) and the local duality theorem, $P_0\widehat{R}_{P_0} \in \operatorname{Att}_{\widehat{R}_{P_0}}(\mathrm{H}^{s-t+1}_{P_0\widehat{R}_{P_0}}(\widehat{\lambda M}_{P_0}))$, where $s = \dim \widehat{R}_{P_0}$. As $\widehat{R}$ is equidimensional, $s + \dim(\widehat{R}/P_0) = d$. Hence, by Theorem 2.7,

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$P_0 \in \operatorname{Att}_{\widehat{R}}(\mathrm{H}^{d-t+1}_{\widehat{\mathfrak{m}}}(\widehat{\lambda M}))$ and so $\mathfrak{p}_0 \in \operatorname{Att}_R(\mathrm{H}^{d-t+1}_{m}(\lambda M))$ by Theorem 2.6(iv), which is a contradiction because $d-n < d-t+1 < d$ and $\mathfrak{p}_0 \in X$. □

It should be mentioned that Theorem 3.3 is new even in the case of cyclic modules.

Corollary 3.4. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and let X be a subset of* Spec R *which is stable under generalization. Assume that I and J are ideals of R which are linked by the zero ideal. If $\mathsf{pd}_R(R/I) < \infty$ or R is Gorenstein, then the following statements are equivalent.*

(i) $X \subseteq S_n(R/I)$.

(ii) $\operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(R/J)) \subseteq \operatorname{Spec} R \setminus X$ *for* $d-n < i < d$.

As a consequence of Theorem 3.3, we have the following result.

Corollary 3.5. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Assume that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in X^{n-1}(R)$. For a horizontally linked R-module M, the following statements are equivalent.*

(i) *$M_{\mathfrak{p}}$ is Cohen–Macaulay for all $\mathfrak{p} \in X^{n-1}(R)$.*

(ii) $\mathsf{depth}\, R_{\mathfrak{p}} \geq n$ *for all* $\mathfrak{p} \in \bigcup_{0<i<d} \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(\lambda M))$.

Proof. Statement (i) is equivalent to saying that $X^{n-1}(R) \subseteq S_d(M)$. Now the assertion is clear by Theorem 3.3. □

The following result is an immediate consequence of Theorems 3.3 and 2.6(ii).

Corollary 3.6. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Assume that M is a horizontally linked R-module and that $\mathsf{G\text{-}dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. The following statements are equivalent.*

(i) *$M_{\mathfrak{p}}$ satisfies (S_n) for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.*

(ii) *$\ell(\mathrm{H}^i_{\mathfrak{m}}(\lambda M))) < \infty$ for $d-n < i < d$, where $\ell(-)$ denotes the length.*

Theorem 3.7. *Let $(R,\mathfrak{m})$ be a formally equidimensional local ring of dimension d which is the homomorphic image of a Cohen–Macaulay ring. Assume that M is a horizontally linked R-module and that n is an integer with $0 < n \leq d$. Let X be a subset of* Spec R *which is stable under generalization such that $\mathsf{G\text{-}dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ and $R_{\mathfrak{p}}$ is Cohen–Macaulay for all $\mathfrak{p} \in X$. If $\operatorname{Att}_R(\mathrm{H}^j_{\mathfrak{m}}(\lambda M)) \subseteq \operatorname{Spec} R \setminus X$ for $d-n < j < d$, then $\operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \subseteq \operatorname{Spec} R \setminus X$ for $0 < i < n$.*

Proof. Assume to the contrary that $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap X$ for some $0 < i < n$. By Theorem 2.7, $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}(\mathrm{H}^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}))$. In particular,

$$\mathrm{H}^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0. \tag{3.7.1}$$

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By Theorem 3.3, $M_{\mathfrak{p}}$ satisfies $\widetilde{S}_n$. In other words, as $R_{\mathfrak{p}}$ is Cohen–Macaulay,

$$\mathsf{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{\dim R_{\mathfrak{p}}, n\}. \tag{3.7.2}$$

On the other hand, as R is equidimensional and catenary,

$$\dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} = d. \tag{3.7.3}$$

If $n \geq \dim R_{\mathfrak{p}}$, then $M_{\mathfrak{p}}$ is maximal Cohen–Macaulay by (3.7.2) and so $\mathrm{H}^j_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ for all $j \neq \dim R_{\mathfrak{p}}$. Therefore, by (3.7.1), $i - \dim R/\mathfrak{p} = \dim M_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ and so $i = d$ by (3.7.3), which is a contradiction. On the other hand, if $n < \dim R_{\mathfrak{p}}$, then $\mathsf{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq n$ by (3.7.2) and so $\mathrm{H}^j_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ for all $j < n$. In particular, $\mathrm{H}^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$, which is a contradiction by (3.7.1). □

Recall that a subset X of $\operatorname{Spec} R$ is called *stable under specialization* if every prime ideal of R containing some prime ideal in X belongs to X. The following is an immediate consequence of Theorem 3.7.

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring which is the homomorphic image of a Cohen–Macaulay ring. Let X be a subset of $\operatorname{Spec} R$ which is stable under specialization such that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec} R \setminus X$. Assume M is a horizontally linked R-module. The following statements are equivalent.*

(i) $\operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \subseteq X$ *for* $0 < i < \dim R$.

(ii) $\operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(\lambda M)) \subseteq X$ *for* $0 < i < \dim R$.

In what follows, we extend Theorem 3.3 to include a change of ring. First we need to recall some definitions and results from [**13**].

Definition 3.9. Let C be a semidualizing R-module. An R-module M is called G_C-*perfect* if $\mathsf{grade}_R(M) = \mathsf{G}_C\text{-}\mathsf{dim}_R(M)$. An R-module M is called G_C-*Gorenstein* if M is G_C-perfect and $\mathsf{Ext}^n_R(M, C)$ is cyclic, where $n = \mathsf{G}_C\text{-}\mathsf{dim}_R(M)$. An ideal I is called G_C-perfect (respectively, G_C-Gorenstein) if R/I is G_C-perfect (respectively, G_C-Gorenstein) as an R-module.

Note that if I is a G_C-Gorenstein ideal of G_C-dimension n, then $\mathsf{Ext}^n_R(R/I, C) \cong R/I$.

Theorem 3.10 (Golod [13, Proposition 5]). *Let K be a semidualizing R-module and let I be a G_K-perfect ideal. Set $C = \mathsf{Ext}^{\mathsf{grade}(I)}_R(R/I, K)$. Then the following statements hold.*

(i) *C is a semidualizing R/I-module.*

(ii) *If M is a R/I-module, then $\mathsf{G}_K\text{-}\mathsf{dim}_R(M) < \infty$ if and only if $\mathsf{G}_C\text{-}\mathsf{dim}_{R/I}(M) < \infty$, and if these dimensions are finite, then $\mathsf{G}_K\text{-}\mathsf{dim}_R(M) = \mathsf{grade}(I) + \mathsf{G}_C\text{-}\mathsf{dim}_{R/I}(M)$.*

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and let C be a semidualizing R-module. Assume that M is an R-module of finite G_C-dimension which is linked*

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by a G_C-Gorenstein ideal $\mathfrak{c}$. Set $\overline{R} = R/\mathfrak{c}$. Suppose that X is a subset of $\operatorname{Spec}\overline{R}$ *which is stable under generalization. The following statements are equivalent.*

(i) $X \subseteq \widetilde{S}_n(M)$.

(ii) $\operatorname{Att}_{\overline{R}}(\mathrm{H}^i_{\mathfrak{m}}(\lambda_{\overline{R}}M)) \subseteq \operatorname{Spec}\overline{R} \setminus X$ *for* $\dim\overline{R} - n < i < \dim\overline{R}$.

Proof. As R is Cohen–Macaulay and $\mathfrak{c}$ is G_C-perfect, $R/\mathfrak{c}$ is Cohen–Macaulay. By Theorem 3.10(ii), $\mathsf{G}\text{-}\dim_{R/\mathfrak{c}}(M) < \infty$. Now the assertion is clear by Theorem 3.3. □

For an R-module M, set $\mathrm{c}(M) = \sup\{i < \dim_R(M) \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}$. In what follows, we describe the attached prime ideals of $\mathrm{H}^{\mathrm{c}(M)}_{\mathfrak{m}}(M)$ for a non-Cohen–Macaulay horizontally linked R-module M. The following result should be compared with [**8**, Corollary 4.5].

Theorem 3.12. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Assume that M is an R-module of finite and positive Gorenstein dimension which is horizontally linked. Set $\mathrm{c} = \mathrm{c}(\lambda M)$. Then*

$$\operatorname{Att}_R(\mathrm{H}^{\mathrm{c}}_{\mathfrak{m}}(\lambda M)) = \{\mathfrak{p} \in S_{d-\mathrm{c}}(M) \setminus S_{d-\mathrm{c}+1}(M) \mid \mathsf{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = d - \mathrm{c}\}.$$

Proof. As $\mathrm{H}^{\mathrm{c}}_{\mathfrak{m}}(\lambda M)$ is an Artinian R-module, it has the structure of an $\widehat{R}$-module. More precisely, it is an Artinian $\widehat{R}$-module. By Theorem 2.6(iv),

$$\operatorname{Att}_R(\mathrm{H}^{\mathrm{c}}_{\mathfrak{m}}(\lambda M)) = \left\{P \cap R \mid P \in \operatorname{Att}_{\widehat{R}}(\mathrm{H}^{\mathrm{c}}_{\mathfrak{m}}(\lambda M))\right\}. \tag{3.12.1}$$

Note that by the local duality theorem and Lemma 3.1, $\widehat{M}$ is $(d-c)$-torsionfree $\widehat{R}$-module. It follows from Theorem 2.6(iii), the local duality theorem and Lemma 3.2 that

$$\operatorname{Att}_{\widehat{R}}(\mathrm{H}^{\mathrm{c}}_{\mathfrak{m}}(\lambda M)) = \operatorname{Ass}_{\widehat{R}}(\mathsf{Ext}^{d-\mathrm{c}}_{\widehat{R}}(\widehat{(\lambda M)}, \omega_{\widehat{R}})) = \operatorname{Ass}_{\widehat{R}}(\mathsf{Ext}^{d-\mathrm{c}}_{\widehat{R}}(\widehat{(\lambda M)}, \widehat{R})). \tag{3.12.2}$$

On the other hand, by [**22**, Proposition 9.A],

$$\operatorname{Ass}_R(\mathsf{Ext}^{d-\mathrm{c}}_R((\lambda M), R)) = \{P \cap R \mid P \in \operatorname{Ass}_{\widehat{R}}(\mathsf{Ext}^{d-\mathrm{c}}_{\widehat{R}}(\widehat{(\lambda M)}, \widehat{R}))\}. \tag{3.12.3}$$

It follows from (3.12.1)–(3.12.3) that

$$\operatorname{Att}_R(\mathrm{H}^{\mathrm{c}}_{\mathfrak{m}}(\lambda M)) = \operatorname{Ass}_R(\mathsf{Ext}^{d-\mathrm{c}}_R((\lambda M), R)). \tag{3.12.4}$$

Set $n = d - c$. As M is n-torsionfree, by [**3**, Proposition 3.6], there exists the exact sequence

$$0 \longrightarrow M \longrightarrow \Omega^{n+1}\mathsf{Tr}\Omega^{n+1}\mathsf{Tr}M \oplus P \longrightarrow \Omega^{n-1}\,\mathsf{Ext}^{n+1}_R(\mathsf{Tr}M, R) \longrightarrow 0, \tag{3.12.5}$$

where P is a projective R-module. Set $X = \Omega^{n+1}\mathsf{Tr}\Omega^{n+1}\mathsf{Tr}M \oplus P$ and $Y = \Omega^{n-1}\,\mathsf{Ext}^{n+1}_R(\mathsf{Tr}M, R)$. Let $\mathfrak{p} \in \operatorname{Ass}_R(\mathsf{Ext}^n_R(\lambda M, R))$. Hence $\mathfrak{p} \in \operatorname{Ass}_R(\mathsf{Ext}^{n+1}_R(\mathsf{Tr}M, R))$ and so $\mathfrak{p} \notin S_{n+1}(M)$ by Theorem 2.3. Note that $\mathsf{depth}\, R_{\mathfrak{p}} > n$. Therefore $\mathsf{depth}_{R_{\mathfrak{p}}}(Y_{\mathfrak{p}}) = n - 1$ and $\mathsf{depth}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) > n$. By localizing the exact sequence (3.12.5) at $\mathfrak{p}$, we conclude that $\mathsf{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = n$.

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Conversely, assume that $\mathfrak{p} \in S_n(M) \setminus S_{n+1}(M)$ and that $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = n$. Hence by property (ii) in § 2.2, $\mathsf{depth}\, R_\mathfrak{p} > n$. By localizing the exact sequence (3.12.5) at $\mathfrak{p}$, we conclude that $\mathsf{depth}_{R_\mathfrak{p}}(Y_\mathfrak{p}) = n-1$ and so $\mathsf{depth}_{R_\mathfrak{p}}(\mathsf{Ext}_R^{n+1}(\mathsf{Tr}M, R)_\mathfrak{p}) = 0$. In other words, $\mathfrak{p} \in \mathrm{Ass}_R(\mathsf{Ext}_R^{n+1}(\mathsf{Tr}M, R))$. Now the assertion follows from (3.12.4). □

For a subset X of $\mathrm{Spec}\, R$, we denote by $\overline{X}$ the closure of X in the Zariski topology.

Corollary 3.13. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Assume that M is an R-module of finite and positive Gorenstein dimension which is horizontally linked. For a positive integer $n < d$, we have*

$$\mathrm{Att}_R(\mathrm{H}^{d-n}_\mathfrak{m}(\lambda M)) \setminus \overline{\bigcup_{d-n<i<d} \mathrm{Att}_R(\mathrm{H}^i_\mathfrak{m}(\lambda M))}$$
$$= \{\mathfrak{p} \in S_n(M) \setminus S_{n+1}(M) \mid \mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = n\}.$$

Proof. First note that by [**7**, Theorem 4.2], for a prime ideal $\mathfrak{p}$ of R we have

$$\mathfrak{p} \in S_n(M) \Longleftrightarrow \mathrm{H}^i_{\mathfrak{p}R_\mathfrak{p}}((\lambda M)_\mathfrak{p}) = 0, \quad \text{for } \dim R_\mathfrak{p} - n < i < \dim R_\mathfrak{p}. \tag{3.13.1}$$

Set $X = \{\mathfrak{p} \in S_n(M) \setminus S_{n+1}(M) \mid \mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = n\}$ and $Y = \overline{\bigcup_{d-n<i<d} \mathrm{Att}_R(\mathrm{H}^i_\mathfrak{m}(\lambda M))}$. Note that $\mathrm{Spec}\, R \setminus Y$ is stable under generalization. Hence, by Theorem 3.3, $\mathrm{Spec}\, R \setminus Y \subseteq S_n(M)$. Now let $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^{d-n}_\mathfrak{m}(\lambda M)) \setminus Y$. Therefore, $\mathfrak{p} \in S_n(M)$. Also, by Theorem 2.7, $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Att}_{R_\mathfrak{p}}(\mathrm{H}^{\dim R_\mathfrak{p} - n}_{\mathfrak{p}R_\mathfrak{p}}((\lambda M)_\mathfrak{p}))$. Hence, by (3.13.1), $\mathrm{c}_{R_\mathfrak{p}}(\lambda_{R_\mathfrak{p}} M_\mathfrak{p}) = \dim R_\mathfrak{p} - n$. It follows from Corollary 3.12 that $\mathfrak{p}R_\mathfrak{p} \notin S_{n+1}(M_\mathfrak{p})$ and $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = n$. In other words, $\mathfrak{p} \in X$.

Conversely, if $\mathfrak{p} \in X$, then $\mathrm{c}_{R_\mathfrak{p}}(\lambda_{R_\mathfrak{p}} M_\mathfrak{p}) = \dim R_\mathfrak{p} - n$ by (3.13.1). It follows from Corollary 3.12 that $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Att}_{R_\mathfrak{p}}(\mathrm{H}^{\dim R_\mathfrak{p} - n}_{\mathfrak{p}R_\mathfrak{p}}((\lambda M)_\mathfrak{p}))$, and so $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^{d-n}_\mathfrak{m}(\lambda M))$ by Theorem 3.3. Assume to the contrary that $\mathfrak{p} \in Y$. Hence there exists a prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\mathfrak{q} \in \mathrm{Att}_R(\mathrm{H}^j_\mathfrak{m}(\lambda M))$ for some integer j with $d - n < j < d$. By Theorem 3.3, $\mathfrak{q}R_\mathfrak{q} \in \mathrm{Att}_{R_\mathfrak{q}}(\mathrm{H}^{j-\dim R/\mathfrak{q}}_{\mathfrak{q}R_\mathfrak{q}}((\lambda M)_\mathfrak{q}))$. As $j < d$, we have $j - \dim R/\mathfrak{q} < \dim R_\mathfrak{q}$. Note that $\mathfrak{q} \in S_n(M)$ and so, by (3.13.1), $j - \dim R/\mathfrak{q} \leq \dim R_\mathfrak{q} - n$. Hence $j \leq d - n$, which is a contradiction. □

4. Linkage and Cohen–Macaulay modules

The first main theorem in the theory of linkage was due to Peskine and Szpiro. They proved that the Cohen–Macaulay property is preserved under linkage over Gorenstein rings. Note that the above statement is no longer true if the base ring is Cohen–Macaulay but not Gorenstein. On the other hand, there is an interesting generalization of the Peskine–Szpiro theorem, due to Martin [**18**]. Let R be a Gorenstein local ring and let I be Cohen–Macaulay ideal linked to J by an ideal $\mathfrak{a}$, where $\mathfrak{a}$ is Cohen–Macaulay and generically Gorenstein. Then J is Cohen–Macaulay if and only if $I + \mathfrak{b}$ is a Cohen–Macaulay ideal with the same height as $\mathfrak{b}$, where $\mathfrak{b}$ is an ideal of R, containing $\mathfrak{a}$, with $\mathfrak{b}/\mathfrak{a}$ equal to a canonical module of $\omega_{R/\mathfrak{a}}$ [**18**, Corolary 2.11].

In the theory of linkage of modules, Martsinkovsky and Strooker generalized the Peskine–Szpiro theorem for stable modules over Gorenstein local rings

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[20, Proposition 8]. In this section, we study the theory of linkage for Cohen–Macaulay modules. As a consequence, we generalize Martin's result to the module case.

A *semidualizing ideal* is an ideal that is semidualizing as an R-module. Let C be a semidualizing R-module. If R is generically Gorenstein, then C can be identified with an ideal of R [32, Proposition 3.1]. Let R be a Cohen–Macaulay local ring and let $\mathfrak{c}$ be a semidualizing ideal. Then $\mathfrak{c}$ is an ideal of height 1 or equals R [32, Proposition 3.2].

Theorem 4.1. *Let R be a generically Gorenstein and let $\mathfrak{c}$ be a semidualizing ideal of R. Assume that M is a horizontally linked R-module. Then the following statements hold.*

(i) $\mathsf{G}_{\mathfrak{c}}\text{-}\mathsf{dim}_R(M) = 0$ *if and only if* $\mathsf{G}_{\mathfrak{c}}\text{-}\mathsf{dim}_R(\mathfrak{c}\lambda M) = 0$.

(ii) *Assume that M has finite $\mathsf{G}_{\mathfrak{c}}$-dimension on $X^{n-1}(R)$. Then M satisfies $\widetilde{S}_n$ if and only if* $\mathsf{Ext}^i_R(\mathfrak{c}\lambda M, \mathfrak{c}) = 0$ *for* $0 < i < n$.

(iii) *Assume that R is Cohen–Macaulay and that* $\mathsf{G}_{\mathfrak{c}}\text{-}\mathsf{dim}_R(M) < \infty$. *Then M is Cohen–Macaulay if and only if $\mathfrak{c}\lambda M$ is so.*

Proof. Let $\cdots \to F_1 \to F_0 \xrightarrow{f} M \to 0$ be the minimal free resolution of M. From the exact sequence $(F_0)^* \xrightarrow{g} \lambda M \to 0$ we get the exact sequence $(F_0)^* \otimes_R \mathfrak{c} \xrightarrow{g\otimes\mathfrak{c}} \lambda M \otimes_R \mathfrak{c} \to 0$. Let $\phi = h \circ (g \otimes \mathfrak{c})$, where h is the natural epimorphism $\lambda M \otimes_R \mathfrak{c} \twoheadrightarrow \mathfrak{c}\lambda M$. We denote by ψ the tensor evaluation isomorphism $(F_0)^* \otimes_R \mathfrak{c} \xrightarrow{\cong} \mathsf{Hom}_R(F_0, \mathfrak{c})$. Dualize the free resolution of M into $\mathfrak{c}$ to obtain the exact sequence

$$0 \to \mathsf{Hom}_R(M, \mathfrak{c}) \to \mathsf{Hom}_R(F_0, \mathfrak{c}) \to \mathsf{Hom}_R(F_1, \mathfrak{c}) \to \mathsf{Tr}_{\mathfrak{c}}(M) \to 0. \tag{4.1.1}$$

Set $\lambda_{\mathfrak{c}}(M) := \mathrm{Coker}(\mathsf{Hom}(f, \mathfrak{c}))$. We denote by α the epimorphism $\mathsf{Hom}_R(F_0, \mathfrak{c}) \twoheadrightarrow \lambda_{\mathfrak{c}}(M)$. Now it is easy to check that $\phi \circ \psi^{-1} \circ \mathsf{Hom}(f, \mathfrak{c}) = 0 = \alpha \circ \psi \circ \iota$, where $\iota : \ker\phi \hookrightarrow (F_0)^* \otimes_R \mathfrak{c}$ is the natural inclusion. Therefore we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\phi & \xrightarrow{\iota} & (F_0)^* \otimes_R \mathfrak{c} & \xrightarrow{\phi} & \mathfrak{c}\lambda M & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & & & \\
0 & \longrightarrow & \mathsf{Hom}_R(M, \mathfrak{c}) & \longrightarrow & \mathsf{Hom}_R(F_0, \mathfrak{c}) & \longrightarrow & \lambda_{\mathfrak{c}}(M) & \longrightarrow & 0
\end{array}$$

with exact rows. Hence we obtain the following isomorphism:

$$\lambda_{\mathfrak{c}}(M) \cong \mathfrak{c}\lambda M. \tag{4.1.2}$$

(i) From the exact sequence (4.1.1) we get the short exact sequence

$$0 \to \lambda_{\mathfrak{c}}(M) \to \mathsf{Hom}_R(F_1, \mathfrak{c}) \to \mathsf{Tr}_{\mathfrak{c}}(M) \to 0. \tag{4.1.3}$$

As M is a syzygy module, it is easy to see that $\mathsf{Ext}^1_R(\mathsf{Tr}_{\mathfrak{c}}(M), \mathfrak{c}) = 0$. Now the assertion follows from property (iv) in § 2.2, (4.1.2) and the exact sequence (4.1.3).

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(ii) By [**8**, Proposition 2.4], M satisfies $\widetilde{S}_n$ if and only if $\mathsf{Ext}^i_R(\mathsf{Tr}_{\mathfrak{c}}(M), \mathfrak{c}) = 0$ for $1 \leq i \leq n$. As M is horizontally linked, it satisfies $\widetilde{S}_1$ and so $\mathsf{Ext}^1_R(\mathsf{Tr}_{\mathfrak{c}}(M), \mathfrak{c}) = 0$. The exact sequence (4.1.3) induces the isomorphism $\mathsf{Ext}^i_R(\lambda_{\mathfrak{c}}(M), \mathfrak{c}) \cong \mathsf{Ext}^{i+1}_R(\mathsf{Tr}_{\mathfrak{c}}M, \mathfrak{c})$ for all $i > 0$. Now the assertion follows from (4.1.2).

(iii) If M is Cohen–Macaulay, then $\mathfrak{c}\lambda M$ is Cohen–Macaulay by property (ii) in § 2.2(ii) and part (i) above.
Conversely, assume $\mathfrak{c}\lambda M$ is Cohen–Macaulay. Suppose to the contrary that M is not Cohen–Macaulay. Hence, $\mathsf{G}_{\mathfrak{c}}\text{-}\mathsf{dim}_R(M) > 0$. Set $n = \min\{i > 0 \mid \mathsf{Ext}^i_R(M, \mathfrak{c}) \neq 0\}$. As $\mathsf{Ext}^i_R(M, \mathfrak{c}) = 0$ for all $0 < i < n$, applying the functor $(-)^\nabla = \mathsf{Hom}_R(-, \mathfrak{c})$ to the minimal free resolution of M implies the exact sequences

$$0 \to \lambda_{\mathfrak{c}}(M) \to (F_1)^\nabla \to \cdots \to (F_n)^\nabla \to \mathsf{Tr}_{\mathfrak{c}}\Omega^{n-1}(M) \to 0, \tag{4.1.4}$$

$$0 \longrightarrow \mathsf{Ext}^n_R(M, \mathfrak{c}) \longrightarrow \mathsf{Tr}_{\mathfrak{c}}\Omega^{n-1}(M). \tag{4.1.5}$$

Now let $\mathfrak{p} \in \mathrm{Ass}_R(\mathsf{Ext}^n_R(M, \mathfrak{c}))$. It follows from the exact sequence (4.1.5) that $\mathsf{depth}_{R_{\mathfrak{p}}}((\mathsf{Tr}_{\mathfrak{c}}\Omega^{n-1}(M)_{\mathfrak{p}}) = 0$. As $\mathsf{Ext}^n_R(M, \mathfrak{c})_{\mathfrak{p}} \neq 0$, by properties (ii) and (iii) in § 2.2, $\mathsf{depth}\, R_{\mathfrak{p}} - \mathsf{depth}(M_{\mathfrak{p}}) = \mathsf{G}_{\mathfrak{c}_{\mathfrak{p}}}\text{-}\mathsf{dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq n$. Since M is a first syzygy, we conclude that $\mathsf{depth}\, R_{\mathfrak{p}} > n$. It follows from the exact sequence (4.1.4) that $\mathsf{depth}_{R_{\mathfrak{p}}}(\lambda_{\mathfrak{c}}(M)_{\mathfrak{p}}) = n$, which is a contradiction, because $\lambda_{\mathfrak{c}}(M) \cong \mathfrak{c}\lambda M$ is a maximal Cohen–Macaulay module by our assumption. □

In what follows, we extend Theorem 4.1 to include a change of ring.

Corollary 4.2. *Let R be a Cohen–Macaulay local ring and let K be a semidualizing R-module. Assume $\mathfrak{a}$ is a G_K-perfect ideal of R which is generically Gorenstein. Set $C = \mathsf{Ext}^m_R(R/\mathfrak{a}, K)$, where $m = \mathsf{grade}(\mathfrak{a})$ and write $C \cong \mathfrak{c}/\mathfrak{a}$. Assume that M and N are R-modules such that $M \underset{\mathfrak{a}}{\sim} N$. Then the following statements hold.*

(i) *M is G_K-perfect if and only if $\mathfrak{c}N$ is G_K-perfect.*

(ii) *Assume that M has finite G_K-dimension. Then M satisfies (S_n) if and only if $\mathsf{Ext}^i_R(\mathfrak{c}N, K) = 0$ for $m < i < n + m$. Also, M is Cohen–Macaulay if and only if $\mathfrak{c}N$ is so.*

Proof. (i) By Theorem 3.10(ii) and [**31**, Lemma 3.16], M is G_K-perfect if and only if $\mathsf{G}_C\text{-}\mathsf{dim}_{R/\mathfrak{a}}(M) = 0$. Note that $\mathsf{grade}(\mathfrak{c}N) = \mathsf{grade}(N)$. Again by using Theorem 3.10(ii) and [**31**, Lemma 3.16], we have that $\mathfrak{c}N$ is G_K-perfect if and only if $\mathsf{G}_C\text{-}\mathsf{dim}_{R/\mathfrak{a}}(\mathfrak{c}N) = 0$. Now the assertion follows from Theorem 4.1(i).

(ii) By Theorem 3.10(ii), $\mathsf{G}_C\text{-}\mathsf{dim}_{R/\mathfrak{a}}(M) < \infty$. As $\mathfrak{a}$ is G_K-perfect, $R/\mathfrak{a}$ is a Cohen–Macaulay ring. By [**13**, Corollary], $\mathsf{Ext}^i_{R/\mathfrak{a}}(\mathfrak{c}N, C) \cong \mathsf{Ext}^{i+m}_R(\mathfrak{c}N, K)$ for all $i \geq 0$. Now the assertion follows from Theorem 4.1(ii), (iii). □

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The following result is a generalization of [18, Corollary 2.11].

Corollary 4.3. *Let R be a Cohen–Macaulay local ring and let K be a semidualizing R-module. Assume $\mathfrak{a}$ is a G_K-perfect ideal of R which is generically Gorenstein. Set $C = \mathsf{Ext}^n_R(R/\mathfrak{a}, K)$, where $n = \mathsf{grade}(\mathfrak{a})$, and write $C \cong \mathfrak{c}/\mathfrak{a}$. Assume that M is an R-module of finite G_K-dimension and that N is a Cohen–Macaulay R-module which is linked to M by the ideal $\mathfrak{a}$. Then M is Cohen–Macaulay if and only if $\mathsf{depth}_R(N/\mathfrak{c}N) \geq \dim(R/\mathfrak{a}) - 1$.*

Proof. Consider the exact sequence

$$0 \to \mathfrak{c}N \to N \to N/\mathfrak{c}N \to 0. \tag{4.3.1}$$

Now the assertion follows from Theorem 4.1(ii) and the exact sequence (4.3.1). □

It should be noted that Theorem B is a special case of Corollary 4.3.

Corollary 4.4. *Let R be a Cohen–Macaulay local ring with canonical module ω_R and let $\mathfrak{a}$ be a Cohen–Macaulay ideal of R which is generically Gorenstein. Write $\omega_{R/\mathfrak{a}} \cong \mathfrak{c}/\mathfrak{a}$. Assume that I and J are ideals of R linked by the ideal $\mathfrak{a}$. The following statements hold.*

(i) *R/I satisfies (S_n) if and only if $\mathrm{H}^i_{\mathfrak{m}}(\mathfrak{c}/\mathfrak{c} \cap J) = 0$ for $\dim(R/\mathfrak{a}) - n < i < \dim(R/\mathfrak{a})$. In particular, R/I is Cohen–Macaulay if and only if $\mathfrak{c}/\mathfrak{c} \cap J$ is so.*

(ii) *Assume R/I is Cohen–Macaulay. Then R/J is Cohen–Macaulay if and only if $\mathsf{depth}_R(R/(I + \mathfrak{c})) \geq \dim R/\mathfrak{a} - 1$.*

Proof. This follows immediately from Corollaries 4.2, 4.3 and the local duality theorem. □

Let n be a positive integer. An R-module M is said to be an nth *syzygy* if there exists an exact sequence

$$0 \to M \to F_0 \to \cdots \to F_{n-1},$$

where $F_0, \ldots, F_{n-1}$ are free modules. By convention, every module is a zeroth syzygy.

Theorem 4.5. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with canonical module ω_R. Assume that M is a horizontally linked R-module and that $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$. Then the following conditions are equivalent.*

(i) *$M \otimes_R \omega_R$ is maximal Cohen–Macaulay.*

(ii) *λM is a $(d+1)$th syzygy module.*

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Proof. By [30, Theorem 10.62], there is a third quadrant spectral sequence,

$$\mathrm{E}_2^{p,q} = \mathsf{Ext}_R^p(\mathsf{Tor}_q^R(M,\omega_R),\omega_R) \Rightarrow \mathsf{Ext}_R^{p+q}(M,R).$$

As $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$, $\mathsf{Tor}_i^R(M,\omega_R)$ has finite length for all $i > 0$ (see § 2.4). Therefore, $\mathrm{E}_2^{p,q} = 0$ when $q > 0$ and $p < d$, and so we obtain the isomorphisms

$$\mathsf{Ext}_R^i(M \otimes_R \omega_R, \omega_R) \cong \mathsf{Ext}_R^i(M,R), \quad \text{for all } 1 \le i \le d. \tag{4.5.1}$$

(i)⇒(ii). It follows from (4.5.1) and our assumption that $\mathsf{Ext}_R^i(M,R) = 0$ for all $1 \le i \le d$. As M is horizontally linked, $M \cong \lambda^2 M \cong \Omega\mathsf{Tr}\lambda M$. Therefore, $\mathsf{Ext}_R^i(\mathsf{Tr}\lambda M, R) = 0$ for all $1 \le i \le d+1$ and so λM is a $(d+1)$th syzygy module by [21, Proposition 11].

(ii)⇒(i). By [7, Proposition 2.6] and our assumption, $\mathsf{Ext}_R^i(M,R)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$ and all $1 \le i \le \mathsf{depth}\, R_\mathfrak{p}$. Hence, by property (iii) in § (2.2), $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ and so $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}((\lambda M)_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R - \{\mathfrak{m}\}$. It follows from [21, Theorem 43] that $\mathsf{Ext}_R^i(\mathsf{Tr}\lambda M, R) = 0$ for all $1 \le i \le d+1$. As M is horizontally linked, $M \cong \Omega\mathsf{Tr}\lambda M$. Therefore, $\mathsf{Ext}_R^i(M,R) = 0$ for all $1 \le i \le d$. Now the assertion is clear by (4.5.1). □

Here is a characterization of Gorenstein rings in terms of linkage.

Proposition 4.6. *Let $(R,\mathfrak{m},k)$ be a Cohen–Macaulay local ring of dimension $d > 1$. Then the following statements are equivalent.*

(i) *R is Gorenstein.*

(ii) *The Cohen–Macaulay property is preserved under horizontal linkage.*

Proof. (i)⇒(ii). This follows from [20, Proposition 8].

(ii)⇒(i). By [4, Corollary 1.2.5], $\Omega^{d+1}k$ is stable and so it is horizontally linked (see § 2.1). It follows from the assumption that $\lambda(\Omega^{d+1}k)$ is maximal Cohen–Macaulay. Hence $\mathsf{depth}_R(\mathsf{Tr}\Omega^{d+1}k) \ge d-1 \ge 1$. Now consider the exact sequence

$$0 \to \mathsf{Ext}_R^{d+2}(k,R) \to \mathsf{Tr}\Omega^{d+1}k \to \Omega\mathsf{Tr}\Omega^{d+2}k \to 0.$$

As $\mathsf{Ext}_R^{d+2}(k,R)$ has finite length and $\mathsf{depth}_R(\mathsf{Tr}\Omega^{d+1}k) > 0$, we conclude that $\mathsf{Ext}_R^{d+2}(k,R) = 0$. In other words, R is Gorenstein. □

5. Duality between local cohomology modules of linked module

Recall that an R-module M of dimension $d \ge 1$ is called a *generalized Cohen–Macaulay* module if $\ell(\mathrm{H}_\mathfrak{m}^i(M)) < \infty$ for all i, $0 \le i \le d-1$, where ℓ denotes the length. Let R be a Gorenstein local ring and let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of R linked by a Gorenstein ideal $\mathfrak{c}$. Assume that $R/\mathfrak{a}$ is generalized Cohen–Macaulay. Schenzel established an interesting duality between local cohomology modules of $R/\mathfrak{a}$ and $R/\mathfrak{b}$. More precisely, he proved that

$$\mathrm{H}_\mathfrak{m}^i(R/\mathfrak{a}) \cong \mathsf{Hom}_R(\mathrm{H}_\mathfrak{m}^{d-i}(R/\mathfrak{b}), \mathrm{E}_R(R/\mathfrak{m})),$$

for $0 < i < d$, where $d = \dim R/\mathfrak{a} = \dim R/\mathfrak{b}$ [33, Corollary 3.3]. Martsinkovsky and Strooker generalized the above result for all generalized Cohen–Macaulay modules linked

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by a Gorenstein ideal [**20**, Theorem 11]. In this section we generalize the above result to modules which satisfy Serre's condition (S_n) on the punctured spectrum of R.

For an ideal $\mathfrak{a}$ of R, we denote by $V(\mathfrak{a})$ the set of all prime ideals of R containing $\mathfrak{a}$. Let X be a closed subset of $\operatorname{Spec} R$. Then $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R. We call such an ideal $\mathfrak{a}$ the *defining ideal* of X. This is uniquely determined up to a radical.

Theorem 5.1. *Let U be an open subset of $\operatorname{Spec} R$ and let M be an R-module such that $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in U$. Assume that $U \subseteq \widetilde{S}_n(M)$ for some $0 < n \leq \mathsf{grade}(\mathfrak{a})$, where $\mathfrak{a}$ is the defining ideal of the complement of U in $\operatorname{Spec} R$. Then $\mathrm{H}^i_\mathfrak{a}(M) \cong \mathsf{Ext}^{i+1}_R(\mathsf{Tr}M, R)$ for $0 \leq i < n$. In particular, $\mathrm{H}^i_\mathfrak{a}(M)$ is finitely generated for $0 \leq i < n$.*

Proof. As $M_\mathfrak{p}$ satisfies $\widetilde{S}_n$, $\mathsf{Ext}^i_R(\mathsf{Tr}M, R)_\mathfrak{p} = 0$ for all $1 \leq i \leq n$ and all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \mathrm{V}(\mathfrak{a})$ by Theorem 2.3. Therefore, $\bigcup_{1\leq i\leq n} \operatorname{Supp}_R(\mathsf{Ext}^i_R(\mathsf{Tr}M, R)) \subseteq \mathrm{V}(\mathfrak{a})$ and so $\mathsf{Ext}^i_R(\mathsf{Tr}M, R)$ is $\mathfrak{a}$-torsion for $1 \leq i \leq n$. Consider the exact sequence

$$0 \longrightarrow \mathsf{Ext}^i_R(\mathsf{Tr}M, R) \longrightarrow \mathsf{Tr}\Omega^{i-1}\mathsf{Tr}M \longrightarrow X_i \longrightarrow 0, \tag{5.1.1}$$

where $X_i \approx \Omega\mathsf{Tr}\Omega^i\mathsf{Tr}M$. Since $\mathsf{Ext}^i_R(\mathsf{Tr}M, R)$ is $\mathfrak{a}$-torsion for $1 \leq i \leq n$, applying the functor $\Gamma_\mathfrak{a}(-)$ to the exact sequence (5.1.1), we get

$$\mathrm{H}^j_\mathfrak{a}(\mathsf{Tr}\Omega^{i-2}\mathsf{Tr}M) \cong \mathrm{H}^j_\mathfrak{a}(\Omega\mathsf{Tr}\Omega^{i-1}\mathsf{Tr}M) \quad \text{for all } j > 0,\ 2 \leq i \leq n+1, \tag{5.1.2}$$

and also

$$\mathsf{Ext}^i_R(\mathsf{Tr}M, R) = \Gamma_\mathfrak{a}(\mathsf{Ext}^i_R(\mathsf{Tr}M, R)) \cong \Gamma_\mathfrak{a}(\mathsf{Tr}\Omega^{i-1}\mathsf{Tr}M) \quad \text{for all } 1 \leq i \leq n, \tag{5.1.3}$$

because $\mathsf{grade}(\mathfrak{a}) > 0$. On the other hand, if $n > 1$, then

$$\mathrm{H}^j_\mathfrak{a}(\mathsf{Tr}\Omega^{i-1}\mathsf{Tr}M) \cong \mathrm{H}^{j+1}_\mathfrak{a}(\Omega\mathsf{Tr}\Omega^{i-1}\mathsf{Tr}M) \quad \text{for all } i \geq 1,\ 0 \leq j < n-1. \tag{5.1.4}$$

Now, by using (5.1.2)–(5.1.4), we obtain the result. □

The following is an immediate consequence of Theorem 5.1.

Corollary 5.2. *Let U be an open subset of $\operatorname{Spec} R$ and let M be a horizontally linked R-module such that $\mathsf{G\text{-}dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in U$. Assume that $U \subseteq \widetilde{S}_n(M)$ for some $1 < n \leq \mathsf{grade}(\mathfrak{a})$, where $\mathfrak{a}$ is the defining ideal of the complement of U in $\operatorname{Spec} R$. Then $\mathrm{H}^i_\mathfrak{a}(M) \cong \mathsf{Ext}^i_R(\lambda M, R)$ for $0 < i < n$.*

Let R be a Cohen–Macaulay local ring. An R-module M is generalized Cohen–Macaulay if and only if $M_\mathfrak{p}$ is a Cohen–Macaulay $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ (see [**34**, Lemmas 1.2 and 1.4]). Therefore the following result can be viewed as a generalization of [**20**, Theorem 10].

Corollary 5.3. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring of dimension $d > 1$ and let M be a horizontally linked R-module. Assume n is a positive integer and that $M_\mathfrak{p}$ satisfies (S_n) for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Then $\mathrm{H}^i_\mathfrak{m}(M) \cong \mathsf{Hom}_R(\mathrm{H}^{d-i}_\mathfrak{m}(\lambda M), \mathrm{E}_R(k))$ for $0 < i < n$, where $\mathrm{E}_R(k)$ is the injective envelope of k.*

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Proof. This follows from Corollary 5.2 and the local duality theorem. □

We end this section with the following result which extends Corollary 5.3 to include a change of ring.

Corollary 5.4. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring, $\mathfrak{c}$ a Gorenstein ideal of R, $\overline{R} = R/\mathfrak{c}$, and $d = \dim \overline{R} > 1$. Assume that M is an R-module which is linked by the ideal $\mathfrak{c}$ and that $M_\mathfrak{p}$ satisfies (S_n) for all $\mathfrak{p} \in \operatorname{Spec} \overline{R} \setminus \{\overline{\mathfrak{m}}\}$. Then $\mathrm{H}^i_\mathfrak{m}(M) \cong \operatorname{Hom}_R(\mathrm{H}^{d-i}_\mathfrak{m}(\lambda_{\overline{R}} M), \mathrm{E}_R(k))$ for $0 < i < n$.*

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