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On a singular eigenvalue problem

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1. Sears and Titchmarsh (1) have formulated an expansion in eigenfunctions which requires a knowledge of the s-zeros of the equation

$$[J_{is}(ka) - J_{-is}(ka)] \cos\beta \cosh \tfrac{1}{2}s\pi - i[J_{is}(ka) + J_{-is}(ka)] \sin\beta \sinh \tfrac{1}{2}s\pi = 0. \tag{1}$$

Here $ka > 0$ is supposed given and β is a real constant such that $0 \leqslant \beta < \pi$. The above equation is encountered when one seeks the eigenfunctions of the differential equation

$$r^2 w_{rr} + r w_r + (k^2 r^2 + s^2) w = 0 \tag{2}$$

on the interval $0 < a \leqslant r < \infty$ subject to the condition of vanishing at $r = a$. Solutions of (2) are the Bessel functions $J_{\pm is}(kr)$ and every solution w of (2) is such that $r^{-\frac{1}{2}}w(r)$ belongs to $L^2(a, \infty)$. Since the problem is of the limit circle type at infinity it is necessary to prescribe a suitable asymptotic condition there to make the eigenfunctions determinate. In the present instance this condition is

$$\lim_{r\to\infty} r^{\frac{1}{2}}[w(r) \cos(kr - \beta - \tfrac{1}{4}\pi) - w'(r) \sin(kr - \beta - \tfrac{1}{4}\pi)] = 0. \tag{3}$$

The solution of (2) which automatically satisfies the boundary condition at $r = a$ is

$$w(r) = J_u(kr)\, Y_u(ka) - J_u(ka)\, Y_u(kr), \tag{4}$$

where $u = is$. On inserting (4) into (3) and utilizing the Hankel asymptotic expansions of the Bessel functions of large argument we find the eigenvalue equation for u:

$$J_u(ka) \sin(\tfrac{1}{2}u\pi - \beta) + Y_u(ka) \cos(\tfrac{1}{2}u\pi - \beta) = 0. \tag{5}$$

The corresponding eigenfunctions are of the form

$$w = c[J_u(kr) \sin(\tfrac{1}{2}u\pi - \beta) + Y_u(kr) \cos(\tfrac{1}{2}u\pi - \beta)],$$

where c is independent of r. For large r these functions are such that

$$w \sim c(2/\pi kr)^{\frac{1}{2}} \sin(kr - \beta - \tfrac{1}{4}\pi).$$

Thus the eigenfunctions possess the same phase and frequency at infinity.

In this paper we shall consider the equation (5) and both the real and imaginary values of u which satisfy it. The related equation which corresponds to the condition of vanishing derivative at $r = a$ is also considered.

In a previous paper (2) the eigenvalue problem associated with (2) was considered for the finite interval $0 < a \leqslant r \leqslant b$ when arbitrary (real) Sturm–Liouville conditions were applied at the end points. The corresponding eigenvalue equation was of the form

$$[H_1 J_u(ka) - ka J'_u(ka)] \cdot [H_2 Y_u(kb) + kb Y'_u(kb)]$$
$$= [H_2 J_u(kb) + kb J'_u(kb)] \cdot [H_1 Y_u(ka) - ka\, Y'_u(ka)] \tag{6}$$

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in which the constants H_1, H_2 were independent of u and k. For this equation it was proved that the real u-zeros were such that $dk/du > 0$ whilst the imaginary zeros $u = is$ were such that $dk/ds < 0$. The type of argument employed to establish these results proceeded directly from the differential equation and the boundary conditions. This method does not appear capable of extension to handle the present singular case due to the appearance of divergent integrals. Instead it is necessary to resort to a method developed by Watson ((3), p. 508) in connexion with an equation like (5) in which the trigonometric functions do not depend on u. This method requires the use of certain integral representations of products of Bessel functions and has been employed by Cochran (4) and Willis (5) to discuss particular cases of (6) above. It should be noted that the results for the singular case cannot be obtained from those for the finite interval by allowing $b \to \infty$. We shall in fact show that for the infinite interval the opposite results are true, viz. $dk/du < 0$ and $dk/ds > 0$.

2. If we set

$$Y_u(x) = [J_u(x)\cos u\pi - J_{-u}(x)]\operatorname{cosec} u\pi \tag{7}$$

we find from (5) the equation

$$[J_u(x) - J_{-u}(x)]\cos \tfrac{1}{2}u\pi \cos\beta - [J_u(x) + J_{-u}(x)]\sin \tfrac{1}{2}u\pi \sin\beta = 0, \tag{8}$$

where $x = ka$. With $u = is$ this is the equation considered by Titchmarsh. It can also be written in the form

$$J_u(x)\cos(\tfrac{1}{2}u\pi + \beta) - J_{-u}(x)\cos(\tfrac{1}{2}u\pi - \beta) = 0. \tag{9}$$

The alternative forms (8), (9) do not apply when u is an integer or zero. For such values it is necessary to refer to the original equation (5).

Since the possible values of u^2 are the eigenvalues of a self adjoint boundary-value problem and are necessarily real it follows that the u-zeros of (5) are real or purely imaginary. There are in fact an infinite number of real u-zeros and an infinite number of purely imaginary u-zeros of (5) for any positive ka.

We shall consider first the imaginary values $u = is$ and write

$$J_{is}(x) = De^{i\psi}, \quad J_{-is}(x) = De^{-i\psi}, \tag{10}$$

where D and ψ are real functions of x and s. The function D cannot vanish for real non-zero s since for such values J_{is} and J_{-is} are independent solutions of (2). Substitution in (8) then yields the equation

$$\tan\psi = \tan\beta \tanh(\tfrac{1}{2}s\pi). \tag{11}$$

We regard this equation as defining a relationship between each root s and x and find, by logarithmic differentiation,

$$\frac{\partial\psi}{\partial s} + \frac{\partial\psi}{\partial x}\frac{dx}{ds} = \frac{\pi \sin 2\psi}{2\sinh(s\pi)}. \tag{12}$$

The ψ-derivatives appearing here can be calculated from (10) and we find that (12) yields the equation

$$\frac{1}{x}\frac{dx}{ds} = \frac{\pi^2 D^2 \sin 2\psi}{2(\sinh s\pi)^2} + \frac{i\pi}{2\sinh s\pi}\left[J_{-is}(x)\frac{\partial J_{is}(x)}{\partial s} - J_{is}(x)\frac{\partial J_{-is}(x)}{\partial s}\right]. \tag{13}$$

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Now, by Watson ((3), p. 444),

$$J_{is}(x)^2+Y_{is}(x)^2=\frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cos 2st\,dt, \tag{14}$$

$$J_{is}(x)\frac{\partial Y_{is}(x)}{\partial s}-Y_{is}(x)\frac{\partial J_{is}(x)}{\partial s}=\frac{-4i}{\pi}\int_0^\infty K_0(2x\sinh t)\,e^{-2ist}\,dt. \tag{15}$$

By means of (7), (10) we can express the terms on the right side of (13) in terms of the above integrals. After some reduction it is found that

$$\frac{dx}{ds}=2x\int_0^\infty K_0(2x\sinh t)\sin 2st\,dt. \tag{16}$$

An integration by parts gives the formula

$$s\frac{dx}{ds}=4x^2\int_0^\infty K_1(2x\sinh t)\cosh t\sin^2 st\,dt. \tag{17}$$

It follows that each function $x(s)$ is monotone increasing in the quadrant $x>0$, $s>0$ since dx/ds is non-negative there.

Further information about the curves of the zeros $x(s)$ can be found by investigating their behaviour for small values of x. Suppose that x^2 is small compared with unity and s so that

$$J_{is}(x)=\frac{(\frac{1}{2}x)^{is}}{\Gamma(1+is)}\left[1-\frac{1}{1+is}\left(\frac{x}{2}\right)^2+O\left(\frac{x^4}{1+s^2}\right)\right]. \tag{18}$$

It follows from this formula and (11) that the zeros of (9) satisfy the approximate equation

$$e^{-2i\psi}(\tfrac{1}{2}x)^{2is}=\frac{\Gamma(1+is)}{\Gamma(1-is)}\left[1-\frac{isx^2}{2(1+s^2)}\right]. \tag{19}$$

When s is small enough we shall employ the expansion

$$\log\frac{\Gamma(1+is)}{\Gamma(1-is)}=-2si\left[\gamma+\sum_{m=1}^{\infty}\frac{(-1)^m s^{2m}\zeta(2m+1)}{2m+1}\right],$$

where ζ is the Riemann zeta function and γ is Euler's constant. The equation (19) may now be expressed in the form

$$\log(x/2)=\frac{\psi-n\pi}{s}-\gamma-\frac{x^2}{4(1+s^2)}-\sum_{m=1}^{\infty}\frac{(-1)^m s^{2m}\zeta(2m+1)}{2m+1}+O(x^4). \tag{20}$$

Here ψ is defined by (11) and n is an integer which must be positive by virtue of the monotonic property. It follows from the preceding formula that $x\to 0$ as $s\to 0$. The corresponding curves approach the origin and have zero slope there.

There exist in addition curves representing imaginary zeros of (5) which do not pass through the origin and for these (20) is not in general applicable. If $s=0$ then $u=0$ and from (5) the corresponding curves meet the x-axis in the points given by the roots of the equation

$$J_0(x)\sin\beta-Y_0(x)\cos\beta=0. \tag{21}$$

There are an infinite number of such curves since the preceding equation possesses an infinite number of real roots.

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The shape of the $x(s)$ curves for s large compared with unity and x^2 may be obtained from (19) by inserting Stirling's formula for the Γ-functions. This yields the approximate equation

$$s \log\left(\frac{2s}{ex}\right) = (n-\tfrac{1}{4})\pi - \psi + \frac{1}{12s} + \frac{sx^2}{4(1+s^2)} + O\left(\frac{x^4}{s^2}\right) + Os^{-5}, \tag{22}$$

where ψ is given by (11).

Accurate information on the zeros $x(s)$ can be obtained by integrating (16) numerically from a specified initial value $x(s_0)$. (A similar procedure was followed by Olver (6) to compute the zeros of equation (25) below.) In the case of those curves which do not approach the origin, the initial values of x at $s = 0$ are the known zeros of (21) and the problem is straightforward. For the curves which approach the origin the integration must be started from an initial point on the appropriate curve. The expansion (20), valid for small x and s, is employed to provide this point. Here s_0 is chosen small enough so that the term $x^2/4(1+s^2)$ is an insignificant contribution to the right-hand side of (20), thus ensuring the accuracy of the starting value. In fact by choosing s_0 sufficiently small, the first few steps of the numerical integration can themselves be verified using (20), providing a check on the numerical process at this end of the curve.

Numerical integration was performed over the range $0 < s < 4$ using formulae of Runge–Kutta type (8). Two such formulae with respective truncation errors $O(h^3)$ and $O(h^5)$, where h is the length of the integration step in s, were used. The value of h was adjusted so that, for a given step of the integration, the values of x calculated from the two formulae were in reasonable agreement. Where they differed, the more accurate of the two values was used to continue the integration. The several evaluations of the integral in (16) which are necessary for each step of the Runge–Kutta process were carried out numerically, using a suitable step in t.

Values of x obtained by numerical integration and corresponding to large enough values of s in the range $0 < s < 4$ were checked using the expansion (22). In all the calculations carried out for the case considered ($\beta = 0$) the check was adequate to three or four significant figures. This not only verifies the actual values in the corresponding region but provides a satisfactory check on the whole integration procedure.

The form of the $x(s)$ curves is illustrated in Figure 1. The curves which approach the origin correspond to positive integral values of n in (20) and (22) and the first four of these are plotted. The numbers $n = 0, -1, -2, \ldots$, give the curves which meet the x-axis in the points which correspond to the zeros of $Y_0(x)$.

The parts of the curves in Figure 1 lying in the range $4 < s < 8$ were drawn by plotting points calculated from the formula

$$s \log\left(\frac{s+z}{x}\right) - z - \frac{c}{z} = (n-\tfrac{1}{4})\pi - \psi + O(z^{-3}),$$

where $z = (s^2+x^2)^{\frac{1}{2}}$, $c = (5s^2-3z^2)/(24z^2)$. This formula, which reduces to (22) when x/s is small enough, is valid for large s uniformly for $0 < x < \infty$. It is derived by substituting the uniform asymptotic expansion ((9), p. 87) for $J_{is}(x)$ of large order into the equation (9).

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When x is large compared with s^2 and unity the zeros of (5) may be obtained by modifying a formula ((3), p. 506) developed by McMahon for the zeros of equation (25) below. We then obtain the asymptotic formula

$$x = \lambda + \frac{4s^2+1}{8\lambda} - \frac{(4s^2+1)(28s^2+31)}{384\lambda^3} + O(\lambda^{-5}),$$

where $\lambda = \beta - (n-\frac{1}{4})\pi$. Here n is a negative integer which for $\beta = 0$ identifies the curves in the way shown in Figure 1.

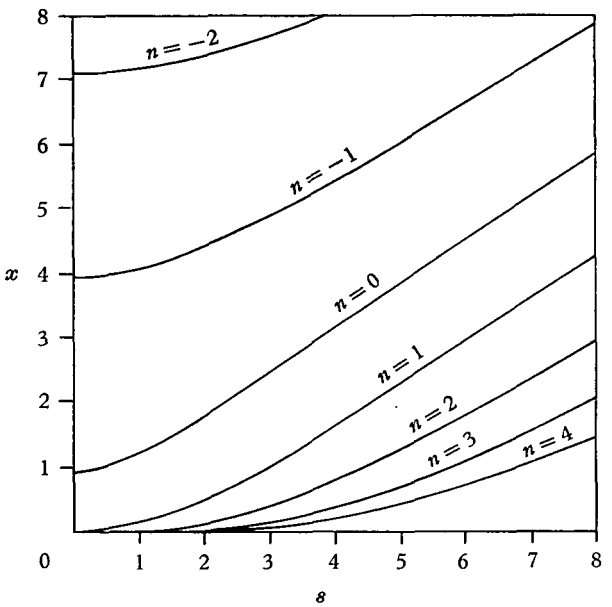


Fig. 1

Now let the curve representing the zero $x(s)$ commence at the point $[0, x(0)]$ where $x(0)$ is a root of (21). Then along this curve we shall prove that

$$x^2 - x(0)^2 \leqslant s^2. \tag{23}$$

In the event that the curve $x(s)$ is one of those which approaches the origin the preceding inequality reduces to $0 < x \leqslant s$.

The result (23) may be obtained from (17) by dividing the range of integration into the two parts $0 \leqslant t \leqslant t_1$, $t_1 \leqslant t < \infty$. In the first interval we use the fact that $|\sin st| \leqslant s \sinh t$ whilst in the second interval we employ $|\sin st| \leqslant 1$. The resulting integrals are exact and we find that

$$\frac{dx}{ds} \leqslant \frac{ps}{x} + \frac{qx}{s},$$

where $p = 1 - \rho^2 K_2(\rho)$, $q = 2K_0(\rho)$ and ρ is any positive number. On integrating the above differential inequality and using (s_1, x_1) as the initial point we find that

$$(1-q)x^2 - ps^2 \leqslant (s/s_1)^{2q}[(1-q)x_1^2 - ps_1^2]$$

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for $0 < x_1 < x$ and $0 < s_1 < s$. If we let $\rho \to \infty$ then $p \to 1$ and $q \to 0$ so that

$$x^2 - s^2 \leqslant x_1^2 - s_1^2.$$

We obtain (23) by taking $s_1 = 0$, $x_1 = x(0)$ in the preceding inequality.

3. The real zeros of (5) may be investigated by employing similar methods. It is evidently sufficient to consider $u \geqslant 0$. Let $x = x(u)$ denote a curve representing such a zero then along this curve it may be shown analogous to (16) that

$$\frac{dx}{du} = -2x\int_0^\infty K_0(2x\sinh t)\sinh 2ut\,dt. \tag{24}$$

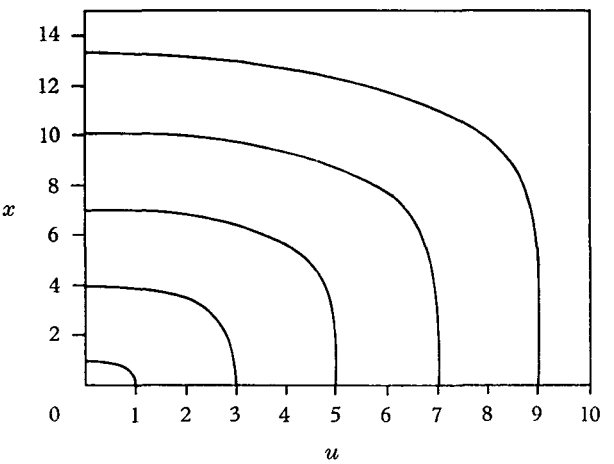


Fig. 2

It follows that $x'(u) < 0$ in the quadrant $x > 0$, $u > 0$ so that each zero $x(u)$ is a decreasing function of u. This result is in contrast to that which applies to the related equation

$$J_u(x)\cos\alpha - Y_u(x)\sin\alpha = 0 \tag{25}$$

in which α is independent of x and u. For this equation it is known ((3), p. 508) that the x-zeros are steadily increasing functions of u.

The curves $x(u)$ representing the zeros of (5) meet the x-axis in the points which are given by the roots of (21). It follows that there are an infinite number of such curves each of which by (24) has zero slope at these points. To determine the behaviour of these curves as $x \to 0$ we first multiply (9) by x^u. It then follows that as $x \to 0$ the curves representing the real zeros of (9) approach the points given by $\cos(\frac{1}{2}u\pi - \beta) = 0$. Therefore $u \to 2n + 1 + 2\beta/\pi$ where n is zero or an integer (which must however be such that $u > 0$).

The general shape of the $x(u)$ curves in the case $\beta = 0$ is depicted in Figure 2. In constructing these curves it is useful to note that, since $\beta = 0$ and $x = ka$, the x-zeros of (5) for odd integral u are simply the zeros of $J_u(x)$. Similarly when u is an even integer the zeros of (5) coincide with those of $Y_u(x)$.

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Although the $x(u)$ curves are different in the two cases, the equations (5), (25) coincide when $x = ka$ and $\alpha = (u-1)\frac{1}{2}\pi - \beta$. Therefore we can obtain information concerning the location and spacing of the x-zeros of (5), for prescribed real values of u, from similar existing results concerning the zeros of (25). Since the latter equation is fully treated for real u from this point of view in Watson's book we shall not consider it further here.

4. If the boundary condition at $r = a$ is $w'(a) = 0$ we obtain instead of (5) the eigenvalue equation

$$J'_u(y)\sin(\tfrac{1}{2}u\pi - \beta) + Y'_u(y)\cos(\tfrac{1}{2}u\pi - \beta) = 0. \tag{26}$$

When $y > 0$ is given this equation possesses an infinite number of real u-zeros as well as an infinite number of purely imaginary u-zeros. It can be shown that the real zeros satisfy the equation

$$(u^2 - y^2)\frac{dy}{du} = 2y\int_0^\infty K_0(2y\sinh t)\,(y^2\cosh 2t - u^2)\sinh 2ut\,dt.$$

It follows that $y'(u)$ is negative when $0 < u < y$, but not necessarily elsewhere. This may be illustrated by considering the particular equation which occurs when $\beta = 0$. In this case it is evident from (26) that $y \to 0$ whenever u tends to an odd integer. To obtain the form of the $y(u)$ curves near their points of intersection with the u-axis we write (26), when $\beta = 0$, in the form

$$[J'_u(y) - J'_{-u}(y)]\operatorname{cosec}(\tfrac{1}{2}u\pi) = 0. \tag{27}$$

We now insert the series expansions of the Bessel functions giving

$$\left(\frac{y}{2}\right)^{2u}\sum_{m=0}^{\infty}\frac{(-1)^m(2m+u)\,y^{2m}}{2^{2m}\Gamma(m+u+1)\,m!} = \sum_{m=0}^{\infty}\frac{(-1)^m(2m-u)\,y^{2m}}{2^{2m}\Gamma(m-u+1)\,m!}.$$

The first n terms in the series on the right possess simple zeros when $u = n$ owing to the poles of the Γ-functions. If we set $u = n + \delta$ where δ is small we find that $\delta = O(y^{2n})$ and that

$$u = n + \frac{2}{n!(n-1)!}\left(\frac{y}{2}\right)^{2n} + O(y^{2n+2}),$$

where n is an odd positive integer. The form of the curves of the real zeros of (26) is shown in Figure 3, and it is seen that dy/du is positive near the points $(n, 0)$.

The imaginary zeros $u = is$ of (26) satisfy the integro-differential equation

$$(y^2 + s^2)\frac{dy}{ds} = 2y\int_0^\infty K_0(2y\sinh t)\,(y^2\cosh 2t + s^2)\sin 2st\,dt.$$

In this case it can be shown after an integration by parts, that $dy/ds > 0$ for $y^2 + s^2 \geqslant \frac{1}{2}$. Also there are an infinite number of curves which approach the origin and for these the following formula, analogous to (20), can be derived:

$$\log\frac{y}{2} = -\gamma + \frac{\psi - (n-\frac{1}{2})\pi}{s} - \frac{(2+s^2)\,y^2}{4(1+s^2)\,s^2} - \sum_{m=1}^{\infty}\frac{(-1)^m s^{2m}\zeta(2m+1)}{2m+1} + O[y^4(\log y)^2],$$

where $n = 1, 2, 3, \ldots$, and ψ is defined by (11).

When both s and y are large it is necessary to derive a formula based on the uniform asymptotic expansion of the function $J'_{is}(y)$. This gives the following equation:

$$s\log\left(\frac{s+z}{y}\right) - z - \frac{c'}{z} = (n - \tfrac{3}{4})\pi - \psi + O(z^{-3}),$$

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where $z = (s^2+y^2)^{\frac{1}{2}}$, $c' = (9z^2-7s^2)/(24z^2)$ and n is a positive or negative integer or zero. The general arrangement of the curves in the (s, y) plane is similar to that shown in figure 1. For the case $\beta = 0$ the values $n = 0, -1, -2, \ldots$, give the curves which approach the origin whilst the values $n = 1, 2, \ldots$, give the curves which meet the y-axis in the points which correspond to the zeros of $Y_0'(y)$.

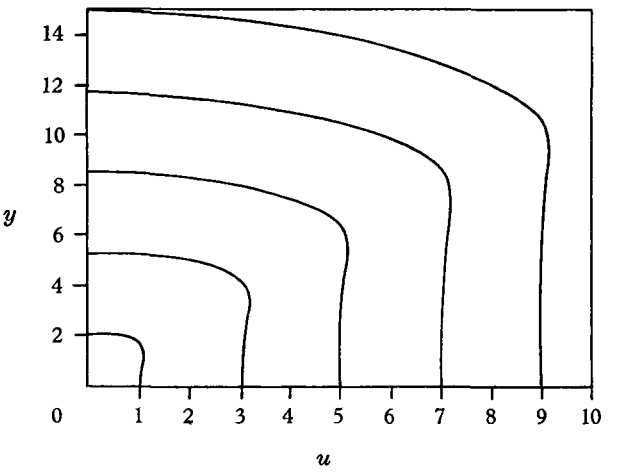


Fig. 3

When y is large compared with s^2 and unity we should employ a formula of the McMahon type. This formula is

$$y = \lambda_1 + \frac{4s^2-3}{8\lambda_1} - \frac{112s^4-328s^2-9}{384\lambda_1^3} + O(\lambda_1^{-5}),$$

where $\lambda_1 = \beta-(n-\frac{3}{4})\pi$, and n is a negative integer.

The large real y-zeros of (26) are given by the preceding formula with $-u^2$ substituted for s^2 therein.

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