A SOLUTION OF $A^5+B^5+C^5 = D^5+E^5+F^5$

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We are concerned to find solutions in rationals, or more generally in polynomials with rational coefficients, of the equation

$$A^5+B^5+C^5 = D^5+E^5+F^5. \tag{1}$$

Since the equation is homogeneous, the distinction between integers and rationals is not here important.

The only previous solution of (1) is given by

$$(x^5+75y^5)^5 = (75y^5-x^5)^5+(x^5+25y^5)^5+(x^5-25y^5)^5+(10x^3y^2)^5+(50xy^4)^5,$$

due to Sastry (2). This is obviously very particular—so much so that it does not even yield any positive solutions of (1). In the present paper I obtain two further solutions of (1), each in three parameters, by a general method due to Norrie (1), combined in each case with a special device. It is likely that there is a solution in four parameters, or at least that there are an infinity of solutions in three parameters, but I see no prospect of making further progress by the methods of this paper.

Throughout this paper all sums will be taken over the three cyclic permutations of the suffices 1, 2, 3.

The general method we use is to assume that the A, B, C, D, E, F are linear expressions in the variable x. We thus have twelve disposable coefficients, and a quintic equation in x. We choose the twelve coefficients so that the quintic reduces, by the vanishing of four of its terms, to a linear equation in x, which must necessarily have a rational solution.

In the first place, let us make the coefficients of x^5, x^4, x^3, x^0 vanish. This involves, for the coefficients of x^5 and x^0, already knowing two solutions of (1); and since the only solutions we have are the trivial symmetric ones, we must take

$$\left.\begin{aligned} A = a_1x+b_2, \quad B = a_2x+b_3, \quad C = a_3x+b_1, \\ D = a_1x+b_3, \quad E = a_2x+b_1, \quad F = a_3x+b_2. \end{aligned}\right\} \tag{2}$$

(It may easily be verified that any other arrangement either is equivalent to this or involves an equation of the type $A = D$.) An immediate consequence of these assumptions is the equation

$$A+B+C = D+E+F, \tag{3}$$

which will hold for any solution of this type. For the coefficients of x^4, x^3 to vanish, we must now satisfy

$$\Sigma a_1^4(b_2-b_3) = \Sigma a_1^3(b_2^2-b_3^2) = 0. \tag{4}$$

Regarded as equations for the ratios $b_1 : b_2 : b_3$ these equations have two solutions; and since one of these is clearly $b_1 = b_2 = b_3$, the other must also be rational. It may be verified that, since a common factor in the b_i is irrelevant, we can take the second solution to be

$$b_1-a_1^4 = b_2-a_2^4 = b_3-a_3^4 = -\frac{\Sigma a_1^3(a_2^8-a_3^8)}{\Sigma a_1^3(a_2^4-a_3^4)}. \tag{5}$$

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The resulting linear equation for x is

$$2x\Sigma a_1^2(b_2^3-b_3^3) = -\Sigma a_1(b_2^4-b_3^4), \tag{6}$$

and (2), (5) and (6) together give a solution of (1) in the three parameters a_1, a_2, a_3. It is clear that we can take out a common factor $(a_1-a_2)(a_2-a_3)(a_3-a_1)$, both in the fraction in (5) and also subsequently in (6), so that on reducing the solution to polynomials these will be at most of degree 30, and it is not unlikely that further common factors can be removed.

The special case $a_1 = 0$ is of particular interest. It now follows from (5) that $a_2b_2 = a_3b_3$, and thus from (2) that

$$A/D = F/B = a_3/a_2, \tag{7}$$

so that in (1) we may further require that two pairs of terms have the same (arbitrary) ratio. In the special case $a_1 = 0, a_2 = 1, a_3 = 2$, for example, we have the solution

$$14760^5 + 12418^5 + 5963^5 = 7380^5 + 11926^5 + 13835^5. \tag{8}$$

The alternative method of solution is to make the coefficients of x^5, x^4, x^3, x^2 vanish. Since this involves one trivial solution of (1), we shall take

$$\left.\begin{aligned} A &= a_1^3x+b_1+c_1, \quad B = a_2^3x+b_2+c_2, \quad C = a_3^3x+b_3+c_3,\\ D &= a_1^3x+b_1-c_1, \quad E = a_2^3x+b_2-c_2, \quad F = a_3^3x+b_3-c_3. \end{aligned}\right\} \tag{9}$$

The reason for taking a_1^3, a_2^3, a_3^3 instead of the more natural a_1, a_2, a_3 will become clear below. For the coefficients of x^4, x^3, x^2 to vanish, we must now have

$$\Sigma a_1^{12}c_1 = \Sigma a_1^9b_1c_1 = \Sigma a_1^6c_1(c_1^2+3b_1^2) = 0. \tag{10}$$

If we eliminate b_1 between the last two equations in (10), using the first one to simplify, we find

$$3a_2^6a_3^6c_2c_3(a_2^3b_3-a_3^3b_2)^2 = a_1^{12}c_1\Sigma a_1^6c_1^3,$$

so that for (10) to give rational values for the ratio $b_1 : b_2 : b_3$ it is necessary and sufficient that

$$3c_1c_2c_3\Sigma a_1^6c_1^3 \tag{11}$$

should be a perfect square. Problems of this sort are in general not easy to attack; but in this particular case if we take

$$\Sigma a_1^2c_1 = 0, \tag{12}$$

then (11) reduces by a well-known result to $(3a_1a_2a_3c_1c_2c_3)^2$. It is clear that the arbitrary constant in the b_i or in the c_i leads to no extra generality, being merely absorbed in the value of x; thus we can take, from (10) and (12),

$$c_1 = (a_2^{10}-a_3^{10})/a_1^2, \quad c_2 = (a_3^{10}-a_1^{10})/a_2^2, \quad c_3 = (a_1^{10}-a_2^{10})/a_3^2, \tag{13}$$

and then to satisfy the remaining parts of (10)

$$b_1 = a_2^5a_3^5/a_1^2, \quad b_2 = a_3^5a_1^5/a_2^2, \quad b_3 = a_1^5a_2^5/a_3^2. \tag{14}$$

The equation for x now becomes

$$20x\Sigma a_1b_1c_1(b_1^2+c_1^2)+\Sigma c_1(c_1^4+10b_1^2c_1^2+5b_1^4) = 0, \tag{15}$$

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and from (9), (13), (14) and (15) we obtain another three-parameter solution of (1), though one which is substantially more complicated than the previous solution. (So far as a cursory inspection reveals, the resulting polynomials would be of degree 64, though this is clearly capable of some further reduction.)

REFERENCES

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CORRIGENDUM

to the paper

CALCULATION OF *S*-MATRIX ELEMENTS*

By J. S. R. CHISHOLM

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In equation (1*b*), $i\partial/\partial p'_\alpha$ and $i\partial/\partial p''_\alpha$ should be replaced by $-(i/2)\,\partial/\partial p'_\alpha$ and $-(i/2)\,\partial/\partial p''_\alpha$. This results in similar changes in the expression (29) for $\mathscr{V}_\alpha$, and in the line following equation (29) the factor -2 should be omitted from the expression

$$-2[i\chi_{p^{(a)}}+i\chi_{p^{(b)}}]_\alpha.$$

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