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ON TWO CLASSICAL LATTICE POINT PROBLEMS

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1. Let $r(n)$ be the number of representations of n as a sum of two squares, $d(n)$ the number of divisors of n, and

$$R(x) = \sum_{n=0}^{x} r(n) = \pi x + \mathrm{P}(x), \tag{1}$$

$$D(x) = \sum_{n=1}^{x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x), \tag{2}$$

where γ is Euler's constant. Thus $\mathrm{P}(x)$ is the error term in the problem of the lattice points of a circle, and $\Delta(x)$ the error term in Dirichlet's divisor problem, or the problem of the lattice points of a rectangular hyperbola.

It is known that, when $x \to \infty$,

$$\mathrm{P}(x) = \Omega_-(x^{\frac{1}{4}} \log^{\frac{1}{2}} x), \tag{3}$$

that is to say that

$$\varliminf_{x\to\infty} \frac{\mathrm{P}(x)}{x^{\frac{1}{4}} \log^{\frac{1}{2}} x} < 0.$$

But on the Ω_+ side it has apparently not been proved even that

$$\varlimsup_{x\to\infty} \frac{\mathrm{P}(x)}{x^{\frac{1}{4}}} = +\infty; \tag{4}$$

the most that is known in this direction is that the left-hand side of (4) is greater than or equal to a certain positive numerical constant.

In the problem of $\Delta(x)$ there is a similar situation, except that the Ω_- problem happens to be the more difficult, and that there is an extra factor $\log\log x$ in the formula corresponding to (3). Thus it has been proved that

$$\Delta(x) = \Omega_+(x^{\frac{1}{4}} \log^{\frac{1}{2}} x \log\log x), \tag{5}$$

but not that

$$\varliminf_{x\to\infty} \frac{\Delta(x)}{x^{\frac{1}{4}}} = -\infty.* \tag{6}$$

* Hardy (1); Landau (4), ii, 240–9.

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The proofs of (3) and (5) depend on Dirichlet's theorem on Diophantine approximation. The object of this note is to prove (4) and (6) by substituting Kronecker's theorem for Dirichlet's theorem. The proof is set out in detail for $\mathrm{P}(x)$ in § 2, and in outline for $\Delta(x)$ in § 3. The possibility of going beyond (4) and (6) is discussed briefly in § 4.

Throughout the paper the symbol O implies an absolute constant and an inequality valid for all specified values of the variables.

2. We divide the proof of (4) into three stages, Theorems R 1, R 2 and R 3.

Theorem R 1. *If*

$$\omega \geqslant 2\eta > 0, \quad X\eta^2 \geqslant 1,$$

and

$$\sigma(u) = u^{-\frac{1}{4}}\mathrm{P}(u^2),$$

then

$$\underline{\sigma}(\omega-\eta, \omega+\eta) < -S_X(\omega) + \frac{A}{\omega} + \frac{A}{X^{\frac{1}{4}}\eta}, \tag{7}$$

$$\overline{\sigma}(\omega-\eta, \omega+\eta) > -S_X(\omega) - \frac{A}{\omega} - \frac{A}{X^{\frac{1}{4}}\eta}, \tag{8}$$

where $\underline{\sigma}(\alpha, \beta)$ and $\overline{\sigma}(\alpha, \beta)$ denote the lower and upper bounds of $\sigma(u)$ in the interval $\alpha \leqslant u \leqslant \beta$, A is a positive absolute constant, and

$$S_X(\omega) = \frac{1}{\pi} \sum_{n<X} \left(1 - \frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right) \frac{r(n)}{n^{\frac{3}{4}}} \cos(2\pi n^{\frac{1}{2}}\omega + \tfrac{1}{4}\pi),$$

the summation being from $n = 1$ onwards.

We start from the formula*

$$\int_a^b \chi(x)\,\mathrm{P}(x)\,dx = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} \int_a^b \chi(x) f(nx)\,dx, \tag{9}$$

where $b > a \geqslant 0$, $\chi(x)$ has a continuous derivative for $a \leqslant x \leqslant b$, and

$$\begin{aligned} f(v) &= \pi v^{\frac{1}{2}} J_1(2\pi v^{\frac{1}{2}}) \\ &= -v^{\frac{1}{4}} \cos(2\pi v^{\frac{1}{2}} + \tfrac{1}{4}\pi) + O(v^{-\frac{1}{4}}) \quad (v > 0). \end{aligned} \tag{10}$$

Under the conditions stated this can be deduced at once from the comparatively easy explicit formula for

$$\mathrm{P}_1(x) = \int_0^x \mathrm{P}(u)\,du,$$

which may be written in the form†

$$\mathrm{P}_1(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^2} f_1(nx) \quad (x \geqslant 0), \tag{11}$$

where

$$f_1(v) = vJ_2(2\pi v^{\frac{1}{2}}) = \int_0^v f(u)\,du = O(v^{\frac{1}{2}}) \quad (v \geqslant 0). \tag{12}$$

* Cf. Ingham (3) for the corresponding stage in the analogous treatment of Littlewood's Ω-theorem in the theory of the distribution of primes.

† Landau (4), ii, 204–6.

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We have only to multiply (11) by $\chi'(x)$, integrate term by term (the series being absolutely and uniformly convergent by (12), since $\Sigma r(n)n^{-\frac{1}{2}}$ is convergent), integrate by parts in each term, and use (11) again with $x = a$ and $x = b$. The argument shows that the infinite series in (9) is absolutely convergent.

Now let $$K(y) = \left(\frac{\sin \frac{1}{2}y}{\frac{1}{2}y}\right)^2, \quad k(x) = \begin{cases} 2\pi(1-|x|) & (|x|<1), \\ 0 & (|x|\geqslant 1), \end{cases}$$

so that
$$k(x) = \int_{-\infty}^{\infty} K(y)\, e^{xyi}\, dy. \tag{13}$$

Then by suitable choice of $\chi(x)$, a and b in (9), and the substitution $x = u^2$, we obtain

$$\int_{\omega-\eta}^{\omega+\eta} K_U(u-\omega)\, u^{-\frac{1}{2}} \mathrm{P}(u^2)\, du = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} \int_{\omega-\eta}^{\omega+\eta} K_U(u-\omega)\, u^{-\frac{1}{2}} f(nu^2)\, du,$$

where $\omega \geqslant 2\eta > 0$, $U > 0$, and $K_U(v) = UK(Uv)$. Substituting from (10), making the transformation $u = \omega + y/U$, and writing $\sigma(u) = u^{-\frac{1}{2}}\mathrm{P}(u^2)$, we deduce

$$\int_{-U\eta}^{U\eta} K(y)\, \sigma\left(\omega + \frac{y}{U}\right) dy = -\frac{1}{\pi} \sum_{1}^{\infty} \frac{r(n)}{n^{\frac{1}{2}}} \int_{-U\eta}^{U\eta} K(y) \cos\left\{2\pi n^{\frac{1}{2}}\left(\omega + \frac{y}{U}\right) + \tfrac{1}{4}\pi\right\} dy$$
$$+ O\left\{\sum_{1}^{\infty} \frac{r(n)}{n^{\frac{1}{2}}} \int_{-U\eta}^{U\eta} |K(y)| \left(\omega + \frac{y}{U}\right)^{-1} dy\right\}. \tag{14}$$

The last term in (14) is $O(\omega^{-1})$, since $\omega + y/U \geqslant \omega - \eta \geqslant \frac{1}{2}\omega$ for $-U\eta \leqslant y \leqslant U\eta$ and

$$\sum_{1}^{\infty} \frac{r(n)}{n^{\frac{1}{2}}}, \quad \int_{-\infty}^{\infty} |K(y)|\, dy$$

are convergent. Further, if the ranges of integration in the first series on the right of (14) are extended to $(-\infty, \infty)$, the error in the nth integral is

$$O\left\{\min\left(\frac{1}{U\eta}, \frac{1}{n^{\frac{1}{2}}\eta}\right)\right\},$$

the first estimate being obtained trivially from $K(y) = O(y^{-2})$ and the second by partial integration from $K'(y) = O(y^{-2})$ and $K(y) = O(y^{-1})$ $(y > 0)$. The total error in this series is therefore

$$O\left(\frac{1}{U\eta} \sum_{n \leqslant U^2} \frac{r(n)}{n^{\frac{1}{2}}} + \frac{1}{\eta} \sum_{n > U^2} \frac{r(n)}{n^{\frac{1}{2}}}\right) = O\left(\frac{1}{U^{\frac{1}{2}}\eta}\right),$$

since $\sum_{n=1}^{x} r(n) = O(x)$. But the nth integral thus modified is

$$\Re \int_{-\infty}^{\infty} K(y)\, e^{\{2\pi n^{\frac{1}{2}}(\omega + y/U) + \frac{1}{4}\pi\}i}\, dy = k\left(\frac{2\pi n^{\frac{1}{2}}}{U}\right) \cos\left(2\pi n^{\frac{1}{2}}\omega + \tfrac{1}{4}\pi\right)$$

by (13), since $k(x)$ is real. Hence we obtain, since $k(x) = 0$ for $x \geqslant 1$,

$$\int_{-U\eta}^{U\eta} K(y)\, \sigma\left(\omega + \frac{y}{U}\right) dy = -\frac{1}{\pi} \sum_{n<X} \frac{r(n)}{n^{\frac{1}{2}}} k\left(\frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right) \cos\left(2\pi n^{\frac{1}{2}}\omega + \tfrac{1}{4}\pi\right) + O\left(\frac{1}{\omega}\right) + O\left(\frac{1}{X^{\frac{1}{2}}\eta}\right),$$

where $X = (U/2\pi)^2$. (15)

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Now, since $K(y) \geqslant 0$, we have

$$\underline{\sigma}(\omega-\eta, \omega+\eta)\int_{-U\eta}^{U\eta} K(y)\,dy \leqslant \int_{-U\eta}^{U\eta} K(y)\,\sigma\left(\omega+\frac{y}{U}\right)dy,$$

where $\underline{\sigma}$ has the meaning explained in the enunciation. Using this in (15), multiplying by the positive number

$$\left(\int_{-U\eta}^{U\eta} K(y)\,dy\right)^{-1} = \frac{1}{k(0)}\left\{1+O\left(\frac{1}{U\eta}\right)\right\} = \frac{1}{k(0)}\left\{1+O\left(\frac{1}{X^{\frac{1}{2}}\eta}\right)\right\},$$

and assuming that $X^{\frac{1}{2}}\eta \geqslant 1$, we obtain, on inserting the value of $k(x)/k(0)$,

$$\underline{\sigma}(\omega-\eta, \omega+\eta) \leqslant -\frac{1}{\pi}\sum_{n<X}\frac{r(n)}{n^{\frac{1}{2}}}\left(1-\frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\cos\left(2\pi n^{\frac{1}{2}}\omega+\tfrac{1}{4}\pi\right) + O\left(\frac{1}{X^{\frac{1}{2}}\eta}\sum_{n<X}\frac{r(n)}{n^{\frac{1}{2}}}\right)+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{X^{\frac{1}{2}}\eta}\right).$$

The first O term here is $O(X^{-\frac{1}{2}}\eta^{-1}X^{\frac{1}{2}}) = O(X^{-\frac{1}{2}}\eta^{-1})$, and the inequality (7) of Theorem R 1 follows. The inequality (8) is deduced from (15) in the same way.

Theorem R 2. *For fixed $X > 1$ we have*

$$\overline{\lim_{\omega\to\infty}}\{-S_X(\omega)\} \geqslant \frac{1}{2^{\frac{1}{2}}\pi}\sum_{n<X}\left(1-\frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{\epsilon_n r(n)}{n^{\frac{1}{2}}},$$

where $S_X(\omega)$ is defined in Theorem R 1, *and*

$$\epsilon_n = \begin{cases} +1 & \text{if } n \not\equiv 0 \pmod 4, \\ -1 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Any integer n in the range $1 \leqslant n < X$ can be expressed uniquely in the form $n = \nu^2 q$, where q is a square-free integer and ν an integer ($1 \leqslant q < X$, $1 \leqslant \nu < X^{\frac{1}{2}}$); and ν is even when and only when $n \equiv 0 \pmod 4$. Now the numbers $q^{\frac{1}{2}}$ are linearly independent (with respect to the field of rational numbers)*. Hence, given $\delta > 0$, we can find, by Kronecker's theorem†, arbitrarily large values of ω such that, for each q ($1 \leqslant q < X$),

$$q^{\frac{1}{2}}\omega = \tfrac{1}{2} + m_q + \delta_q \quad (m_q \text{ an integer}, |\delta_q| < \delta), \tag{16}$$

and therefore, for each $n = \nu^2 q$ ($1 \leqslant n < X$),

$$\begin{aligned} -\cos\left(2\pi n^{\frac{1}{2}}\omega+\tfrac{1}{4}\pi\right) &= -\cos\left(\pi\nu + 2\pi\nu\delta_q + \tfrac{1}{4}\pi\right) \\ &= (-1)^{\nu-1}\cos\left(2\pi\nu\delta_q+\tfrac{1}{4}\pi\right) > \epsilon_n\cos\tfrac{1}{4}\pi - 2\pi n^{\frac{1}{2}}\delta, \end{aligned}$$

since $(-1)^{\nu-1} = \epsilon_n$, $|\cos\alpha - \cos\beta| \leqslant |\alpha-\beta|$ for real α and β, and $|\nu\delta_q| < n^{\frac{1}{2}}\delta$.

* It is not easy to find an explicit proof of this fact in the classical literature. A proof of a more comprehensive result will shortly be published by A. S. Besicovitch in the *J. London Math. Soc.*

† See, for example, Hardy and Wright(2), Chapter XXIII (Theorem 444).

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Hence

$$\overline{\lim_{\omega\to\infty}}\frac{1}{\pi}\sum_{n<X}\left(1-\frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{r(n)}{n^{\frac{3}{4}}}\{-\cos(2\pi n^{\frac{1}{2}}\omega+\tfrac{1}{4}\pi)\}\geqslant\frac{1}{\pi}\sum_{n<X}\left(1-\frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{r(n)}{n^{\frac{3}{4}}}\left(\frac{\epsilon_n}{2^{\frac{1}{2}}}-2\pi n^{\frac{1}{2}}\delta\right),$$

and the result follows on making $\delta\to 0$.

Theorem R 3. *When* $X\to\infty$,

$$\sum_{n<X}\left(1-\frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{\epsilon_n r(n)}{n^{\frac{3}{4}}}\sim\tfrac{4}{3}\pi X^{\frac{1}{2}}, \tag{17}$$

where ϵ_n *is defined in Theorem* R 2.

We have first, for $x>0$,

$$\begin{aligned}\sum_{n\leqslant x}\epsilon_n r(n)&=\sum_{n\leqslant x}r(n)-2\sum_{4m\leqslant x}r(4m)\\&=\sum_{n\leqslant x}r(n)-2\sum_{m\leqslant\frac{1}{4}x}r(m)^*\\&=\pi x-2(\tfrac{1}{4}\pi x)+O(x^{\frac{1}{2}})=\tfrac{1}{2}\pi x+\rho(x),\end{aligned}$$

say. Hence the left-hand side of (17) is, for $X>1$,

$$\begin{aligned}&\int_1^X\left(1-\frac{x^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{1}{x^{\frac{3}{4}}}d\{\tfrac{1}{2}\pi x+\rho(x)\}+O(1)\\&\quad=\tfrac{1}{2}\pi\int_1^X\left(1-\frac{x^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{1}{x^{\frac{3}{4}}}dx-\int_1^X\rho(x)\,d\left\{\left(1-\frac{x^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{1}{x^{\frac{3}{4}}}\right\}+O(1)\\&\quad=\tfrac{4}{3}\pi X^{\frac{1}{2}}+O\left(\int_1^X x^{\frac{1}{2}}(x^{-\frac{1}{2}}+x^{-\frac{3}{4}}X^{-\frac{1}{2}})\,dx\right)+O(1)=\tfrac{4}{3}\pi X^{\frac{1}{2}}+O(1).\end{aligned}$$

Proof of (4). In Theorem R 1 (8) take a fixed $X>1$, make $\omega\to\infty$, and choose $\eta=\eta(\omega)$ so that $\eta\to\infty$, $0<\eta\leqslant\frac{1}{2}\omega$. We deduce that

$$\begin{aligned}\overline{\lim_{u\to\infty}}\,\sigma(u)&\geqslant\overline{\lim_{\omega\to\infty}}\{-S_X(\omega)\}\\&\geqslant\frac{1}{2^{\frac{1}{2}}\pi}\sum_{n<X}\left(1-\frac{n^{\frac{1}{2}}}{X^{\frac{1}{2}}}\right)\frac{\epsilon_n r(n)}{n^{\frac{3}{4}}}\end{aligned}$$

by Theorem R 2. But the right-hand side is arbitrarily large with X, by Theorem R 3. Hence

$$\overline{\lim_{u\to\infty}}\,\sigma(u)=+\infty,$$

and this is equivalent to (4) in virtue of the definition of $\sigma(u)$.

* $r(4m)=r(m)$, since the integers x, y in any representation $4m=x^2+y^2$ must both be even.

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3. The steps in the parallel proof of (6) are as follows.

Theorem D 1. *If* $\omega \geqslant 2\eta > 0, X\eta^2 \geqslant 1,$

and $\tau(u) = u^{-\frac{1}{4}}\Delta(u^2),$

then
$$\underline{\tau}(\omega-\eta, \omega+\eta) < T_X(\omega) + \frac{A}{\omega} + \frac{A\log(X+2)}{X^{\frac{1}{4}}\eta},$$
$$\overline{\tau}(\omega-\eta, \omega+\eta) > T_X(\omega) - \frac{A}{\omega} - \frac{A\log(X+2)}{X^{\frac{1}{4}}\eta},$$

where
$$T_X(\omega) = \frac{1}{2^{\frac{1}{4}}\pi}\sum_{n<X}\left(1-\frac{n^{\frac{1}{4}}}{X^{\frac{1}{4}}}\right)\frac{d(n)}{n^{\frac{3}{4}}}\sin(4\pi n^{\frac{1}{4}}\omega + \tfrac{1}{4}\pi),$$

and the other notation is analogous to that of Theorem R 1.

Theorem D 2. *For fixed* $X > 1$ *we have*
$$\varlimsup_{\omega\to\infty} T_X(\omega) \leqslant -\frac{1}{2\pi}\sum_{n<X}\left(1-\frac{n^{\frac{1}{4}}}{X^{\frac{1}{4}}}\right)\frac{\epsilon_n d(n)}{n^{\frac{3}{4}}},$$

where ϵ_n *is defined in Theorem* R 2.

Theorem D 3. *When* $X \to \infty,$
$$\sum_{n<X}\left(1-\frac{n^{\frac{1}{4}}}{X^{\frac{1}{4}}}\right)\frac{\epsilon_n d(n)}{n^{\frac{3}{4}}} \sim (4\log 2)\, X^{\frac{1}{4}}.$$

The proof of Theorem D 1 is similar to that of Theorem R 1. The fundamental identity, corresponding to (9), is
$$\int_a^b \chi(x)\,\Delta(x)\,dx = \frac{1}{2^{\frac{1}{4}}\pi}\sum_{n=1}^{\infty}\frac{d(n)}{n}\int_a^b \chi(x)\,g(nx)\,dx,$$
where $b > a \geqslant 0$, $\chi(x)$ has a continuous derivative for $a \leqslant x \leqslant b$, and
$$g(v) = v^{\frac{1}{4}}\sin(4\pi v^{\frac{1}{4}} + \tfrac{1}{4}\pi) + O(v^{-\frac{1}{4}}) \quad (v > 0).$$

Theorem D 2 is proved by the argument of Theorem R 2, except that, since we now have $4\pi n^{\frac{1}{4}}\omega$ in place of $2\pi n^{\frac{1}{4}}\omega$, we choose ω to make the numbers $2q^{\frac{1}{4}}\omega$ (instead of $q^{\frac{1}{4}}\omega$) approximately congruent to $\frac{1}{2}$ (mod 1).

Theorem D 3 is not quite so simple as Theorem R 3, and we give the proof in full. We have first, for $x > 0$,
$$\sum_{n\leqslant x}\epsilon_n d(n) = \sum_{n\leqslant x} d(n) - 2\sum_{4m\leqslant x} d(4m). \tag{18}$$

Now
$$d(2m) = 2d(m) - d(\tfrac{1}{2}m), \tag{19}$$
where $d(\frac{1}{2}m) = 0$ if $\frac{1}{2}m$ is not an integer. For the divisors of $2m$ may be enumerated as d and $2d$, where d runs through the divisors of m; and any divisor of $2m$ will

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be counted just once, unless it is one of the $d(\frac{1}{2}m)$ even divisors of m, in which case it will be counted twice (once as a d and once as a $2d$). Applying (19) twice, we obtain

$$d(4m) = 2d(2m) - d(m) = 3d(m) - 2d(\tfrac{1}{2}m),$$

whence, by (18),

$$\sum_{n\leqslant x} \epsilon_n d(n) = \sum_{n\leqslant x} d(n) - 6 \sum_{4m\leqslant x} d(m) + 4 \sum_{4m\leqslant x} d(\tfrac{1}{2}m)$$

$$= D(x) - 6D(\tfrac{1}{4}x) + 4D(\tfrac{1}{8}x).$$

Since $D(x) = x\log x + \beta x + O(x^{\frac{1}{2}})$, where $\beta = 2\gamma - 1$, it follows that

$$\sum_{n\leqslant x} \epsilon_n d(n) = H(x\log x + \beta x) + Kx + O(x^{\frac{1}{2}}),$$

where $$H = 1 - \tfrac{6}{4} + \tfrac{4}{8} = 0, \quad K = \tfrac{6}{4}\log 4 - \tfrac{4}{8}\log 8 = \tfrac{3}{2}\log 2.$$

The proof now proceeds as before.

4. If we wished to go further and replace (4), for example, by

$$\mathrm{P}(x) = \Omega_+\{x^{\frac{1}{2}}\phi(x)\} \tag{20}$$

with an explicit $\phi(x)$ tending to infinity with x, we should have to find an explicit function $\Phi_\delta(X)$ such that the relations (16) of Theorem R 2 hold for some (large) ω not exceeding $\Phi_\delta(X)$ (X large, δ small). In the corresponding treatment of the Ω_- problem an upper bound for ω of the form $\Phi_\delta(X) = e^{cX}(c = c_\delta > 0)$ is provided at once by Dirichlet's theorem, which we apply in order to make all the numbers $n^{\frac{1}{2}}\omega$ ($1 \leqslant n < X$) approximately congruent to 0 (mod 1) and therefore $S_X(\omega)$ approximately equal to $S_X(0)$; and (3) then follows easily from (7) since $S_X(0) > CX^{\frac{1}{2}}$ ($C > 0$) for large X*. But in our problem, where Kronecker's theorem is involved, any upper bound $\Phi_\delta(X)$ for ω must depend on the degree of linear independence of the numbers $q^{\frac{1}{2}}$ ($1 \leqslant q < X$, q square-free), and a straightforward application of known theorems† would give a much smaller $\phi(x)$ in (20) than the $\log^{\frac{1}{2}} x$ of (3). Similar remarks apply to the $\Delta(x)$ problem, but it would now be desirable to modify the procedure of theorems D 2 and D 3 in order to preserve the extra logarithmic factor corresponding to the $\log\log x$ in (5).

* Cf. Ingham (3) for details of the corresponding treatment of Littlewood's Ω-theorem.

† For appropriate forms of Kronecker's theorem see Landau (5), Thomas (6). A crude measure of the degree of linear independence of the numbers $q^{\frac{1}{2}}$ in the form

$$\sum_{1\leqslant q<X} l_q q^{\frac{1}{2}} \geqslant \epsilon(X, L) > 0,$$

where the l_q are any rational integers satisfying $|l_q| \leqslant L$, but not all 0, may be deduced from the fact that the product of the numbers $\sum_{1\leqslant q<X} (\pm l_q) q^{\frac{1}{2}}$ (with all possible combinations of the signs $\pm$) is a non-zero rational integer and therefore at least 1 in absolute value.

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