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Dimensions of compact invariant sets of some expanding maps

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Abstract. We study the Hausdorff dimension and measures of full Hausdorff dimension for a compact invariant set of an expanding non-conformal map on the torus given by an integer-valued diagonal matrix. The Hausdorff dimension of a 'general Sierpiński carpet' was found by McMullen and Bedford and the uniqueness of the measure of full Hausdorff dimension in some cases was proved by Kenyon and Peres. We extend these results by using compensation functions to study a general Sierpiński carpet represented by a shift of finite type. We give some conditions under which a general Sierpiński carpet has a unique measure of full Hausdorff dimension and study the properties of the unique measure.

1. *Introduction*

Many natural problems in dynamics are concerned with the following kinds of questions. Let T be a continuous expanding map on a Riemannian manifold M, and K be a T-invariant compact subset. What is the Hausdorff dimension of K? Is there an ergodic measure of full Hausdorff dimension? If so, is it unique? What are the dynamical properties of any measure(s) of full Hausdorff dimension? A useful summary of this area may be found in Gatzouras and Peres [**5**], where these questions are explicitly formulated.

The simplest examples are the middle-third Cantor set and the Sierpiński carpet. These are compact invariant sets for the conformal maps $x \mapsto 3x \bmod 1$ and $(x, y) \mapsto (3x \bmod 1, 3y \bmod 1)$, respectively (maps that dilate all directions uniformly). The questions above are well understood in the conformal setting (see Gatzouras and Peres [**5**] for an overview and history). For the non-conformal setting, only a few cases are understood (see [**1, 5, 9, 12**]). In this paper we study the non-conformal case and, in particular, consider non-conformal expanding maps of the torus given by $T(x, y) = (lx \bmod 1, my \bmod 1)$, $l > m \geq 2$, $l, m \in \mathbb{N}$. Bedford [**1**] and McMullen [**12**] independently answered the question on Hausdorff dimension for compact T-invariant subsets of the following form. Consider a subset of the torus constructed in the

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following way. At the first step, draw $l-1$ vertical lines and $m-1$ horizontal lines in the unit square to get lm congruent rectangles. Shade some of the rectangles in the square and erase the parts that are not shaded. At the second step, in each shaded rectangle draw again $l-1$ vertical lines and $m-1$ horizontal lines to get lm congruent rectangles, and shade again the corresponding (smaller) rectangles as in the first step. Erase parts that are not shaded at this step. Repeating this process, we get a compact T-invariant set which we call an *NC carpet* (for non-conformal). Suppose that R is the set of rectangles chosen to be shaded at the first step. McMullen [**12**] showed, by finding a Bernoulli measure of full Hausdorff dimension, that the Hausdorff dimension of the set is given by

$$\log_m \left(\sum_{j=0}^{m-1} t_j^{\log_l m} \right), \tag{1}$$

where t_j is the number of members of R in the jth row in the unit square. Kenyon and Peres [**9**] showed that for an NC carpet there is a unique measure of full Hausdorff dimension. Using a coding map constructed by a Markov partition for T, one obtains a symbolic representation of the carpet which is a full shift on finitely many symbols.

In this paper, we consider a more general set whose symbolic representation is a shift of finite type rather than a full shift. We call such a set an *SFT-NC carpet*. We will give partial answers to the questions above in this setting. Our results give (under some technical conditions) a formula for the Hausdorff dimension as well as uniqueness and the Bernoulli property for the measure of full Hausdorff dimension. These positive results narrow the possibilities where an example of non-uniqueness of the measure of full Hausdorff dimension might be found (see §7).

Using the Ledrappier–Young formula and a Markov partition, Gatzouras and Peres [**5**] translated the problem into one concerning factor maps $\pi : (X, \sigma_X) \to (Y, \sigma_Y)$ between symbolic dynamical systems: one seeks measures that maximize, for a fixed $\alpha > 0$ determined by the Lyapunov exponents, the weighted entropy functional $h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)$. Shin [**18**] showed that if there is a *saturated compensation function* $G \circ \pi$ (see the paragraph below Theorem 2.3 for the definition of the factor map π), then for any $\alpha > 0$ the set of all shift-invariant measures μ on X that maximize $h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)$ is the set of equilibrium states for the function $(\alpha/(\alpha+1))G \circ \pi$. A saturated compensation function helps us make some progress on the problem, giving us a systematic way to approach the problem. We will answer questions about SFT-NC carpets for which a saturated compensation function exists (when they are represented in symbolic dynamics, see p. 286). Besides the results of Gatzouras and Peres and of Shin, key ingredients of our approach are the work on grid functions by Hofbauer [**7**] and Markley and Paul [**10**], and on functions in the Bowen class by Walters [**24**] and on g-measures by Coelho and Quas [**4**].

Let X, Y be topologically mixing shifts of finite type, and let $\pi : X \to Y$ be a factor map, which as usual we may assume to be a 1-block map (see [**11**]). We begin by considering, in §3, the case where the alphabet of Y is $\{1, 2\}$. Unless Y is trivial, there is a singleton clump (some symbol in the alphabet for Y has only one preimage in the alphabet for X), so we assume that $\pi^{-1}\{1\} = \{1\}$. In Theorem 3.1 we find a saturated compensation function which is not necessarily in the Bowen class. In Theorem 4.1 we find many situations in which there is a saturated compensation function in the Bowen class.

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Examples are provided in §5. In the first example, a saturated compensation function is not in the Bowen class but is a *grid function* in the sense of Markley and Paul [**10**]. We show that the measure of full dimension is Bernoulli, using a theorem of Coelho and Quas (Theorem 2.2).

In §6, we extend the results of §§3 and 4 to the case where the alphabet of Y is $\{1, \ldots, n\}$, $n \geq 2$. The approach we use to establish uniqueness of the equilibrium state is again based on determining when a saturated compensation function is a grid function or is in the Bowen class. Let B be the transition matrix of $\pi^{-1}\{2, 3, \ldots, k\}$. We construct a saturated compensation function and show the uniqueness of the equilibrium state for it under some conditions (see Theorem 6.1 for the case $h_{\text{top}}(X_B) = 0$ and Theorem 6.2 for the case $h_{\text{top}}(X_B) > 0$ that make a compensation function have the form of a grid function). This approach is generalized in Proposition 6.4 to find more saturated compensation functions that have unique equilibrium states. Then we study some particular situations that are not covered by the results mentioned so far, namely the case where the matrix B has a certain block form (Theorem 6.6) or a triangular form (Theorem 6.8). Theorem 6.6 treats the case where the graph of Y has no arrow among symbols other than 1. We see in Theorem 6.8 that if B is reducible and all the irreducible components are topologically mixing (with some additional conditions), then there is a saturated compensation function which is in the Bowen class.

2. *Background*

We review briefly the setup and previous results that we will need. For notation or terminology not explained here, see [**11, 14, 15, 22**]. Write $\Sigma_n^+ = \{1, 2, \ldots, n\}^{\mathbb{N}}$ and $\Sigma_n = \{1, 2, \ldots, n\}^{\mathbb{Z}}$. We use standard terminology for symbolic dynamics from [**11**]; in particular, $[x_0, \ldots, x_{n-1}]$ denotes the cylinder set $\{w : w_0 = x_0, \ldots, w_{n-1} = x_{n-1}\}$. If X is a compact metric space and $T : X \to X$ is a homeomorphism, let $M(X, T)$ denote the space of all T-invariant Borel probability measures on X. For each $\mu \in M(X, T)$, $h_\mu(T)$ denotes the measure-theoretic *entropy* of T with respect to μ, and let $h_{\text{top}}(X)$ be the topological entropy of T. For $f \in C(X)$ and $n \geq 1$, let

$$(S_n f)(x) = \sum_{k=0}^{n-1} f(T^k(x)).$$

The *topological pressure* P of T is given by

$$P(T, f) = \sup\left\{ h_\mu(T) + \int f \, d\mu \;\middle|\; \mu \in M(X, T) \right\},$$

and $\mu \in M(X, T)$ is an *equilibrium state* for f if $P(T, f) = h_\mu(T) + \int f \, d\mu$. If T is clear from the context, we write $P_X(f)$. Conditions under which a potential function f has a unique equilibrium state μ and the properties of the unique equilibrium states are considered throughout this paper.

Let (X, σ) be a one-sided topologically mixing shift of finite type. A measure $\mu \in M(X, \sigma)$ is a (*Bowen*) *Gibbs measure* corresponding to $f \in C(X)$ if there are constants $C_1, C_2 > 0$ and $P > 0$ such that

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$$C_1 \le \frac{\mu([x_0 x_1 \dots x_{n-1}])}{\exp(-Pn + (S_n f)(x))} \le C_2$$

for every $x \in X$ and $n \ge 1$.

For each $f \in C(X)$, the *Perron–Frobenius operator*, $L_f : C(X) \to C(X)$, is defined by $(L_f \phi)(x) = \sum_{y \in \sigma^{-1}(\{x\})} e^{f(y)} \phi(y)$.

We say that $f \in C(X)$ satisfies the RPF *condition* if there exist $\lambda > 0$, $h \in C(X)$ with $h > 0$, and $\nu \in M(X, \sigma)$ for which $L_f h = \lambda h$, $L_f^* \nu = \lambda \nu$, $\int h \, d\nu = 1$, and

$$\lim_{m \to \infty} \left\| \lambda^{-m} L_f^m \phi - h \int \phi \, d\nu \right\|_\infty = 0$$

for all $\phi \in C(X)$. It is well known that if f satisfies the RPF condition then $\mu = \nu h$ is the unique equilibrium state for f (see [**7**]).

Walters [**24**] introduced a class $\mathrm{Bow}(X, \sigma)$ of functions that contains the functions with summable variation, all of which have unique equilibrium states. Let

$$\mathrm{var}_n(f) = \sup\{|f(x) - f(y)| : x, y \in X, x_i = y_i \text{ for all } 0 \le i \le n-1\}.$$

Then $\mathrm{Bow}(X, \sigma) = \{f \in C(X) \mid \sup_{n \ge 1} \mathrm{var}_n(S_n f) < \infty\}$.

THEOREM 2.1. [**24**] *If $f \in \mathrm{Bow}(X, \sigma)$, then f has a unique equilibrium state μ, and the natural extension of (X, σ, μ) is measure-theoretically isomorphic to a Bernoulli system.*

THEOREM 2.2. [**4**] *Suppose that g is a continuous function from Σ_n^+ to $(0, 1)$ and $\sum_{i=1}^n g(ix) = 1$ for all $x \in \Sigma_n^+$. For each $i = 1, 2, \dots$, let $a_i = (n/2) \mathrm{var}_i(g)$. Suppose that there exists $r \ge 1$ such that*

$$\sum_{k=r}^{\infty} \prod_{i=r}^{k} (1 - a_i) = \infty.$$

Then there is a unique g-measure μ_g corresponding to g, and the natural extension of $(\Sigma_n^+, \sigma, \mu_g)$ is isomorphic to a Bernoulli system.

The definition of g-measure can be found in the paper of Coelho and Quas [**4**]. Theorem 2.2 is also valid for the case of shifts of finite type.

Based on the potential functions Hofbauer [**7**] considered, Markley and Paul [**10**] extended his ideas, introducing *grid functions*. Let (X, σ) be a subshift of (Σ_n, σ). Consider a partition $\mathcal{P} = \{\rho(X), M_0, M_1, \dots\}$ of Σ_n^+ satisfying the following conditions:

(a) $\rho(X) = \{x \in \Sigma_n^+ \mid \text{there exists } w \in X \text{ with } x = w_0 w_1 w_2 \dots\}$;

(b) each M_n is closed and open for $n = 0, 1, 2, \dots$;

(c) $\lim_{n \to \infty} M_n = \rho(X)$, where the limit is taken with respect to the Hausdorff metric on the closed subsets of Σ_n^+;

(d) there exists $K_0 \in \mathbb{N}$ such that if B is any cylinder set whose length is larger than K_0 and such that $B \cap \rho(X) = \emptyset$, then there exists $j(B)$ such that $B \subset M_{j(B)}$;

(e) for each positive integer s, there is a length $l_0(s) > K_0$ such that $j(B) \ge s$ whenever $B = B' b_l = b_0 b_1 \dots b_{l-1} b_l$ is a block with $l \ge l_0(s)$, $B \cap \rho(X) = \emptyset$ and $B' \cap \rho(X) \ne \emptyset$.

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Then

$$\mathcal{G}(X, \mathcal{P}) = \left\{ g = \sum_{n=0}^{\infty} a_n \chi_{M_n} \;\middle|\; \lim_{n\to\infty} a_n = 0 \right\} \subset C(\Sigma_n^+)$$

is called the set of *grid functions* associated with (X, σ) and the partition $\mathcal{P}$.

THEOREM 2.3. [**10**] *For $g \in \mathcal{G}(X, \mathcal{P})$, g satisfies the RPF condition if and only if $P(g) > h_{\text{top}}(X)$. If $P(g) > h_{\text{top}}(X)$, then the unique equilibrium state for g is νh, where ν and h are found by the RPF condition. The function h is constant on any cylinder set C whose length is greater than K_0 and such that $C \cap \rho(X) = \emptyset$.*

If (X, σ) is a one-sided topologically mixing shift of finite type and $f : X \to \mathbb{R}$ is a continuous function, then L_f^* has an eigenmeasure corresponding to a positive eigenvalue. Hence Theorem 2.3 remains valid in the case of topologically mixing shifts of finite type.

Let $S : X \to X$, $T : Y \to Y$ be continuous maps on compact metric spaces. A map $\pi : (X, S) \to (Y, T)$ is called a *factor map* if it is a continuous surjection with $\pi \circ S = T \circ \pi$. Denote by $P_X(S, \cdot)$ and $P_Y(T, \cdot)$ the topological pressure functionals of S and T, respectively. Compensation functions were introduced by Boyle and Tuncel [**3**] and studied by Walters [**23**] in connection with a relative pressure. A function $F \in C(X)$ is a *compensation function* for (S, T, π) if

$$P_X(S, F + \phi \circ \pi) = P_Y(T, \phi) \quad \text{for all } \phi \in C(Y).$$

If $F = G \circ \pi \in C(X)$ with $G \in C(Y)$, then $G \circ \pi$ is a *saturated compensation function*.

THEOREM 2.4. [**23**] *Let (X, σ_X), (Y, σ_Y) be subshifts, and let π be a factor map. For each $n \geq 1$ and $y \in Y$, let $D_n(y)$ consist of one point from each non-empty set $\pi^{-1}(y) \cap [i_0 i_1 \ldots i_{n-1}]$. For $G \in C(Y)$, $G \circ \pi$ is a compensation function for π if and only if*

$$\int_Y \limsup_{n\to\infty} \frac{1}{n} [\log(e^{(S_n G)(y)} \cdot |D_n(y)|)] \, d\nu = 0$$

for all $\nu \in M(Y, \sigma_Y)$.

The following theorems have key roles in helping to solve the problems under consideration.

THEOREM 2.5. (Special case of the Ledrappier–Young formula [**9**]) *Let $2 \leq m < l$, m, $l \in \mathbb{N}$. Let S be defined on the 2-torus X by $S(x, y) = (lx \bmod 1, my \bmod 1)$. Let μ be an ergodic S-invariant measure on X. Let π be the projection of X onto the y-axis, let T be defined on Y by $Ty = my \bmod 1$, and let $\pi\mu = \mu\pi^{-1}$. Then*

$$\dim_H \mu = \frac{1}{\log l} h_\mu(S) + \left(\frac{1}{\log m} - \frac{1}{\log l} \right) h_{\pi\mu}(T).$$

THEOREM 2.6. [**18**] *Let (X, σ_X), (Y, σ_Y) be subshifts and let π be a factor map. Suppose that π has a saturated compensation function $G \circ \pi$, $G \in C(Y)$. For $\alpha \geq 0$, the set of all shift-invariant measures μ on X which maximize*

$$\phi_\alpha(\mu) = h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)$$

is the set of equilibrium states of $(\alpha/(\alpha + 1))G \circ \pi \in C(X)$. In particular, it contains an ergodic measure.

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We now give the definitions of NC carpets and SFT-NC carpets. Fix two positive integers l and m, $l > m \geq 2$. Let T be the endomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by $T(x, y) = (lx \bmod 1, my \bmod 1)$. Let

$$\mathcal{P} = \left\{ \left[\frac{i}{l}, \frac{i+1}{l} \right] \times \left[\frac{j}{m}, \frac{j+1}{m} \right] \,\middle|\, 0 \leq i \leq l-1, 0 \leq j \leq m-1 \right\}$$

be the natural Markov partition for T. Label $[i/l, (i+1)/l] \times [j/m, (j+1)/m]$, $0 \leq i \leq l-1$, $0 \leq j \leq m-1$, by the symbol (i, j). Define (Σ_{lm}^+, σ) to be the full shift on these lm symbols. Consider the coding map $\chi : \Sigma_{lm}^+ \to T^2$, defined by

$$\chi(\{(x_k, y_k)\}_{k=1}^{\infty}) = \left(\sum_{k=1}^{\infty} \frac{x_k}{l^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right).$$

Let $R = \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$ be a subalphabet of the labels of $\mathcal{P}$. The *NC carpet* $K(T, R)$ is defined by

$$K(T, R) = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{l^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) \,\middle|\, (x_k, y_k) \in R \text{ for all } k \right\}.$$

It is a compact T-invariant subset of the torus. Denote by A a transition matrix among the members of R, so that A is an $r \times r$ matrix with entries 0 or 1. The *SFT-NC carpet* $K(T, R, A)$ is defined by

$$K(T, R, A) = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{l^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) \,\middle|\, (x_k, y_k) \in R,\ A_{(x_k, y_k)(x_{k+1}, y_{k+1})} = 1 \text{ for all } k \right\}.$$

Now let (X, σ) be the subshift on symbols of the members of R with the transition matrix A as above. Let π be the projection map to the y-axis. Let $Y = \pi(X)$. Using Theorems 2.5 and 2.6, we get the following corollary.

COROLLARY 2.7. *Suppose that there is a saturated compensation function $G \circ \pi \in C(X)$ with $G \in C(Y)$ for $\pi : X \to Y$. Then the Hausdorff dimension of the SFT-NC carpet $K(T, R, A)$ is given by*

$$\dim_H K(T, R, A) = \frac{P((\alpha/(\alpha+1))G \circ \pi)}{\log m}.$$

Suppose in addition that $(\alpha/(\alpha+1))G \circ \pi$ satisfies the RPF condition. Then the Hausdorff dimension of the carpet is given by $\log_m \lambda$, where λ is found by the RPF condition.

Proof. Let μ be an ergodic T-invariant measure on $K(T, R, A)$. Using the natural coding map, there is a measure $\bar{\mu}$ on X mapped to μ. Using the Ledrappier–Young formula and the fact that coding map is bounded to one (see [**13**]),

$$\begin{aligned} \dim_H \mu &= \frac{1}{\log l} h_\mu(T) + \left(\frac{1}{\log m} - \frac{1}{\log l} \right) h_{\pi\mu}(y \to my) \\ &= \frac{1}{\log l} [h_{\bar{\mu}}(\sigma_X) + (\log_m l - 1) h_{\pi\bar{\mu}}(\sigma_Y)]. \end{aligned}$$

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By the proof of Theorem 2.6 [**18**], for $\alpha > 0$,

$$P_X\left(\frac{\alpha}{\alpha+1}G\circ\pi\right) = \frac{1}{\alpha+1}\sup_{\mu\in M(X,\sigma_X)}\{h_\mu(\sigma_X)+\alpha h_{\pi\mu}(\sigma_Y)\}. \tag{2}$$

Letting $\alpha = \log_m l - 1$ and using a theorem of Gatzouras and Peres [**6**], we get

$$\begin{aligned}\dim_H K(T,R,A) &= \sup\{\dim_{\mathrm{H}}\mu \mid \mu(K(T,R,A))=1,\ \mu \text{ is } T\text{-invariant and ergodic}\}\\ &= \frac{1}{\log l}\sup_{\mu\in M(X,\sigma_X)}\{h_\mu(\sigma_X)+(\log_m l-1)h_{\pi\mu}(\sigma_Y)\}\\ &= \frac{1}{\log l}(\alpha+1)P\left(\frac{\alpha}{\alpha+1}G\circ\pi\right) = \frac{P((\alpha/(\alpha+1))G\circ\pi)}{\log m}.\end{aligned}$$

For the second part, we show that $P((\alpha/(\alpha+1))G\circ\pi) = \log\lambda$. Let $\varphi = (\alpha/(\alpha+1))G\circ\pi$. Let $\lambda > 0$, $h \in C(X)$ be obtained by the RPF condition. Let

$$\bar{\varphi} = (\alpha/(\alpha+1))G\circ\pi + \log h - \log h\circ\sigma - \log\lambda.$$

Then μ is the g-measure for $g = e^{\bar{\varphi}}$ [**7**], and so the same arguments as in [**21**, proof of Corollary 3.3(i)] give us the result. □

Specializing to NC carpets, we have an alternative, dynamical proof of the formula given by McMullen [**12**]: the Hausdorff dimension for each NC carpet $K(T,R)$ is given by formula (1).

3. *Main result— part 1*

To formulate and prove our main results, Theorem 3.1 and Theorem 4.1, we will employ the following setting (Setting (A)) and convention, and we will need the following additional hypothesis [C].

Setting (A). Fix $r = 3, 4, \ldots$. Let $X \subset \{1, 2, \ldots, r\}^{\mathbb{N}}$ be a topologically mixing shift of finite type with positive entropy, $Y = \{1,2\}^{\mathbb{N}}$ or $Y \subset \{1,2\}^{\mathbb{N}}$ a shift of finite type with positive entropy, and $\pi : X \to Y$ a one-block factor map such that $\pi^{-1}\{1\} = \{1\}$. Let A be the transition matrix of X and let B be the $(r-1)\times(r-1)$ submatrix of A corresponding to the indices $2, 3, \ldots, r$ (giving the transitions among the symbols in $\pi^{-1}\{2\}$). Denote by X_B the shift of finite type determined by B. Let $0 \le \tau < 1$.

Convention. For a block $y_0 \ldots y_{k-1}$ of length k on $\{1,2\}^{\mathbb{N}}$, we define $|\pi^{-1}[y_0 \ldots y_{k-1}]|$ to be the number of blocks of length k in X mapped to $y_0 \ldots y_{k-1}$ under π. We define $|\pi^{-1}[12^n1]| = 1$ if $\pi^{-1}[12^n1] = \emptyset$.

Condition [C]. Let $\overline{M}_n = \pi^{-1}[2^n1]$ for $n \ge 1$ and $\overline{Z} = \pi^{-1}\{2^\infty\}$. We make the following assumptions:

(1) $\overline{Z}$ is a one-sided shift of finite type such that if T is the transition matrix of $\overline{Z}$, Σ_T is the two-sided shift of finite type on $\{2, \ldots, r\}^{\mathbb{Z}}$ determined by T, and $\Sigma_T|_+$ is the projection of Σ_T to a one-sided subshift, then $\Sigma_T|_+ = \overline{Z}$;

(2) $\lim_{n\to\infty}\overline{M}_n = \overline{Z}$, where the limit is taken in the Hausdorff metric.

THEOREM 3.1. *Let Setting (A) hold, and suppose that* $h_{\mathrm{top}}(X_B) = 0$.

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(1) *Suppose that there exists $n \geq 1$ such that $B^n = 0$. Then there is a compensation function which is locally constant.*

(2) *Suppose for every n that $B^n \neq 0$ and that the following two conditions hold:*

(i)
$$\lim_{n\to\infty} \frac{|\pi^{-1}[12^{n-1}1]|}{|\pi^{-1}[12^n 1]|} = 1;$$

(ii)
$$h_{\mathrm{top}}(X_B) = \lim_{n\to\infty} \frac{\log |\pi^{-1}[2^n]|}{n} = 0.$$

Then:

(a) *there exists a compensation function $G \circ \pi \in C(X)$ such that $G \in C(Y)$;*

(b) *$\tau G \circ \pi$ has a unique equilibrium state μ;*

(c) *under Condition [C], (σ, μ) is exact, hence strongly mixing;*

(d) *the unique equilibrium state μ is Gibbs if and only if $\sup_n |\pi^{-1}[12^n 1]| < \infty$;*

(e) *if the unique equilibrium state μ is Gibbs, then the natural extension of (σ, μ) is isomorphic to a Bernoulli system.*

Remark 3.2. The hypothesis that X be topologically mixing is not necessary for (2)(a).

3.1. *Proof of (2)(a).* Define $G : Y \to \mathbb{R}$ by

$$G(y) = \begin{cases} \log(|\pi^{-1}[12^{k-1}1]|/|\pi^{-1}[12^k 1]|) & \text{if } y \in [2^k 1],\ k \geq 2, \\ \log(1/|\pi^{-1}[121]|) & \text{if } y \in [21], \\ 0 & \text{if } y \in [1] \text{ or } y = 2^\infty. \end{cases} \tag{3}$$

We show that $G \circ \pi$ is a compensation function for π by showing that

$$\int_Y \limsup_{n\to\infty} \frac{1}{n} [\log(e^{(SnG)(y)} |D_n(y)|)]\, d\nu = 0 \quad \text{for all } \nu \in M(Y, \sigma_Y) \tag{4}$$

(see Therorem 2.4 for the definition of D_n). For $i = 1, 2$, let

$$E_i = \{y \in Y \mid y = y_0 \dots y_{p-1} i^\infty \text{ for some } p \geq 1,\ y \neq i^\infty\}.$$

Then for any $\nu \in M(Y, \sigma_Y)$, $\nu(E_1) = \nu(E_2) = 0$ because ν is a σ_Y-invariant measure. It is enough to show that

$$\limsup_{n\to\infty} \frac{1}{n} [\log(e^{(SnG)(y)} |D_n(y)|)] = 0 \quad \text{for all } y \in Y \setminus (E_1 \cup E_2). \tag{5}$$

3.1.1. *Case 1(a).* We consider the case where $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$. Let $n > 2$ be fixed and consider the first n states $y_0 y_1 \dots y_{n-1}$ of $y \in Y$. Let $y_0 = 2$ and $y_{n-1} = 1$. Fix such a $y \in Y$. Then there exist $k_1, k_1', \dots, k_l, k_l' \geq 1$ (where k_i for $2 \leq i \leq l$, k_i' for $1 \leq i \leq l$, and l depend on y and n, and k_1 depends on y) such that $k_1 + k_1' + \dots + k_l + k_l' = n$ and $y = 2^{k_1} 1^{k_1'} \dots 2^{k_l} 1^{k_l'} \dots$. Then

$$|D_n(y)| = |\pi^{-1}[2^{k_1}1]||\pi^{-1}[12^{k_2}1]| \dots |\pi^{-1}[12^{k_l}1]|$$

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and

$$(S_nG)(y) = \log \frac{1}{|\pi^{-1}[12^{k_1}1]||\pi^{-1}[12^{k_2}1]|\dots|\pi^{-1}[12^{k_l}1]|}.$$

Therefore,

$$|D_n(y)|e^{(S_nG)(y)} = \frac{|\pi^{-1}[2^{k_1}1]|}{|\pi^{-1}[12^{k_1}1]|},$$

which is a constant depending on y. Hence (5) holds for such y.

3.1.2. *Case 1(b).* Let n be fixed as for case 1(a). Consider $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$ and $y_0 = y_{n-1} = 2$. Fix $y \in Y$. Then there exist $k_1, k_1', \dots, k_{l-1}, k_{l-1}', k_l \geq 1$ (where k_i for $1 \leq i \leq l$, k_i' for $1 \leq i \leq l-1$, and l depend on y and n), and there exists $t \geq 0$ (where t depends on y and n), such that $k_1 + k_1' + \dots + k_{l-1} + k_{l-1}' + k_l = n$ and $y = 2^{k_1}1^{k_1'}\dots 2^{k_{l-1}}1^{k_{l-1}'}2^{k_l}2^t1\dots$. Depending on the values of k_l and t associated with y and n, n falls into one of four subsequences of $\mathbb{N}$ (see below). We will show that the estimate for the limsup holds along each of these four subsequences of ns, and hence holds along the full sequence.

Now

$$|D_n(y)| \leq |\pi^{-1}[2^{k_1}1]||\pi^{-1}[12^{k_2}1]|\dots|\pi^{-1}[12^{k_{l-1}}1]||\pi^{-1}[12^{k_l}]|$$

and

$$(S_nG)(y) = \log \frac{|\pi^{-1}[12^t1]|}{|\pi^{-1}[12^{k_1}1]||\pi^{-1}[12^{k_2}1]|\dots|\pi^{-1}[12^{k_{l-1}}1]||\pi^{-1}[12^{k_l+t}1]|}. \tag{6}$$

If we let $C(y) = |\pi^{-1}[2^{k_1}1]|/|\pi^{-1}[12^{k_1}1]|$, then

$$|D_n(y)|e^{(SnG)(y)} \leq C(y)\frac{|\pi^{-1}[12^t1]||\pi^{-1}[12^{k_l}]|}{|\pi^{-1}[12^{k_l+t}1]|}. \tag{7}$$

Now fix $\epsilon > 0$. Then by (2)(i), there exists $N \in \mathbb{N}$ such that

$$(1-\epsilon)|\pi^{-1}[12^{t+1}1]| \leq |\pi^{-1}[12^t1]| \leq (1+\epsilon)|\pi^{-1}[12^{t+1}1]|$$

for all $t \geq N$. Therefore, in the case where $t = t(y,n) \geq N$, $k_l = k_l(y,n) \geq 1$,

$$|\pi^{-1}[12^t1]| \leq (1+\epsilon)^{k_l}|\pi^{-1}[12^{t+k_l}1]|,$$

and so

$$\frac{|\pi^{-1}[12^t1]|}{|\pi^{-1}[12^{t+k_l}1]|} \leq (1+\epsilon)^{k_l} \leq (1+\epsilon)^n. \tag{8}$$

For the same fixed $\epsilon > 0$, by (2)(ii), there exists $\overline{N} \in \mathbb{N}$ such that

$$|\pi^{-1}[2^k]| \leq e^{k\epsilon} \quad \text{for all } k \geq \overline{N}. \tag{9}$$

For $1 \leq k \leq \overline{N}-1$, if we let $\beta = \max_{1\leq k\leq \overline{N}-1}(\log|\pi^{-1}[2^k]|)/k$, then

$$|\pi^{-1}[2^k]| \leq e^{\beta\overline{N}} \quad \text{for all } 1 \leq k \leq \overline{N}-1. \tag{10}$$

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Now consider n such that $t = t(y, n) \geq N$ and $k_l = k_l(y, n) \geq \overline{N}$. By (8), (9), $|\pi^{-1}[12^{k_l}]| \leq |\pi^{-1}[2^{k_l}]|$, and $k_l \leq n$, we have for such n that

$$|D_n(y)|e^{(SnG)(y)} \leq C(y)(1+\epsilon)^n |\pi^{-1}[12^{k_l}]| \leq C(y)(1+\epsilon)^n e^{n\epsilon}.$$

Taking the increasing subsequence $\{n_i\}_{i=1}^{\infty}$ of such n,

$$\begin{aligned}\limsup_{i\to\infty} \frac{1}{n_i} \log(e^{(Sn_iG)(y)}|D_{n_i}(y)|) &\leq \limsup_{i\to\infty} \frac{n_i \log(1+\epsilon) + n_i\epsilon}{n_i} \\ &= \log(1+\epsilon) + \epsilon < 2\epsilon. \end{aligned} \tag{11}$$

Now consider n such that $t = t(y, n) \geq N$ and $1 \leq k_l = k_l(y, n) < \overline{N}$. By (8), (10), $|\pi^{-1}[12^{k_l}]| \leq |\pi^{-1}[2^{k_l}]|$, and $k_l < \overline{N}$, we have for such n that

$$|D_n(y)|e^{(SnG)(y)} \leq C(y)(1+\epsilon)^n |\pi^{-1}[12^{k_l}]| \leq C(y)(1+\epsilon)^n e^{\beta\overline{N}}.$$

Taking the increasing sequence $\{n_i\}_{i=1}^{\infty}$ of such n,

$$\begin{aligned}\limsup_{i\to\infty} \frac{1}{n_i} \log(e^{(Sn_iG)(y)}|D_{n_i}(y)|) &\leq \limsup_{i\to\infty} \frac{n_i \log(1+\epsilon)}{n_i} \\ &= \log(1+\epsilon) < \epsilon. \end{aligned} \tag{12}$$

Now consider n such that $0 \leq t = t(y, n) < N$ and $k_l = k_l(y, n) \geq \overline{N}$. Let $M = \max_{0\leq t\leq N-1} |\pi^{-1}[12^t 1]|$. Then by using (9) and $1 \leq |\pi^{-1}[12^{t+k_l}1]|$, referring to (7), we get for such n that

$$|D_n(y)|e^{(SnG)(y)} \leq C(y)M|\pi^{-1}[2^{k_l}]| \leq C(y)Me^{n\epsilon}.$$

Thus taking the increasing subsequence $\{n_i\}_{i=1}^{\infty}$ of such n,

$$\limsup_{i\to\infty} \frac{1}{n_i} \log(e^{(Sn_iG)(y)}|D_{n_i}(y)|) \leq \epsilon. \tag{13}$$

Consider n such that $0 \leq t = t(y, n) < N$ and $1 \leq k_l = k_l(y, n) < \overline{N}$. By (10), for such n,

$$|D_n(y)|e^{(SnG)(y)} \leq C(y)M|\pi^{-1}[2^{k_l}]| \leq C(y)Me^{\beta\overline{N}}.$$

Thus for the increasing subsequence $\{n_i\}_{i=1}^{\infty}$ of such n,

$$\limsup_{i\to\infty} \frac{1}{n_i} \log(e^{(Sn_iG)(y)}|D_{n_i}(y)|) \leq 0. \tag{14}$$

By (11) through (14), for $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$ and $y_0 = y_{n-1} = 2$,

$$\limsup_{n\to\infty} \frac{1}{n}[\log(e^{(SnG)(y)}|D_n(y)|)] < 2\epsilon.$$

By cases 1(a) and 1(b), for $y_0 = 2$ and $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$, we get

$$\limsup_{n\to\infty} \frac{1}{n}[\log(e^{(SnG)(y)}|D_n(y)|)] \leq 0.$$

Note that from above in case 1(a) we have that $|D_n(y)|e^{(SnG)(y)} \geq 1$ for y with $y_{n-1} = 1$. By the definition of limsup, for y such that $y_0 = 2$ and $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$, we also have that

$$\limsup_{n\to\infty} \frac{1}{n}[\log(e^{(SnG)(y)}|D_n(y)|)] \geq 0.$$

Therefore, we get (5) for all $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$ and $y_0 = 2$.

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3.1.3. *Case 1(c).* Let $n > 2$ be fixed, as for case 1(a). Consider $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$ and $y_0 = y_{n-1} = 1$. Let $y = 1^k 2^{k_1} 1^{k'_1} \dots 2^{k_l} 1^{k'_l} \dots$ as in case 1(a), where $k + k_1 + k'_1 + \dots + k_l + k'_l = n$. Then we get $|D_n(y)| e^{(SnG)(y)} = 1$.

3.1.4. *Case 1(d).* Let $n > 2$ be fixed, as in case 1(a). Consider $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$ and $y_0 = 1$ and $y_{n-1} = 2$. Let $y = 1^k 2^{k_1} 1^{k'_1} \dots 1^{k'_{l-1}} 2^{k_l} 2^t 1 \dots$, as in case 1(b), where $k + k_1 + k'_1 + \dots + k'_{l-1} + k_l = n$. Then we get

$$|D_n(y)| e^{(SnG)(y)} = \frac{|\pi^{-1}[12^t 1]||\pi^{-1}[12^{k_l}]|}{|\pi^{-1}[12^{k_l+t}1]|} \quad (C(y) \text{ is replaced by } 1).$$

By cases 1(c) and 1(d), we have (5) for for all $y \notin E_1 \cup E_2 \cup \{1^\infty, 2^\infty\}$, and $y_0 = 1$.

3.1.5. *Case 2.* We consider the case where $y \in \{1^\infty, 2^\infty\}$. If $y = 2^\infty$, we use $1 \le |D_n(y)| \le |\pi^{-1}[2^n]|$ and $(S_n G)(y) = 0$ for all n. For $y = 1^\infty$, we use $|D_n(y)| = 1$ and $(S_n G)(y) = 0$.

Therefore, by cases 1(a) through 1(d) and 2, (5) is satisfied for all $y \in Y \setminus (E_1 \cup E_2)$.

3.2. *Proof of (1).* Define G as in (3). Since $B^n = 0$ for some $n \ge 1$, there exists $n_1 > 0$ such that, using our convention, $|\pi^{-1}[12^n 1]| = 1$ for all $n \ge n_1$. Thus G is locally constant and (5) holds for $y \in Y \setminus (E_1 \cup E_2 \cup \{2^\infty\})$.

Remark 3.3. Shin [**19**] gave a sufficient condition for the existence of a saturated compensation function for the case where $X \subset \{1, 2, \dots, r\}^{\mathbb{Z}}$ is a topologically mixing shift of finite type, $Y = \{1, 2\}^{\mathbb{Z}}$ or Y is the golden mean subshift, and $\pi^{-1}\{1\} = \{1\}$. The compensation function is defined on the *two-sided* shift space Y. The construction of a compensation function on *one-sided* subshifts in this paper is slightly different, and our sufficient conditions for the existence of a saturated compensation function are also slightly different. But the approach here to showing the existence of a saturated compensation function by using Theorem 2.4 is the same as the approach in [**19**], and there may be relations between these two compensation functions.

3.3. *Proof of (2)(b).* Let $Z = \{2^\infty\}$. Define $M_0 = [1]$ and $M_n = [2^n 1]$ for all $n \ge 1$. Then $\rho(Z) = \{2^\infty\}$. Consider the partition $\mathcal{P} = \{\rho(Z), M_0, M_1 \dots\}$ of Y. Then, using hypothesis (2)(i), $G \in \mathcal{G}(Z, \mathcal{P})$ (see the paragraph below Theorem 2.2). We will use three theorems from [**16, 19**], and [**23**], and formula (2) in [**18**, proof of Theorem 2.6]), which yield a corollary that helps to prove the uniqueness of the equilibrium state for $(\alpha/(\alpha + 1))G \circ \pi$.

THEOREM 3.4. [**23**] *Suppose that (X, σ_X), (Y, σ_Y) are subshifts and $\pi : (X, \sigma_X) \to (Y, \sigma_Y)$ is a factor map. Suppose $F \in C(X)$ is a compensation function. Let $\mu \in M(X, \sigma_X)$ and $\phi \in C(Y)$. Then μ is an equilibrium state of $F + \phi \circ \pi$ if and only if $\pi\mu$ is an equilibrium state of ϕ and μ is a relative equilibrium state of F over $\pi\mu$.*

THEOREM 3.5. [**16**] *Let (X, σ_X) be an irreducible shift of finite type, (Y, σ_Y) a subshift, and $\pi : (X, \sigma_X) \to (Y, \sigma_Y)$ a factor map. Suppose that there is a symbol a of Y whose*

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inverse image is a singleton, which is also denoted by a. Then every ergodic measure on Y which assigns positive measure to $[a]$ has a unique preimage of maximal relative entropy.

THEOREM 3.6. [**19**] *Let $X \subset \{1, 2, \ldots, r\}^{\mathbb{Z}}$ be a shift of finite type, $Y = \{1, 2\}^{\mathbb{Z}}$ a full shift or $Y \subset \{1, 2\}^{\mathbb{Z}}$ the golden mean shift, and $\pi : X \to Y$ a factor map such that $\pi^{-1}\{1\} = \{1\}$. Suppose $G \in C(Y)$ and $G \circ \pi$ is a compensation function. Then*

$$G(2^\infty) = -\limsup_n((\log |\pi^{-1}[2^n]|)/n).$$

Note that Theorem 3.6 is valid when X is a one-sided shift of finite type.

COROLLARY 3.7. *Fix $r = 3, 4, \ldots$. Let $X \subset \{1, 2, \ldots, r\}^{\mathbb{N}}$ be a topologically mixing shift of finite type, $Y = \{1, 2\}^{\mathbb{N}}$ or $Y \subset \{1, 2\}^{\mathbb{N}}$ a shift of finite type, and $\pi : X \to Y$ a one-block factor map such that $\pi^{-1}\{1\} = \{1\}$. Suppose that there is a saturated compensation function $G \circ \pi$ for π. Then, for any $\alpha > 0$, if $-(1/(\alpha + 1))G$ has a unique equilibrium state, then $(\alpha/(\alpha + 1))G \circ \pi$ has a unique equilibrium state.*

Proof. Let ν be an ergodic equilibrium state for $-(1/(\alpha + 1))G$. Let μ be a preimage of ν with maximal relative entropy. We first show that μ is a relative equilibrium state of $G \circ \pi$ over ν. Using the relative variational principle [**23**],

$$\begin{aligned}\int P(\sigma_X, \pi, G \circ \pi)\, d\nu &= \sup\left\{h_\mu(\sigma_X|_{\sigma_Y}) + \int G \circ \pi\, d\mu \;\middle|\; \mu \in M(X, \sigma_X), \pi\mu = \nu\right\}\\ &= \sup\left\{h_\mu(\sigma_X) - h_{\pi\mu}(\sigma_Y) + \int G \circ \pi\, d\mu \;\middle|\; \mu \in M(X, \sigma_X), \pi\mu = \nu\right\}\\ &= \sup\{h_\mu(\sigma_X) \mid \mu \in M(X, \sigma_X), \pi\mu = \nu\} - h_\nu(\sigma_Y) + \int G\, d\nu.\end{aligned}$$

Therefore, μ is a preimage of maximal entropy if and only if it is a relative equilibrium state of $G \circ \pi$ over ν. By Theorem 3.4, using $\phi = -(1/(\alpha + 1))G$ and $F = G \circ \pi$, μ is an equilibrium state of $(\alpha/(\alpha + 1))G \circ \pi$.

Next we show that $\nu([1]) > 0$. Assume $\nu([1]) = 0$. Since $\nu(2^\infty) = 1$,

$$\begin{aligned}P_Y\left(-\frac{1}{\alpha + 1}G\right) &= h_\nu(\sigma_Y) - \int \frac{1}{\alpha + 1}G\, d\nu = -\frac{1}{\alpha + 1}G(2^\infty)\\ &= \frac{1}{\alpha + 1}h_{\text{top}}(X_B) \quad \text{(by Theorem 3.6).}\end{aligned}$$

Take the Shannon–Parry measure $\mu_{\max}$. Since $X_B \subsetneq X$ and X is topologically mixing, $h_{\text{top}}(X) > h_{\text{top}}(X_B)$. Using formula (2) and the definition of compensation function,

$$\begin{aligned}P_Y\left(-\frac{1}{\alpha + 1}G\right) &= P_X\left(\frac{\alpha}{\alpha + 1}G \circ \pi\right)\\ &\geq \frac{1}{\alpha + 1}h_{\mu_{\max}}(\sigma_X) = \frac{1}{\alpha + 1}h_{\text{top}}(X) > \frac{1}{\alpha + 1}h_{\text{top}}(X_B). \qquad (15)\end{aligned}$$

This is a contradiction. Now ν has a unique preimage of maximal relative entropy by Theorem 3.5.

By hypothesis ν is the unique equilibrium state for $-(1/(\alpha + 1))G$. Suppose now that there are two equilibrium states μ_1, μ_2 for $(\alpha/(\alpha + 1))G \circ \pi$. Then, by the above, they are distinct preimages of maximal relative entropy of ν, and this is a contradiction. □

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We now conclude the proof of (2)(b). By Corollary 3.7, it is enough to show that $-(1/(\alpha+1))G$ has a unique equilibrium state. Since $-(1/(\alpha+1))G \in \mathcal{G}(Z, \mathcal{P})$, using (15) and $h_{\text{top}}(Z) = 0$,

$$P_Y\left(-\frac{1}{\alpha+1}G\right) > h_{\text{top}}(Z).$$

By Theorem 2.3, $-(1/(\alpha+1))G$ has a unique equilibrium state.

Remark 3.8. If $Y = \Sigma_2^+$ and G is as above, then tG has a unique equilibrium state for any $t \in \mathbb{R}$ by [**7**].

3.4. *Proof of (2)(d) and (e).*

LEMMA 3.9. *Suppose that (X, σ) is a one-sided topologically mixing shift of finite type. Let $\varphi \in C(X)$. Then an equilibrium state for φ is Gibbs if and only if $\varphi \in \text{Bow}(X, \sigma)$. In particular, if an equilibrium state for φ is Gibbs, then it is the unique equilibrium state.*

Proof. We follow the arguments in [**7**]. Suppose that there is an equilibrium state μ for φ which is Gibbs. Then there exist λ, C_1, $C_2 > 0$ such that

$$C_1 \le \frac{\mu([x_0x_1 \dots x_{n-1}])}{\lambda^{-n}e^{(S_n\varphi)(x)}} \le C_2$$

for every $x \in X$, $n \in \mathbb{N}$. For x, $x' \in [x_0x_1 \dots x_{n-1}]$, we have

$$C_1 e^{(S_n\varphi)(x)} \le \lambda^n \mu([x_0x_1 \dots x_{n-1}]) \le C_2 e^{(S_n\varphi)(x')}.$$

Taking logarithms, we get

$$|(S_n\varphi)(x) - (S_n\varphi)(x')| \le \log\frac{C_2}{C_1}.$$

Therefore, $\sup_{n\ge1} \text{var}_n(S_n\varphi) \le \log(C_2/C_1)$ and so $\varphi \in \text{Bow}(X, \sigma)$.

Conversely, suppose that $\varphi \in \text{Bow}(X, \sigma)$. Let $\lambda = e^{P_X(\varphi)}$. Then there exists a unique measure $\nu \in M(X, \sigma)$ with $L_\varphi^*\nu = \lambda\nu$, and the unique equilibrium state for φ is $\mu = \nu h$, where $h : X \to [a, b]$ is a measurable function, $0 < a < b$, $L_\varphi h = \lambda h$ and $\int h\, d\nu = 1$ [**24**]. Since the shift of finite type X with transition matrix A is topologically mixing and of positive entropy, there is a number k such that $A^k > 0$. Let r be the number of symbols in the alphabet for X. Then, by [**24**], ν satisfies

$$\frac{1}{\exp(\text{var}_n(S_n\varphi))\lambda^k \exp(k\|\varphi\|)} \le \frac{\nu([x_0x_1 \dots x_{n-1}])}{\lambda^{-n}\exp((S_n\varphi)(x))} \le r^k \exp(\text{var}_n(S_n\varphi))\frac{\exp(k\|\varphi\|)}{\lambda^k}.$$

Since $\varphi \in \text{Bow}(X, \sigma)$, we have $\text{var}_n(S_n\varphi) \le L$ for all n for some L. Therefore ν is Gibbs and so $\mu = \nu h$ is Gibbs. □

LEMMA 3.10. *Under the assumptions of (2)(i) and (ii), $|\pi^{-1}[12^n1]|$ is bounded for all n if and only if the unique equilibrium state of $\varphi = \tau G \circ \pi$ is Gibbs.*

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Proof. Suppose first that $|\pi^{-1}[12^n1]|$ is not bounded. Assume that the unique measure μ is Gibbs. Consider $x \in X$ such that $\pi(x) \in [2^n1]$ and $x' \in X$ such that $\pi(x') = 2^\infty$. Then $\pi(x), \pi(x') \in [2^n]$ and $|(S_n\varphi)(x) - (S_n\varphi)(x')| = \tau \log |\pi^{-1}[12^n1]|$, referring to (6) and the fact that $(S_n\varphi)(x') = 0$ for all n, and $\mathrm{var}_n(S_n\varphi)$ is not bounded. This is a contradiction.

Conversely, suppose that $|\pi^{-1}[12^n1]| \le K$ for all n. We show $\varphi = \tau G \circ \pi \in \mathrm{Bow}(X, \sigma)$, that is, $\sup_{n\ge1} \mathrm{var}_n(S_n\varphi) < \infty$.

Case 1. Consider x and x' such that $\pi(x_i) = \pi(x'_i)$ for all $0 \le i \le n-1$ and $\pi(x_{n-1}) = \pi(x'_{n-1}) = 2$.

(i) Suppose that $\pi(x) = 1^{k_1}2^{l_1}1^{k_2}2^{l_2}\ldots1^{k_t}2^\beta1\ldots$, where $k_1 + l_1 + \cdots + l_{t-1} + k_t = k < n$, $k + \beta \ge n$, $\pi(x_{n-1}) = 2$, and $\pi(x') = 1^{k_1}2^{l_1}1^{k_2}2^{l_2}\ldots1^{k_t}2^\gamma1\ldots$, where $k_1 + l_1 + \cdots + l_{t-1} + k_t = k < n$, $k + \gamma \ge n$, $\pi(x'_{n-1}) = 2$. Direct computation shows that

$$|(S_n\varphi)(x) - (S_n\varphi)(x')| = \tau\left|\log \frac{|\pi^{-1}[12^{\beta-(n-k)}1]||\pi^{-1}[12^\gamma1]|}{|\pi^{-1}[12^\beta1]||\pi^{-1}[12^{\gamma-(n-k)}1]|}\right|. \tag{16}$$

Since $1 \le |\pi^{-1}[12^n1]| \le K$ for all n, clearly, for any β, γ, n, k above,

$$0 \le |(S_n\varphi)(x) - (S_n\varphi)(x')| \le 2\tau \log K. \tag{17}$$

(ii) Suppose that $\pi(x) = 1^{k_1}2^{l_1}1^{k_2}2^{l_2}\ldots1^{k_t}2^\beta1\ldots$, where $k_1 + l_1 + \cdots + l_{t-1} + k_t = k < n$, $k + \beta \ge n$, $\pi(x_{n-1}) = 2$, and $\pi(x') = 1^{k_1}2^{l_1}1^{k_2}2^{l_2}\ldots1^{k_t}2^\infty$. Then

$$|(S_n\varphi)(x) - (S_n\varphi)(x')| = \tau\left|\log \frac{|\pi^{-1}[12^{\beta-(n-k)}1]|}{|\pi^{-1}[12^\beta1]|}\right| \le \tau \log K. \tag{18}$$

Case 2. Consider x and x' such that $\pi(x_i) = \pi(x'_i)$ for all $0 \le i \le n-1$ and $\pi(x_{n-1}) = \pi(x'_{n-1}) = 1$, or x and x' such that $\pi(x_i) = \pi(x'_i)$ for all $i \ge 0$. Then

$$|(S_n\varphi)(x) - (S_n\varphi)(x')| = 0, \tag{19}$$

($\varphi(x_{n-1}\ldots) - \varphi(x'_{n-1}\ldots) = 0$ for $\pi(x_{n-1}) = \pi(x'_{n-1}) = 1$).

By (17), (18), and (19), $\tau G \circ \pi \in \mathrm{Bow}(X, \sigma)$. Now use Lemma 3.9. □

3.5. *Proof of (2)(c).* Note that G is a grid function but $G \circ \pi$ is not necessarily a grid function. Let $\overline{M}_0 = \pi^{-1}[1] = [1]$ and $\overline{M}_n = \pi^{-1}[2^n1]$ for $n \ge 1$. Let $\overline{Z} = \pi^{-1}\{2^\infty\}$. Recall Condition [C]. Let W be the two-sided shift of finite type which is the natural extension of $\overline{Z}$. Then we have $W = \Sigma_T$, and so $\rho(W) = W|_+ = \Sigma_T|_+ = \overline{Z}$ by Condition [C](1). Thus $\mathcal{P} = \{\rho(W), \overline{M}_0, \overline{M}_1, \overline{M}_2, \ldots\}$ is a partition of X for which $G \circ \pi \in \mathcal{G}(W, \mathcal{P})$.

We show first that in this case, for any $0 \le \tau < 1$, $\tau G \circ \pi$ satisfies the RPF condition. Let $\tau = \alpha/(\alpha+1)$, $\alpha > 0$. Define $\varphi = (\alpha/(\alpha+1))G \circ \pi$. Since $\varphi \in \mathcal{G}(W, \mathcal{P})$, it is enough to show that $P_X(\varphi) > h_{\mathrm{top}}(W)$. Note that $h_{\mathrm{top}}(W) = h_{\mathrm{top}}(\overline{Z}) = 0$ because $h_{\mathrm{top}}(X_B) = 0$. By formulas (2) and (15), if we denote the Shannon–Parry measure by $\mu_{\max}$,

$$P_X(\varphi) \ge \frac{1}{\alpha+1} h_{\mu_{\max}}(\sigma_X) = \frac{1}{\alpha+1} h_{\mathrm{top}}(X) > 0.$$

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Hence φ satisfies the RPF condition by Theorem 2.3. Let ν, h, λ be given by the RPF condition. By [**7**], the unique equilibrium state μ for φ is $\mu = \nu h$. If we let

$$\overline{\varphi} = (\alpha/(\alpha+1))G \circ \pi + \log h - \log h \circ \sigma - \log \lambda,$$

then μ is the g-measure for $g = e^{\overline{\varphi}}$ [**7**]. Since $L_{\overline{\varphi}}^n f \to \int f \, d\mu$ uniformly for all $f \in C(X)$ [**7**], by using the same arguments as in [**21**, proof of Theorem 3.2], we conclude that (σ, μ) is an exact endomorphism, hence strongly mixing.

4. *Main result—part 2*

We next consider the case where $h_{\text{top}}(X_B) > 0$, under Setting (A).

THEOREM 4.1. *Let Setting (A) in §3 hold. Suppose that $h_{\text{top}}(X_B) > 0$ and that the following two conditions hold:*

(i)′ *there exists $a > 1$ such that*

$$\lim_{n\to\infty} \frac{|\pi^{-1}[12^{n-1}1]|}{|\pi^{-1}[12^n 1]|} = \frac{1}{a};$$

(ii)′

$$h_{\text{top}}(X_B) = \limsup_{n\to\infty} \frac{\log |\pi^{-1}[2^n]|}{n} = \log a.$$

Then:

(a) *there exists a compensation function $G \circ \pi \in C(X)$ such that $G \in C(Y)$;*

(b) *$\tau G \circ \pi$ has a unique equilibrium state μ;*

(c) *under Condition [C], (σ, μ) is exact, hence strongly mixing;*

(d) *the unique equilibrium state μ is Gibbs if and only if there exist $K_1, K_2 > 0$ such that*

$$K_1 \le \frac{a^n}{|\pi^{-1}[12^n 1]|} \le K_2$$

for all $n \ge 1$;

(e) *if the unique equilibrium state μ is Gibbs, then the natural extension of (σ, μ) is isomorphic to a Bernoulli system.*
In particular, if B is irreducible with $h_{\text{top}}(X_B) > 0$, we have the following: if (i)′ *is satisfied, then* (a), (b), (c) *above hold and*

(f) *the unique equilibrium state μ is Gibbs and the natural extension of (σ, μ) is isomorphic to a Bernoulli system.*

Remarks 4.2.

(1) The hypothesis that X be topologically mixing is not necessary for (a).

(2) If B is primitive, then (i)′ is automatically satisfied.

Proof. We give only an outline of the proof because it is similar to that of Theorem 3.1. Note first that we no longer have $G(2^\infty) = 0$ because $h_{\text{top}}(X_B) > 0$. Define $G : Y \to \mathbb{R}$ as in formula (3), but replacing 0 at 2^∞ by $-\log a$. We modify cases 1(b) and 1(d) of 3.1 to take into account that $a > 1$. We find upper bounds for

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$|\pi^{-1}[12^t1]||\pi^{-1}[12^{k_l}]|/|\pi^{-1}[12^{k_l+t}1]|$ by using (i)′ and (ii)′. For (i)′, fix $\epsilon > 0$ small enough so that $\epsilon + 1/a < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\left(\frac{1}{a} - \epsilon\right)|\pi^{-1}[12^{t+1}1]| \le |\pi^{-1}[12^t1]| \le \left(\frac{1}{a} + \epsilon\right)|\pi^{-1}[12^{t+1}1]|$$

for all $t \ge N$. Therefore, in the case where $t = t(y, n) \ge N$ and $k_l = k_l(y, n) \ge 1$,

$$\frac{|\pi^{-1}[12^t1]|}{|\pi^{-1}[12^{t+k_l}1]|} \le \left(\frac{1}{a} + \epsilon\right)^{k_l}.$$

For (b), by Corollary 3.7, it is enough to prove that $(-1/(1+\alpha))(G + \log a)$ has a unique equilibrium state. Since $(-1/(1+\alpha))(G + \log a) \in \mathcal{G}(Z, \mathcal{P})$, where $Z, \mathcal{P}$ are defined as in §3.3, it is enough to show that

$$P_Y\left(-\frac{1}{\alpha+1}(G + \log a)\right) > h_{\mathrm{top}}(Z)(= 0). \tag{20}$$

Using (2), (15), and $h_{\mathrm{top}}(X) > h_{\mathrm{top}}(X_B)$, we get (20).

For (c), we show $\tau(G + \log a) \circ \pi$ satisfies the RPF condition for $0 \le \tau < 1$, as in §3.5.

For (d), we replace $G(2^\infty) = 0$ by $G(2^\infty) = -\log a$ and repeat the arguments as in the proof of (2)(d) in §3.4.

Conclusion (e) follows by Lemma 3.9.

Next consider the case where B is irreducible. Then (ii)′ is satisfied. (i)′ is satisfied if B is primitive. To see this, let

$$V_1 = \{c \in \{2, 3, \ldots, r\} \mid A_{1c} \ne 0\} \quad \text{and} \quad V_2 = \{d \in \{2, 3, \ldots, r\} \mid A_{d1} \ne 0\}.$$

Since $|\pi^{-1}[12^n1]| = \sum_{c \in V_1, d \in V_2} (B_{c,d})^{n-1}$, apply the Perron–Frobenius theorem to find the growth rate of $(B_{c,d})^{n-1}$. For (f), we use the following consequence of the Perron–Frobenius theorem.

LEMMA 4.3. *Let p be the period of the irreducible matrix B. There exist N, A_1, $A_2 > 0$ such that*

$$A_1 a^{np+l} \le |\pi^{-1}[12^{np+l+1}1]| \le A_2 a^{np+l}$$

for all $l \in \{0, 1, \ldots, p-1\}$, $n \ge N$.

Now we apply Lemmas 3.10 and 3.9. □

5. *Examples for Theorems 3.1 and 4.1*

We apply Theorems 3.1 and 4.1 to study SFT-NC carpets.

Example 5.1. Let T be the toral endomorphism given by $T(x, y) = (3x \bmod 1, 2y \bmod 1)$. Let

$$\mathcal{P} = \left\{ \left[\frac{i}{3}, \frac{i+1}{3}\right] \times \left[\frac{j}{2}, \frac{j+1}{2}\right] \,\middle|\, 0 \le i \le 2, 0 \le j \le 1 \right\}$$

be the natural Markov partition for T. Let $R = \{(1, 0), (0, 1), (2, 1)\}$. Then we get the NC carpet $K(T, R)$ given in Figure 1. Let A be the transition matrix among the members of R

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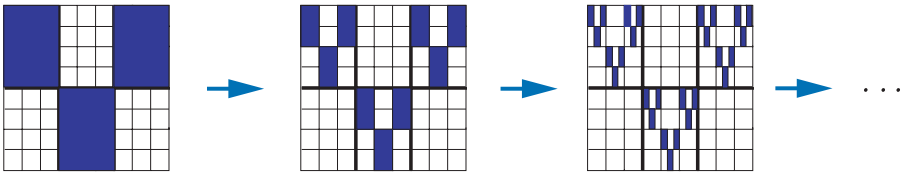


FIGURE 1. NC carpet $K(T, R)$ in Example 5.1.

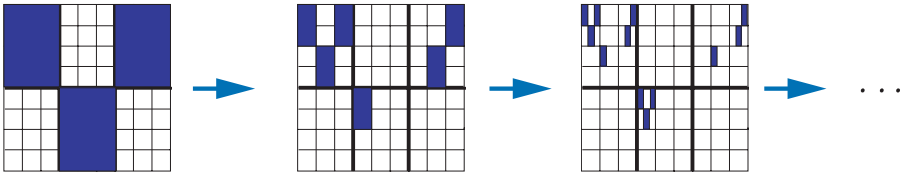


FIGURE 2. SFT-NC carpet $K(T, R, A)$ in Example 5.1.

given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then the SFT-NC carpet $K(T, R, A)$ is defined by

$$K(T, R, A) = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{3^k}, \sum_{k=1}^{\infty} \frac{y_k}{2^k} \right) \,\middle|\, (x_k, y_k) \in R,\ A_{(x_k, y_k)(x_{k+1}, y_{k+1})} = 1 \text{ for all } k \right\}$$

(see Figure 2).

Let $X = \Sigma_A^+$ and let π be the factor map $\pi(x_k, y_k) = y_k$. Set $Y = \pi(X)$. Then Y is a shift of finite type on the two symbols 0, 1 with the transition matrix

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

If we denote (1, 0) by 1, (0, 1) by 2, and (2, 1) by 3, then $\pi(1) = 0$, $\pi(2) = \pi(3) = 1$ (see Figure 3). Let $\alpha = \log_2 3 - 1$. Then the transition matrix B of symbols in $\pi^{-1}\{1\}$ is given by

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Now we apply Theorem 3.1. Since we have $|\pi^{-1}[01^n0]| = n$ and $h_{\text{top}}(X_B) = 0$, Theorem 3.1 (2)(i) and (ii) are satisfied. There is a saturated compensation function $G \circ \pi$, where $G : Y \to \mathbb{R}$ is defined by

$$G(y) = \begin{cases} \log((n-1)/n) & \text{if } y \in [1^n0],\ n \geq 2, \\ 0 & \text{if } y \in [0] \cup [10] \cup \{1^\infty\}. \end{cases}$$

$(\alpha/(\alpha+1))G \circ \pi$ has a unique equilibrium state, $\overline{\mu}$. Therefore, the corresponding measure μ on $K(T, R, A)$ is the unique measure of full Hausdorff dimension. Since

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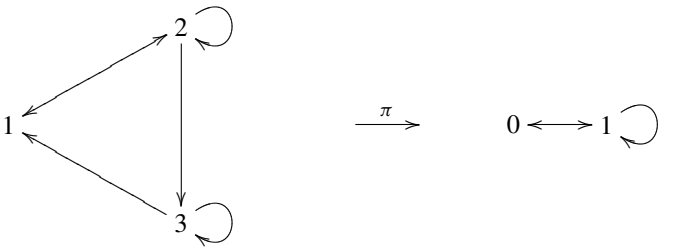


FIGURE 3. X, Y, and π in Example 5.1.

Condition [C] is satisfied, (T, μ) is exact, and hence strongly mixing. The unique measure μ is not Gibbs, because $\sup_n |\pi^{-1}[01^n0]| = \infty$. The RPF condition is satisfied, and by Corollary 2.7,

$$\dim_H K(T, R, A) = \frac{P((\alpha/(\alpha+1))G \circ \pi)}{\log 2} = \log_2 \lambda,$$

where λ is the spectral radius of $L_{(\alpha/(\alpha+1))G\circ\pi}$.

We now approximate $P((\alpha/(\alpha+1))G \circ \pi)$, which gives an approximation of λ. Let $\beta = \alpha/(\alpha+1)$ and $\varphi = \beta G \circ \pi$. Consider a lower bound for

$$\sum_{x_0x_1\ldots x_{3m-1}} \exp\Big(\sup_{y\in[x_0x_1\ldots x_{3m-1}]} (S_{3m}\varphi)(y)\Big).$$

First note that the number of allowable cylinder sets of the form $[x_0x_1 \ldots x_{m-1}]$ in Σ_A^+ of length m is $3 \cdot 2^{m-1}$. We consider the cylinder set $[x_0x_1 \ldots x_{3m-1}]$, where 221, 331 or 231 appears k times in total somewhere in $x_0x_1 \ldots x_{3m-1}$. So there exist $i_1, i_2, \ldots, i_k$ such that $x_{i_l}x_{i_l+1}x_{i_l+2} \in \{221, 331, 231\}$ for $l = 1, 2, \ldots, k$, $1 \le k \le m$. If $y \in [x_0x_1 \ldots x_{3m-1}]$ and if 221, 331, or 231 appears in $x_0x_1 \ldots x_{3m-1}$ a total of k times, then direct computation shows that

$$(S_{3m}\varphi)(y) \ge \beta[k \log(1/2) + (3m - 3k) \log(2/3)].$$

Using this, we have a lower bound

$$\exp\Big(\sup_{y\in[x_0x_1\ldots x_{3m-1}]} (S_{3m}\varphi)(y)\Big) \ge \left(\frac{2}{3}\right)^{3m\beta}.$$

This implies by direct computation that

$$P(\varphi) \ge \lim_{m\to\infty} \frac{\log[3 \cdot 2^{3m-1} \cdot (2/3)^{3m\beta}]}{3m} \ge \log((2/3)^\beta \cdot 2) > 0.$$

Therefore, we get $\lambda \ge (2/3)^\beta \cdot 2$.

Next we show that the natural extension of (T, μ) is isomorphic to a Bernoulli system. Since the unique measure is a g-measure, we use Theorem 2.2 with the approximation of λ from above.

Let $h \in C(X)$ be an eigenfunction of the Perron–Frobenius operator with corresponding eigenvalue λ. Then the unique measure μ is a g-measure for $g = e^{\varphi+\log h-\log h\circ\sigma-\log\lambda}$. Note that g satisfies the hypothesis of Theorem 2.2.

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Applying Theorem 2.2, it is enough to show that

$$\sum_{n=r}^{\infty}\prod_{i=r}^{n}\left(1-\frac{3}{2}\operatorname{var}_i(g)\right)=\infty \quad \text{for some } r\geq 1,$$

where

$$g(x)=\frac{e^{\varphi(x)}h(x)}{(h\circ\sigma)(x)\lambda}.$$

We show this by the following three steps.

Step 1. Show that there is a $K_0\geq 1$ such that $\operatorname{var}_i(\log g)\leq \log(i/(i-1))^{2\beta}$ for $i\geq K_0+1$.

Step 2. Show that $\operatorname{var}_i(g)\leq((i/(i-1))^{2\beta}-1)/\lambda$ for $i\geq K_0+1$.

Step 3. Show that there exists $r\geq 1$ such that

$$\sum_{n=r}^{\infty}\prod_{i=r}^{n}\left(1-\frac{3}{2}\operatorname{var}_i(g)\right)=\infty \quad \text{for some } r\geq 1.$$

Step 3 easily follows from step 2.

Suppose that $\operatorname{var}_i(g)\leq((i/(i-1))^{2\beta}-1)/\lambda$ for $i\geq K_0+1$. Then

$$1-\frac{3}{2}\operatorname{var}_k(g)\geq 1-\frac{3}{2}\left(\left(\frac{k}{k-1}\right)^{2\beta}-1\right)\frac{1}{\lambda}$$

for $k\geq K_0+1$. Since $\lambda>3/2$ (by the approximation of λ above), there exists $K_0'\in\mathbb{N}$ such that

$$1-\frac{3}{2}\left(\left(\frac{k}{k-1}\right)^{2\beta}-1\right)\frac{1}{\lambda}\geq\left(\frac{k-1}{k}\right)^{2\beta}$$

for all $k\geq K_0'$. Now take $\overline{K}_0=\max\{K_0+1,\ K_0'\}$. Then

$$\prod_{i=\overline{K}_0}^{n}\left(1-\frac{3}{2}\operatorname{var}_i(g)\right)\geq\left(\frac{\overline{K}_0-1}{n}\right)^{2\beta}.$$

Therefore, since $2\beta<1$,

$$\sum_{n=\overline{K}_0}^{\infty}\prod_{i=\overline{K}_0}^{n}\left(1-\frac{3}{2}\operatorname{var}_i(g)\right)\geq\sum_{n=\overline{K}_0}^{\infty}\left(\frac{\overline{K}_0-1}{n}\right)^{2\beta}=\infty.$$

Step 2 also follows from step 1. Thus we show step 1. Let $\overline{\varphi}=\log g$. Then

$$\operatorname{var}_i(\overline{\varphi})=\sup\left\{\left|\varphi(x)-\varphi(y)+\log\frac{h(x)h(\sigma y)}{h(\sigma x)h(y)}\right| : x_k=y_k, \text{ for all } 0\leq k\leq i-1\right\}.$$

By Theorem 2.3, we know that h is constant on any cylinder set B of length greater than or equal to K_0 such that $B\cap\{2^k3^\infty,\ 2^\infty\mid k\geq 0\}=\emptyset$ for some K_0. Define $M_\infty=\{2^k3^\infty,\ 2^\infty\mid k\geq 0\}$. Use this fact to show that $\operatorname{var}_i(\overline{\varphi})\leq\log(i/(i-1))^{2\beta}$ for $i=K_0+1$. Using the properties of h, we have the following three cases.

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(I) $[x_0x_1\dots x_{K_0}]\cap M_\infty=\emptyset$ and $[x_1x_2\dots x_{K_0}]\cap M_\infty=\emptyset$.

(II) $[x_0x_1\dots x_{K_0}]\cap M_\infty\neq\emptyset$ and $[x_1x_2\dots x_{K_0}]\cap M_\infty\neq\emptyset$. In this case, $[x_0x_1\dots x_{K_0}]\in [2^{K_0+1}]\cup[3^{K_0+1}]\cup[2^{k_1}3^{k_2}]$, $k_1+k_2=K_0+1$.

(III) $[x_0x_1\dots x_{K_0}]\cap M_\infty=\emptyset$ and $[x_1x_2\dots x_{K_0}]\cap M_\infty\neq\emptyset$. In this case, $[x_0x_1\dots x_{K_0}]\in [12^{K_0}]\cup[12^{k_1}3^{k_2}]$ with $k_1+k_2=K_0$, $k_1\geq 1$.

Clearly for cylinder sets of type (I), $h(x)=h(y)$ and $h(\sigma x)=h(\sigma y)$. Therefore,

$$\begin{aligned}\mathrm{var}_{K_0+1}(\overline{\varphi})|_{\text{cylinder sets of type I}} &\leq \sup\{|\varphi(x)-\varphi(y)| : x_k=y_k \text{ for all } 0\leq k\leq K_0\}\\ &\leq \log\left(\frac{K_0+1}{K_0}\right)^{2\beta}.\end{aligned}$$

For cylinder sets of type (II), we can similarly show that

$$\mathrm{var}_{K_0+1}(\overline{\varphi})|_{\text{cylinder sets of type II}}\leq \log\left(\frac{K_0+1}{K_0}\right)^{2\beta},$$

by showing that

$$\left|\varphi(x)-\varphi(y)+\log\frac{h(x)h(\sigma y)}{h(\sigma x)h(y)}\right|\leq \log\left(\frac{K_0+1}{K_0}\right)^{2\beta} \tag{21}$$

for $x,\ y\in[x_0x_1\dots x_{K_0}]\in[2^{K_0+1}]\cup[3^{K_0+1}]\cup[2^{k_1}3^{k_2}]$.

We will need to consider the following cylinder sets of type (II).

(i) $x,\ y\in[2^{K_0+1}]$,
- (a) $x\in[2^n1]$, $y\in[2^m1]$, $n,m\geq K_0+1$;
- (b) $x\in[2^n1]$, $y=2^\infty$, $n\geq K_0+1$;
- (c) $x\in[2^n1]$, $y\in[2^m3^k1]$, $n,m\geq K_0+1$, $k\geq 1$;
- (d) $x\in[2^n1]$, $y=2^m3^\infty$, $n,m\geq K_0+1$;
- (e) $x=2^\infty$, $y=2^n3^\infty$, $n\geq K_0+1$;
- (f) $x=2^n3^\infty$, $y=2^m3^\infty$, $n,m\geq K_0+1$;
- (g) $x\in[2^n3^k1]$, $y=2^\infty$, $n\geq K_0+1$, $k\geq 1$;
- (h) $x\in[2^n3^k1]$, $y=2^m3^\infty$, $n,m\geq K_0+1$, $k\geq 1$;
- (i) $x\in[2^n3^{k_1}1]$, $y\in[2^m3^{k_2}1]$, $n,m\geq K_0+1$, $k_1,k_2\geq 1$;

(ii) $x,\ y\in[3^{K_0+1}]$,
- (a) $x\in[3^n1]$, $y\in[3^m1]$, $n,m\geq K_0+1$;
- (b) $x\in[3^n1]$, $y=3^\infty$, $n\geq K_0+1$;

(iii) $x,\ y\in[2^{k_1}3^{k_2}]$, $k_1+k_2=K_0+1$, $k_1,k_2\geq 1$,
- (a) $x\in[2^{k_1}3^l1]$, $y\in[2^{k_1}3^m1]$, $l,m\geq k_2$;
- (b) $x\in[2^{k_1}3^l1]$, $y=2^{k_1}3^\infty$, $l\geq k_2$.

To show (21), the idea is to write $h(x)$, $(h\circ\sigma)(x)$, $h(y)$, $(h\circ\sigma)(y)$ as infinite series by using the property that h is an eigenfunction of the Perron–Frobenius operator with corresponding eigenvalue λ. For simplicity, we only consider the case (i)(b) with $n=K_0+1$. Let $x\in[2^{K_0+1}1]$, $y=2^\infty$. We show that

$$|\overline{\varphi}(x)-\overline{\varphi}(y)|\leq\log\left(\frac{K_0+1}{K_0}\right)^{2\beta}.$$

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Similar arguments work for the other cases. Since

$$|\overline{\varphi}(x) - \overline{\varphi}(y)| = \left|\log\left(\frac{K_0}{K_0+1}\right)^{\beta} + \log\frac{h(x)}{h(\sigma x)}\right|,$$

it is enough to show that

$$1 \le \frac{h([2^{K_0+1}1])}{h([2^{K_0}1])} \le \log\left(\frac{K_0+1}{K_0}\right)^{\beta}.$$

Using the Perron–Frobenius operator,

$$h([2^{K_0}1]) = \frac{1}{\lambda}\left[h([12^{K_0}1]) + h([2^{K_0+1}1])\left(\frac{K_0}{K_0+1}\right)^{\beta}\right].$$

Similarly,

$$h([2^{K_0+1}1]) = \frac{1}{\lambda}\left[h([12^{K_0+1}1]) + h([2^{K_0+2}1])\left(\frac{K_0+1}{K_0+2}\right)^{\beta}\right].$$

Note that $h([12^{K_0}1]) = h([12^{K_0+1}1])$, because $[12^{K_0}] \cap M_\infty = \emptyset$. Let $C = h([12^{K_0}1])$. Repeating this argument,

$$h([2^{K_0}1]) = \frac{C}{\lambda}\sum_{i=0}^{n-1}\frac{{K_0}^{\beta}}{\lambda^i(K_0+i)^{\beta}} + \frac{1}{\lambda^n}\left(\frac{K_0}{K_0+n}\right)^{\beta} h([2^{K_0+n}1]) \quad \text{for all } n \ge 1.$$

Letting $n \to \infty$,

$$h([2^{K_0}1]) = \frac{C}{\lambda}\sum_{i=0}^{\infty}\frac{{K_0}^{\beta}}{\lambda^i(K_0+i)^{\beta}}.$$

Similarly,

$$h([2^{K_0+1}1]) = \frac{C}{\lambda}\sum_{i=0}^{\infty}\frac{(K_0+1)^{\beta}}{\lambda^i(K_0+1+i)^{\beta}}.$$

This implies that

$$\frac{h(x)}{h(\sigma x)} = \frac{h([2^{K_0+1}1])}{h([2^{K_0}1])} \le \log\left(\frac{K_0+1}{K_0}\right)^{\beta}.$$

It is easy to see by direct computation, using the infinite series, that

$$1 \le \frac{h(x)}{h(\sigma x)}.$$

Now the result follows.

For the cylinder sets of type (III), we use similar arguments; find the value of h on a cylinder set by using the Perron–Frobenius operator.

From now on we give symbolic dynamical examples without presenting the particular SFT-NC carpets from which they arise. Note that the carpets always exist by defining T appropriately.

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Example 5.2. Let $X \subset \{1, 2, 3, 4\}^{\mathbb{N}}$ and $Y = \{1, 2\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given by Figure 4. Define π by $\pi(1) = 1, \pi(2) = \pi(3) = \pi(4) = 2$. Then

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

A and B are both primitive. Thus there is a compensation function $G \circ \pi \in \mathrm{Bow}(X, \sigma)$. Condition [C] is satisfied. Therefore, (σ, μ) is exact, and hence strongly mixing.

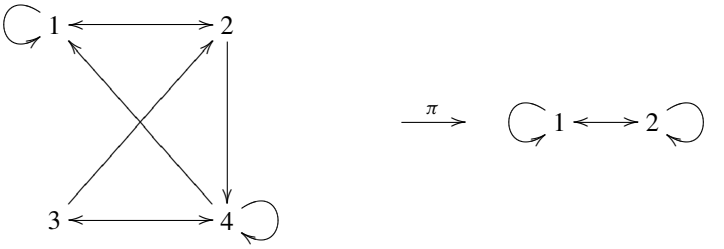


FIGURE 4. X, Y, and π in Example 5.2.

6. *Some extensions of Theorems 3.1 and 4.1*

We generalize Theorems 3.1 and 4.1 to the case where the alphabet of Y can have more than two symbols. In this section, for an allowable word $y_0 y_1 \dots y_{n-1} 1$ of length $(n+1)$ in Y with $y_i \neq 1$ for $0 \leq i \leq n-1$, we define $|\pi^{-1}[1 y_0 y_1 \dots y_{n-1} 1]| = |\pi^{-1}[y_0 y_1 \dots y_{n-1} 1]| \geq 1$ if $\pi^{-1}[1 y_0 y_1 \dots y_{n-1} 1] = \emptyset$.

Setting (B). Fix $k = 3, 4, \dots$. Let $r_2, \dots, r_k \in \mathbb{N}$, let a_i^j $(j = 2, \dots, k;\ i = 1, \dots, r_k)$ be symbols, and let $X \subset \{1, a_1^2, \dots, a_{r_2}^2, a_1^3, \dots, a_{r_3}^3, \dots, a_1^k, \dots, a_{r_k}^k\}^{\mathbb{N}}$ be a topologically mixing shift of finite type with positive entropy. Let $Y = \{1, 2, \dots, k\}^{\mathbb{N}}$ or $Y \subset \{1, 2, \dots, k\}^{\mathbb{N}}$ be a shift of finite type with positive entropy and $\pi : X \to Y$ a one-block factor map such that $\pi^{-1}\{1\} = \{1\}$ and $\pi^{-1}\{i\} = \{a_1^i, \dots, a_{r_i}^i\}$ for $2 \leq i \leq k$. Let A be the transition matrix of X, let B be the submatrix of A corresponding to the indices $a_1^2, \dots, a_{r_k}^k$ (giving the transitions among symbols in $\pi^{-1}\{2, \dots, k\}$) and let B' be the transition matrix of $\{2, 3, \dots, k\}$. Assume that for all n, $B^n \neq 0$. Denote by X_B the shift of finite type determined by B and by $Y_{B'}$ the shift of finite type determined by B'. Let $0 \leq \tau < 1$.

Condition [C′]. (1) If $\Sigma_{B'}$ is the two-sided shift of finite type on $\{2, 3, \dots, k\}$ determined by B', and $\Sigma_{B'}|_+$ is the projection of $\Sigma_{B'}$ onto a one-sided shift of finite type, then $\Sigma_{B'}|_+ = Y_{B'}$.

(2) The ratio

$$\frac{|\pi^{-1}[1 y_1 y_2 \dots y_{n-1} 1]|}{|\pi^{-1}[1 y_0 y_1 \dots y_{n-1} 1]|}$$

is constant in n, for all allowable words $y_0 \dots y_{n-1} 1$ of length $(n+1)$ in Y such that no $y_i = 1$.

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(3) Let $S = \{s \in \{2, \ldots, k\} \mid s1 \text{ is allowed in } Y\}$. For any $a \in \{2, \ldots, k\}$, there exist $s \in S$, $K \in \mathbb{N}$, $y_0 \ldots y_{K-1}$ a word of length K in $Y_{B'}$ (where K is independent of a and s) such that $ay_0 \ldots y_{K-1}s$ is allowable in $Y_{B'}$.

THEOREM 6.1. *Let Setting (B) hold and suppose the following hypothesis I is satisfied.*

I. $h_{\text{top}}(X_B) = 0$ *and the following condition holds: for each* $y \in Y_{B'}$,

$$\lim \frac{|\pi^{-1}[1y_1y_2 \ldots y_{n-1}1]|}{|\pi^{-1}[1y_0y_1 \ldots y_{n-1}1]|} = 1,$$

where the limit is taken along the sequence of $n \to \infty$ *for which* $y_{n-1}1$ *is allowable in* Y. *Then:*

(a) *there exists a compensation function* $G \circ \pi \in C(X)$ *such that* $G \in C(Y)$;
(b) *under Condition [C′], G is a grid function;*
(c) *if G is a grid function constructed in (b),* $\tau G \circ \pi$ *has a unique equilibrium state.*

Proof. For (a), define $G : Y \to \mathbb{R}$ as

$$G(y) = \begin{cases} \log(|\pi^{-1}[1y_1y_2 \ldots y_{n-1}1]|/|\pi^{-1}[1y_0y_1 \ldots y_{n-1}1]|) & \text{if } y \in [y_0y_1 \ldots y_{n-1}1], \\ & y_i \neq 1,\ n \geq 2, \\ \log(1/|\pi^{-1}[1y_01]|) & \text{if } y \in [y_01], \\ 0 & \text{if } y \in [1] \cup Y_{B'}. \end{cases}$$

We can show that (5) holds for all $y \in Y \setminus E_1$.

For (b), note first that G is not always a grid function when Y has more than two symbols. Define $M_0 = [1]$. For $n \geq 1$, let $M_n = \cup[y_0 \ldots y_{n-1}1]$, where the union is taken over all allowable words $y_0y_1 \ldots y_{n-1}1$ of length $n+1$ in Y such that $y_i \neq 1$ for $0 \leq i \leq n-1$. Let $\mathcal{P} = \{\rho(\Sigma_{B'}), M_0, M_1, M_2, \ldots\}$. Then $G \in \mathcal{G}(\Sigma_{B'}, \mathcal{P})$.

For (c), note that we cannot use Theorem 3.6, and so we modify the proof of (2)(b) in §3.3, using $h_{\text{top}}(Y_{B'}) = h_{\text{top}}(\Sigma^+_{B'}) = h_{\text{top}}(\Sigma_{B'}) = 0$, and $G \equiv 0$ on $Y_{B'}$. □

THEOREM 6.2. *Let Setting (B) hold and suppose the following hypothesis II is satisfied.*

II. $h_{\text{top}}(X_B) = \log a$, $a > 1$, *and the following condition holds: for each* $y \in Y_{B'}$,

$$\lim \frac{|\pi^{-1}[1y_1y_2 \ldots y_{n-1}1]|}{|\pi^{-1}[1y_0y_1 \ldots y_{n-1}1]|} = \frac{1}{a},$$

where the limit is taken along the sequence of $n \to \infty$ *for which* $y_{n-1}1$ *is allowed in* Y.

Then:

(a) *there exists a compensation function* $G \circ \pi \in C(X)$ *such that* $G \in C(Y)$;
(b) *under Condition [C′],* $G + \log a$ *is a grid function;*
(c) *if* $G + \log a$ *is a grid function constructed in (b),* $\tau G \circ \pi$ *has a unique equilibrium state.*

Remark 6.3. Hypothesis II seems to hold only in cases where X_B is reducible. We study the case where X_B is reducible in the next two theorems, using different approaches to prove the existence of a saturated compensation function and uniqueness for it.

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Proof. For (a), define G as in Theorem 6.1 by replacing 0 by $-\log a$ on $Y_{B'}$. For each $n = 1, 2, \ldots$ and $y \in Y$, denote $F_n(y)$ by a set consisting of exactly one point from each non-empty cylinder $[x_0 \ldots x_{n-1}] \subset \pi^{-1}[y_0 \ldots y_{n-1}]$. We prove that for all $y \in Y \setminus E_1$,

$$\limsup_{n\to\infty} \frac{1}{n}[\log(e^{(S_n G)(y)} \cdot |F_n(y)|)] \le 0 \tag{22}$$

and

$$\limsup_{n\to\infty} \frac{1}{n}[\log(e^{(S_n G)(y)} \cdot |D_n(y)|)] \ge 0. \tag{23}$$

For $y \in Y_{B'} \cup \{y \in Y \setminus Y_{B'} : y = y_0 \ldots y_{p-1} b$ for some $p \ge 1$, $b \in Y_{B'}\}$, we use a theorem of Petersen and Shin [**17**]. For such y, instead of showing (23), we show that

$$\limsup_{n\to\infty} \frac{1}{n} \log[\Sigma_{x\in F_n(y)} e^{(S_n(G\circ\pi))(x)}] \ge 0 \tag{24}$$

by finding a lower bound of $\Sigma_{x\in F_n(y)} e^{(S_n(G\circ\pi))(x)}$.

For (b), Let M_n and $\mathcal{P}$ be as in the proof of Theorem 6.1(b). Then $G + \log a \in \mathcal{G}(\Sigma_{B'}, \mathcal{P})$.

For (c), we modify the proof of Theorem 3.1(2)(b). We show first that if $-(1/(\alpha+1))G + (\alpha/(\alpha+1)) \log a$ has a unique equilibrium state, then $(\alpha/(\alpha+1))(G + \log a) \circ \pi$ has a unique equilibrium state and hence so does $\tau G \circ \pi$, where $\tau = \alpha/(\alpha+1)$. Let $\varphi = -(1/(\alpha+1))G + (\alpha/(\alpha+1)) \log a$ and $\overline{\varphi} = (\alpha/(\alpha+1))(G + \log a) \circ \pi$. By Theorem 3.4, μ is an equilibrium state of $\overline{\varphi}$ if and only if $\pi\mu$ is an equilibrium state of φ and μ is a relative equilibrium state of $G \circ \pi$ over $\pi\mu$. Let ν be the unique (ergodic) equilibrium state for φ and let μ be a preimage of ν with maximal relative entropy. Recall that μ is a preimage of maximal entropy if and only if it is a relative equilibrium state of $G \circ \pi$ over ν.

Next we will show that $\nu([1]) > 0$. If $\nu([1]) = 0$, then $\nu(Y_{B'}) = 1$. Since $G(Y_{B'}) = -\log a$, using the variational principle,

$$\begin{aligned} P_Y(\varphi) &= h_\nu(\sigma_Y) + \int_{Y_{B'}} \left(-\frac{1}{\alpha+1} G + \frac{\alpha}{\alpha+1} \log a \right) d\nu \\ &= h_\nu(\sigma_Y) + \log a \ge \log a. \end{aligned} \tag{25}$$

Since μ is an equilibrium state of $\overline{\varphi}$ and $\mu(X_B) = 1$,

$$\begin{aligned} P_X(\overline{\varphi}) &= h_\mu(\sigma_X) + \int_{X_B} \left(\frac{\alpha}{\alpha+1} G \circ \pi + \frac{\alpha}{\alpha+1} \log a \right) d\mu \\ &= h_\mu(\sigma_X) \le h_{\mathrm{top}}(X_B) = \log a. \end{aligned} \tag{26}$$

Since $G \circ \pi$ is a compensation function, we have $P_Y(\varphi) = P_X(\overline{\varphi})$, and so by using (25) and (26), we get $P_Y(\varphi) = P_X(\overline{\varphi}) = \log a$. Now by formula (2) and $h_{\mathrm{top}}(X) > \log a$, taking the Shannon–Parry measure $\mu_{\max}$ on X, we get

$$\begin{aligned} P_X(\overline{\varphi}) &= P_X\left(\frac{\alpha}{\alpha+1} G \circ \pi \right) + \frac{\alpha}{\alpha+1} \log a \\ &= \frac{1}{\alpha+1} \sup_{\mu\in M(X,\sigma_X)} \{h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)\} + \frac{\alpha}{\alpha+1} \log a \\ &\ge \frac{1}{\alpha+1} h_{\mu_{\max}}(\sigma_X) + \frac{\alpha}{\alpha+1} \log a = \frac{1}{\alpha+1} h_{\mathrm{top}}(X) + \frac{\alpha}{\alpha+1} \log a > \log a. \end{aligned}$$

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This is a contradiction. Therefore, $\nu([1]) > 0$. By using the same arguments as in §3.3, we conclude that if φ has a unique equilibrium state, then $\overline{\varphi}$ has a unique equilibrium state.

Thus it is enough to show that φ has a unique equilibrium state. If we let $\tilde{\varphi} = -(1/(\alpha+1))(G + \log a)$, then $P_Y(\tilde{\varphi}) \geq h_{\mathrm{top}}(Y_{B'})$ by the variational principle. Since $\tilde{\varphi}$ is a grid function, if the strict inequality occurs, we are done by Theorem 2.3. If not, then by [**10**, Corollary 3.8], exactly one of the following happens:

(i) the set of equilibrium states for $\tilde{\varphi}$ is the set of measures of maximal entropy for $\sigma_{Y_{B'}}$; or

(ii) the set of equilibrium states is the closed convex hull of measures of maximal entropy for $\sigma_{Y_{B'}}$ and the unique (ergodic) equilibrium state ν for $\tilde{\varphi}$ such that $\nu(Y_{B'}) = 0$.

We show that (i) does not happen. Assume that a measure $\bar{\nu}$ of maximal entropy for $\sigma_{Y_{B'}}$ is an equilibrium state for $\tilde{\varphi}$. Let $\bar{\mu}$ be a preimage of $\bar{\nu}$ with maximal relative entropy. Then $\bar{\mu}$ is a relative equilibrium state of $G \circ \pi$ over $\bar{\nu}$. Therefore, by Theorem 3.4, $\bar{\mu}$ is an equilibrium state for $(\alpha/(\alpha+1))G \circ \pi$, and so

$$\begin{aligned} P_X\left(\frac{\alpha}{\alpha+1} G \circ \pi\right) &= \frac{1}{\alpha+1} \sup_{\mu \in M(X,\sigma_X)} \{h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)\} \\ &= \frac{1}{\alpha+1} h_{\bar{\mu}}(\sigma_X) + \frac{\alpha}{\alpha+1} h_{\bar{\nu}}(\sigma_Y) \\ &= \frac{1}{\alpha+1} h_{\bar{\mu}}(\sigma_X) + \frac{\alpha}{\alpha+1} h_{\mathrm{top}}(Y_{B'}). \end{aligned}$$

Using the definition of compensation function and $G(Y_{B'}) = -\log a$, we get

$$\begin{aligned} P_X\left(\frac{\alpha}{\alpha+1} G \circ \pi\right) = P_Y\left(-\frac{1}{\alpha+1} G\right) &= h_{\bar{\nu}}(\sigma_{Y_{B'}}) + \frac{1}{\alpha+1} \log a \\ &= h_{\mathrm{top}}(Y_{B'}) + \frac{1}{\alpha+1} \log a. \end{aligned}$$

Therefore, $h_{\bar{\mu}}(\sigma_X) - \log a = h_{\mathrm{top}}(Y_{B'})$. Since $\bar{\nu}(Y_{B'}) = 1$, we have $\bar{\mu}(X_B) = 1$, and so using $h_{\bar{\mu}}(\sigma_X) \leq \log a$, we get $h_{\mathrm{top}}(Y_{B'}) = 0$. Thus $P_Y(\tilde{\varphi}) = h_{\mathrm{top}}(Y_{B'}) = 0$. However, taking the Shannon–Parry measure $\mu_{\max}$ on X and using formula (2) and $h_{\mathrm{top}}(X) > \log a$, we get

$$\begin{aligned} P_Y(\tilde{\varphi}) &= P_X\left(\frac{\alpha}{\alpha+1} G \circ \pi\right) - \frac{1}{\alpha+1} \log a \\ &= \frac{1}{\alpha+1} \sup_{\mu \in M(X,\sigma_X)} \{h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)\} - \frac{1}{\alpha+1} \log a \\ &\geq \frac{1}{\alpha+1} (h_{\mu_{\max}}(\sigma_X) - \log a) = \frac{1}{\alpha+1} (h_{\mathrm{top}}(X) - \log a) > 0. \end{aligned}$$

This is a contradiction. Therefore, a measure of maximal entropy for $\sigma_{Y_{B'}}$ is not an equilibrium state for $\tilde{\varphi}$. Therefore, (ii) happens. Then there is a unique equilibrium state ν for $\tilde{\varphi}$ such that $\nu(Y_{B'}) = 0$. □

The following proposition follows from the preceding proof of (c).

PROPOSITION 6.4. *Fix $k = 2, 3, \ldots$. Let $X \subset \{1, 2, \ldots, r\}^{\mathbb{N}}$ be a topologically mixing shift of finite type with positive entropy, $Y = \{1, 2, \ldots, k\}^{\mathbb{N}}$ or $Y \subset \{1, 2, \ldots, k\}^{\mathbb{N}}$ a shift*

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of finite type with positive entropy, and $\pi : X \to Y$ a one-block factor map. Suppose there exist a partition $\mathcal{P} = \{A, A_1, A_2 \ldots\}$ of Y, a two-sided subshift (Z, σ) such that $\rho(Z) = A$ (see the paragraph below Theorem 2.2), $a_n \in \mathbb{R}$ for $n \in \mathbb{N}$, and a function $G : Y \to \mathbb{R}$ defined by

$$G(y) = \begin{cases} a_n & \text{if } y \in A_n, \\ -h_{\mathrm{top}}(\pi^{-1}(A)) & \text{if } y \in A, \end{cases}$$

such that $G + h_{\mathrm{top}}(\pi^{-1}(A)) \in \mathcal{G}(\mathcal{P}, Z)$ and $G \circ \pi$ is a saturated compensation function. Then for each $\alpha > 0$, $-(1/(\alpha+1))G$ has a unique equilibrium state and the preimages of the unique equilibrium state with maximal relative entropy are equilibrium states for $(\alpha/(\alpha+1))G \circ \pi$.

Remark 6.5. The conditions when the hypotheses above hold need to be studied. Saturated compensation functions in Theorems 3.1 and 4.1 are examples for Proposition 6.4.

In the next two theorems, we consider the case where B is reducible.

THEOREM 6.6. *Let $X \subset \{1, a_1^2, \ldots, a_{r_2}^2, a_1^3, \ldots, a_{r_3}^3\}^{\mathbb{N}}$ be a topologically mixing shift of finite type with positive entropy, $Y = \{1, 2, 3\}^{\mathbb{N}}$ or $Y \subset \{1, 2, 3\}^{\mathbb{N}}$ a shift of finite type with positive entropy, and $\pi : X \to Y$ a one-block factor map such that $\pi^{-1}\{1\} = \{1\}$ and $\pi^{-1}\{i\} = \{a_1^i, \ldots, a_{r_i}^i\}$ for $i = 2, 3$. Suppose that 23 and 32 are not allowable words in Y. Let B_2 be the submatrix of A corresponding to the indices $a_1^2, \ldots, a_{r_2}^2$ (giving the transitions among $\pi^{-1}\{2\}$) and let B_3 be the submatrix of A corresponding to the indices $a_1^3, \ldots, a_{r_3}^3$ (giving the transitions among $\pi^{-1}\{3\}$). Denote by X_{B_2} the shift of finite type determined by B_2 and by X_{B_3} the shift of finite type determined by B_3. Suppose, for all n, that $(B_i)^n \neq 0$ for each $i = 2, 3$. Let B' be the transition matrix of $\pi^{-1}\{2, 3\}$.*

(1) *Suppose that for each $i = 2, 3$ there exists $b_i \geq 1$ such that*

$$\lim_{n\to\infty} \frac{|\pi^{-1}[1i^{n-1}1]|}{|\pi^{-1}[1i^n1]|} = \frac{1}{b_i} \quad \text{and} \quad h_{\mathrm{top}}(X_{B_i}) = \log b_i.$$

Then there exists a compensation function $G \circ \pi \in C(X)$ such that $G \in C(Y)$.

(2) *Suppose in addition that $b_i > 1$ for $i = 2, 3$ in (1) and there exist $K_1(i), K_2(i) > 0$ such that*

$$K_1(i) \leq \frac{b_i^n}{|\pi^{-1}[1i^n1]|} \leq K_2(i) \quad \text{for all } n.$$

Then $G \circ \pi \in \mathrm{Bow}(X, \sigma)$. Hence $\tau G \circ \pi \in \mathrm{Bow}(X, \sigma)$ for all $\tau \in \mathbb{R}$.

Remarks 6.7.

(1) If B_2 and B_3 are both primitive, then (1) and (2) are automatically satisfied. If B_2 and B_3 are irreducible, we still need (1) for $G \circ \pi$ to be in $\mathrm{Bow}(X, \sigma)$. These cases are also covered by Theorem 6.8.

(2) The hypothesis of (2) is applicable to some cases not covered in Theorem 6.8 when X_B is reducible and two distinct communicating classes for X_B are mapped to the same symbol (see Example 7.5).

(3) We can generalize Theorem 6.6 for the case where the matrix of $\pi^{-1}\{2, 3, \ldots, k\}$ is a direct sum of matrices of $\pi^{-1}\{i\}$ for $i = 2, 3, \ldots, k$.

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Proof. For (1), define $G : Y \to \mathbb{R}$ as

$$G(y) = \begin{cases} \log(|\pi^{-1}[1i^{n-1}1]|/|\pi^{-1}[1i^n1]|) & \text{if } y \in [i^n1],\ n \geq 2,\ i = 2, 3, \\ \log(1/|\pi^{-1}[1i1]|) & \text{if } y \in [i1],\ i = 2, 3, \\ -\log b_i & \text{if } y = i^\infty,\ i = 2, 3, \\ 0 & \text{if } y \in [1]. \end{cases}$$

For (2), we modify Lemma 3.10, noting that $\pi(x_i)$ in the proof of the lemma is 1, 2, or 3. □

We now consider another particular setting in which B is reducible.

THEOREM 6.8. *Let $X \subset \{1, a_1^2, \ldots, a_{r_2}^2, a_1^3, \ldots, a_{r_3}^3\}^{\mathbb{N}}$ be a topologically mixing shift of finite type with positive entropy, $Y = \{1, 2, 3\}^{\mathbb{N}}$ or $Y \subset \{1, 2, 3\}^{\mathbb{N}}$ a shift of finite type with positive entropy, and $\pi : X \to Y$ a one-block factor map such that $\pi^{-1}\{1\} = \{1\}$ and $\pi^{-1}\{i\} = \{a_1^i, \ldots, a_{r_i}^i\}$ for $i = 2, 3$. Let B be the transition submatrix of A for the shift of finite type on the symbols in $\pi^{-1}\{2, 3\}$ and suppose B is reducible. For $i = 2, 3$, let B_i be the transition submatrix of A for the shift on the symbols in $\pi^{-1}\{i\}$. Suppose that B_2 and B_3 are the irreducible components of B. Assume that each B_i is primitive or $B_i = [0]$ for each i. Then there exists a saturated compensation function $G \circ \pi \in C(X)$ such that $G \in C(Y)$ and, for all $\tau \in \mathbb{R}$, $\tau G \circ \pi \in \mathrm{Bow}(X, \sigma)$.*

Remark 6.9. For the general case where the alphabet of Y has more than three symbols, we can find a measurable function $G : Y \to \mathbb{R}$ such that $G \circ \pi$ satisfies (4) and $\sup_{n \geq 1} \mathrm{var}_n(S_n(G \circ \pi)) < \infty$. Such a G is continuous except on a set of measure zero with respect to every σ_Y-invariant measure. This example could be studied in connection with Question 4 in §8.

Proof. Without loss of generality, assume that 23 is allowable in Y and B_i is primitive for $i = 2, 3$. Let B' be the transition matrix for the shift of finite type on symbols in $\{2, 3\}$ and let b_i be corresponding maximum eigenvalue. Define E_1 and E_2 by

$$E_1 = \{y \in Y \mid y = y_0 \ldots y_{p-1} i^\infty \text{ for some } p \geq 1,\ y \neq i^\infty \text{ for } i = 1, 2, 3\}$$

and $E_2 = \{i^\infty \mid i = 1, 2, 3.\}$. Define $G : Y \to \mathbb{R}$ by (defining $|\pi^{-1}[12^03^m1]| = |\pi^{-1}[13^m1]|$, $|\pi^{-1}[12^01]| = |\pi^{-1}[13^01]| = 1$)

$$G(y) = \begin{cases} \log(|\pi^{-1}[12^{n-1}3^m1]|/|\pi^{-1}[12^n3^m1]|) & \text{if } y \in [2^n3^m1],\ n, m \geq 1, \\ \log(|\pi^{-1}[1i^{n-1}1]|/|\pi^{-1}[1i^n1]|) & \text{if } y \in [i^n1],\ i = 2, 3,\ n \geq 1, \\ \lim_{m\to\infty} \log(|\pi^{-1}[12^{n-1}3^m1]|/|\pi^{-1}[12^n3^m1]|) & \text{if } y = 2^n3^\infty,\ n \geq 1, \\ -\log b_i & \text{if } y = i^\infty,\ i = 2, 3, \\ 0 & \text{if } y \in [1]. \end{cases}$$

Note that G is continuous. We claim that for all $y \in Y \setminus E_1$ we have (22) and (23). For $y \in E_2$, (22) and (23) hold easily. Let $y \notin E_1 \cup E_2$. Then (23) holds by using similar arguments as in the proof of Theorem 3.1(2)(a). To show (22), notice first that by hypothesis there are two communicating classes C_1, C_2 in X_B such that there is no arrow in the graph of X from members of C_1 to members C_2. Note that we can write

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$y = (1^{l_1})s_1 1^{l_2} s_2 1^{l_3} s_3 \ldots$, where each s_i is a word of length $n(i)$ in $Y_{B'}$, and $l_i \geq 1$. Let $s_i = y_0^i \ldots y_{n(i)-1}^i$ for each i. Using the Perron–Frobenius theorem and the above fact, we find bounds for $|\pi^{-1}[1s_i 1]|$, $|\pi^{-1}[1y_0^i \ldots y_k^i]|$, and $|\pi^{-1}[1y_{k+1}^i \ldots y_{n(i)-1}^i 1]|$ for any $k \geq 0$. Straightforward computations show that $e^{(S_n G)(y)}|F_n(y)|$ is bounded uniformly. To show that $G \circ \pi \in \mathrm{Bow}(X, \sigma)$, we use similar arguments as in the second part of the proof of Lemma 3.10. □

7. *Examples for Theorems 6.1, 6.2, 6.6, and 6.8*

We first give examples that illustrate each theorem in §6 and then present examples for which we do not know the uniqueness of the equilibrium state for a saturated compensation function.

Example 7.1. This is an example for Theorem 6.1. Let $X \subset \{1, 2, 3, 4, 5\}^{\mathbb{N}}$ be the shift of finite type with the transition matrix A given below. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, and $\pi(4) = \pi(5) = 3$. Let B be the transition matrix of $\pi^{-1}\{2, 3\}$ and B' be the transition matrix of $Y \subset \{1, 2, 3\}^{\mathbb{N}}$. Then

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then $h_{\mathrm{top}}(X_B) = 0$, $|\pi^{-1}[12^n 1]| = |\pi^{-1}[13^n 1]| = n + 1$, and $|\pi^{-1}[12^k 3^l 1]| = k + l + 1$ for all $k, l \geq 1$. Hypothesis I of Theorem 6.1 is satisfied. There is a compensation function $G \circ \pi \in C(X)$, where $G \in C(Y)$ is defined by

$$G(y) = \begin{cases} \log(n/(n+1)) & \text{if } y \in [2^n 1] \cup [3^n 1] \cup [2^k 3^l 1] \text{ for } n \geq 1,\ k + l = n, \\ 0 & \text{if } y \in [1] \cup \{2^\infty, 3^\infty, 2^k 3^\infty k \geq 1\}. \end{cases}$$

Condition [C′] is satisfied, and so for each $0 \leq \tau < 1$, $\tau G \circ \pi$ has a unique equilibrium state. The unique equilibrium state μ is not Gibbs, by the first part of the proof of Lemma 3.10. Moreover, (σ, μ) is an exact endomorphism, hence strongly mixing. To see this, use the fact that $\tau G \circ \pi$ is a grid function and apply Theorem 2.3 to show that $\tau G \circ \pi$ satisfies the RPF condition.

Example 7.2. This is an example for Theorem 6.1(a) but not for (b). However, $G \circ \pi$ is still a grid function and we find a unique equilibrium state for it. Let $X \subset \{1, 2, 3, 4, 5\}^{\mathbb{N}}$ and $Y \subset \{1, 2, 3\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given in Figure 5. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, and $\pi(4) = \pi(5) = 3$. Then $h_{\mathrm{top}}(X_B) = 0$, $|\pi^{-1}[12^n 1]| = 2$, $|\pi^{-1}[13^n 1]| = n + 1$, and $|\pi^{-1}[12^k 3^n 1]| = n + 1$ for all n, k. Hypothesis I of Theorem 6.1 is satisfied. There is a compensation function $G \circ \pi \in C(X)$, where $G \in C(Y)$ is defined by

$$G(y) = \begin{cases} \log(n/(n+1)) & \text{if } y \in [3^n 1] \text{ for } n \geq 1, \\ 0 & \text{if } y \in [2^n] \cup [1] \cup \{3^\infty\}, \text{ for } n \geq 2, \\ -\log 2 & \text{if } y \in [21]. \end{cases}$$

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Condition [C′] is not satisfied, so G is not a grid function associated with $\Sigma_{B'}$ and $\mathcal{P}$ defined as in the proof of Theorem 6.1, but for each $0 \le \tau < 1$, τG is still a grid function with a different partition. Notice that $\tau G \circ \pi$ is also a grid function. Using Theorem 2.3, there exists a unique equilibrium state for it. The unique equilibrium state μ is not Gibbs, by Lemma 3.10. By the proof of Theorem 3.1(2)(c), (σ, μ) is an exact endomorphism, hence strongly mixing.

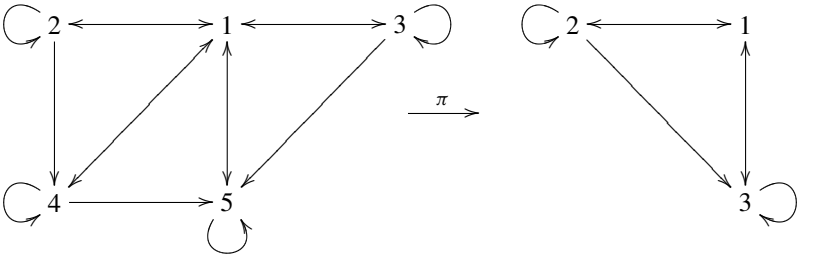


FIGURE 5. X, Y and π in Example 7.2.

Example 7.3. This is an example to which we cannot apply Theorem 6.1, but we still have the existence of a saturated compensation function and uniqueness for it. Let $X \subset \{1, 2, 3, 4, 5\}^{\mathbb{N}}$ and $Y \subset \{1, 2, 3\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given in Figure 6. Define the factor map π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, and $\pi(4) = \pi(5) = 3$. Since $h_{\text{top}}(X_B) = 0$, $|\pi^{-1}[12^n 1]| = n + 1$, $|\pi^{-1}[13^n 1]| = 2$, and $|\pi^{-1}[12^k 3^n 1]| = k + 1$ for all n, k,

$$\lim_{n\to\infty} \frac{|\pi^{-1}[12^{n-1}1]|}{|\pi^{-1}[12^n 1]|} = \lim_{n\to\infty} \frac{|\pi^{-1}[13^{n-1}1]|}{|\pi^{-1}[13^n 1]|} = 1,$$

but

$$\lim_{n\to\infty} \frac{|\pi^{-1}[12^{k-1}3^n 1]|}{|\pi^{-1}[12^k 3^n 1]|} = \frac{k}{k+1} \neq 1.$$

Thus hypothesis I of Theorem 6.1 is not satisfied. Using the proof of Theorem 6.1(a), we can find G that satisfies (5), but it is not continuous on Y. Such a G is defined by

$$G(y) = \begin{cases} \log(n/(n+1)) & \text{if } y \in [2^n 1] \text{ for } n \ge 1, \\ \log(k/(k+1)) & \text{if } y \in [2^k 3^n 1] \text{ for } n \ge 1,\ k \ge 1, \\ 0 & \text{if } y \in [1] \cup [3] \cup \{2^\infty, 2^k 3^\infty\},\ k \ge 0. \end{cases}$$

We now modify G to find a continuous saturated compensation function. Noting that $|\pi^{-1}[12^k 3^n]| = k + 1$ for any n, replacing 0 by $\log(k/(k+1))$ at $2^k 3^\infty$, we get

$$\tilde{G}(y) = \begin{cases} \log(n/(n+1)) & \text{if } y \in [2^n 1] \cup [2^n 3] \text{ for } n \ge 1, \\ 0 & \text{if } y \in [1] \cup [3] \cup \{2^\infty\}. \end{cases}$$

Then $\tilde{G} \circ \pi$ is a saturated compensation function. Since for $0 \le \tau < 1$, $\tau \tilde{G} \circ \pi$ is a grid function, using Theorem 2.3, there exists a unique equilibrium state for it. The unique equilibrium state μ is not Gibbs. (σ, μ) is an exact endomorphism, hence strongly mixing.

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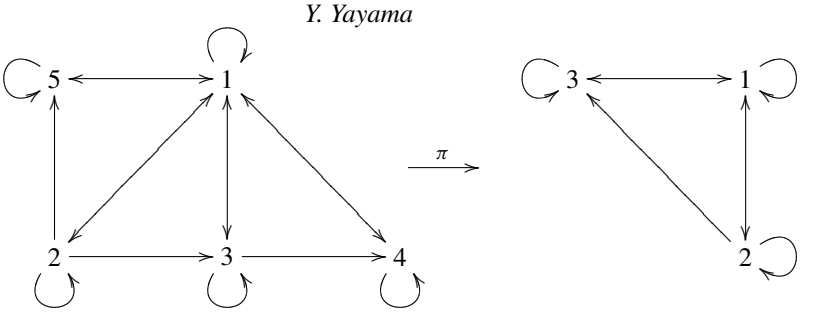


FIGURE 6. X, Y, and π in Example 7.3.

Example 7.4. This is an example for Theorem 6.2 (and Theorem 6.6(2)). Let $X \subset \{1, 2, 3, 4, 5\}^{\mathbb{N}}$ and $Y = \{1, 2, 3\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given in Figure 7, and define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, and $\pi(4) = \pi(5) = 3$. Let $a = (1 + \sqrt{5})/2$. Then

$$\lim_{n\to\infty} \frac{|\pi^{-1}[12^{n-1}1]|}{|\pi^{-1}[12^n 1]|} = \lim_{n\to\infty} \frac{|\pi^{-1}[13^{n-1}1]|}{|\pi^{-1}[13^n 1]|} = \frac{1}{a} \quad \text{and} \quad h_{\text{top}}(X_B) = \log a.$$

Define G by (defining $|\pi^{-1}[12^0 1]| = 1$)

$$G(y) = \begin{cases} \log(|\pi^{-1}[12^{n-1}1]|/|\pi^{-1}[12^n 1]|) & \text{if } y \in [2^n 1] \cup [3^n 1] \text{ for } n \geq 1, \\ -\log a & \text{if } y \in \{2^\infty, 3^\infty\}, \\ 0 & \text{if } y \in [1]. \end{cases}$$

Since Condition [C′] is satisfied, $\tau G \circ \pi$ has a unique equilibrium state for $0 \leq \tau < 1$. Note that this is also an example to illustrate Theorem 6.6(2). Thus $G \circ \pi \in \text{Bow}(X, \sigma)$.

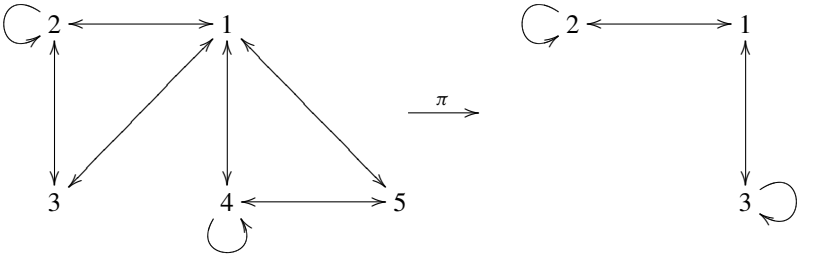


FIGURE 7. X, Y, and π in Example 7.4.

Example 7.5. This is an example for Theorem 6.6(2). Let $X \subset \{1, 2, 3, 4, 5, 6, 7, 8\}^{\mathbb{N}}$ be the shift of finite type with the transition matrix A given below. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = \pi(4) = \pi(5) = \pi(6) = 2$, $\pi(7) = \pi(8) = 3$. Note that B is reducible and all members in two distinct communicating classes $\{2, 3\}$ and $\{4, 5, 6\}$ are mapped to the same symbol.

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$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $h_{\text{top}}(X_{B_2}) = \log 3$ and $|\pi^{-1}[12^n1]| = 2^n + 3^n + 3^{n-1}(1 - (2/3)^{n-1})$. X_{B_3} is primitive, and (1) and (2) of Theorem 6.6 are satisfied for $i = 2, 3$. Thus there is a saturated compensation function $G \circ \pi \in \text{Bow}(X, \sigma)$.

Example 7.6. This is an example for Theorem 6.8. Let $X \subset \{1, 2, 3, 4, 5, 6\}^{\mathbb{N}}$ be the shift of finite type with the transition matrix A given below ($A^5 > 0$). Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, and $\pi(4) = \pi(5) = \pi(6) = 3$.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

There are two communicating classes for X_B, $C_1 = \pi^{-1}\{2\} = \{2, 3\}$ and $C_2 = \pi^{-1}\{3\} = \{4, 5, 6\}$. For each $i = 1, 2$, the irreducible matrix corresponding to C_i is primitive. Hence there is a saturated compensation function $G \circ \pi \in C(X)$, $G \in C(Y)$, such that for all $\tau \in \mathbb{R}$, $\tau G \circ \pi \in \text{Bow}(X, \sigma)$.

We now turn to examples for which we do not know the uniqueness of the equilibrium state for $\tau G \circ \pi$ for any $\tau \neq 0$.

Example 7.7. Let $X \subset \{1, 2, 3, 4, 5, 6\}^{\mathbb{N}}$ and $Y = \{1, 2, 3\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given in Figure 8. Define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, and $\pi(4) = \pi(5) = \pi(6) = 3$. Since $h_{\text{top}}(X_B) = 0$, $|\pi^{-1}[12^n1]| = n + 1$, and $|\pi^{-1}[13^n1]| = 2n + 1$ for all n, hypothesis I of Theorem 6.1 is satisfied, but not (b). Note also that Theorem 6.6(1) is applicable. $G \in C(Y)$ is defined by

$$G(y) = \begin{cases} \log(n/(n+1)) & \text{if } y \in [2^n1] \text{ for } n \geq 1, \\ \log((2n-1)/(2n+1)) & \text{if } y \in [3^n1] \text{ for } n \geq 1, \\ 0 & \text{if } y \in [1] \cup \{2^\infty, 3^\infty\}. \end{cases}$$

G seems not to look like a grid function for any partition, but G is a sum of two grid functions. Thus we do not know whether for all $\tau \in (0, 1)$, $\tau G \circ \pi$ has a unique equilibrium state or not.

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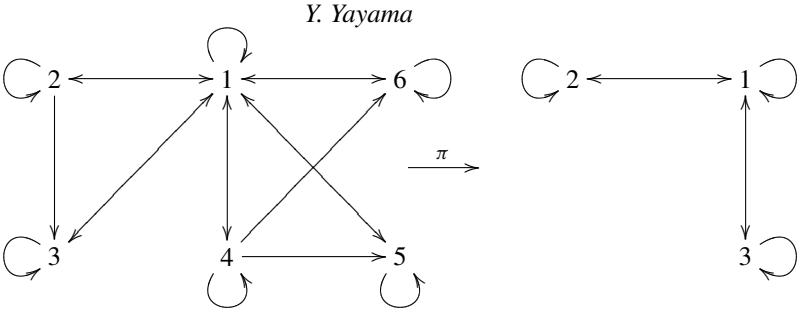


FIGURE 8. X, Y and π in Example 7.7.

Example 7.8. Let $X \subset \{1, 2, 3, 4, 5\}^{\mathbb{N}}$ and $Y \subset \{1, 2, 3\}^{\mathbb{N}}$ be the shifts of finite type determined by the transitions given in Figure 9, and define π by $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, and $\pi(4) = \pi(5) = 3$. Let $a = (1 + \sqrt{5})/2$. Then

$$\lim_{n\to\infty} \frac{|\pi^{-1}[12^{n-1}1]|}{|\pi^{-1}[12^n 1]|} = 1, \quad \lim_{n\to\infty} \frac{|\pi^{-1}[13^{n-1}1]|}{|\pi^{-1}[13^n 1]|} = \frac{1}{a},$$

and

$$h_{\mathrm{top}}(X_{B_2}) = 0 \quad \text{and} \quad h_{\mathrm{top}}(X_{B_3}) = 1/a.$$

The hypothesis of (1) of Theorem 6.6 is satisfied but not the hypothesis of (2). We define G by (defining $|\pi^{-1}[13^0 1]| = 1$)

$$G(y) = \begin{cases} \log(n/(n+1)) & \text{if } y \in [2^n 1] \text{ for } n \geq 1, \\ \log(|\pi^{-1}[13^{n-1}1]|/|\pi^{-1}[13^n 1]|) & \text{if } y \in [3^n 1] \text{ for } n \geq 1, \\ -\log a & \text{if } y = \{3^\infty\}, \\ 0 & \text{if } y \in [1] \cup \{2^\infty\}. \end{cases}$$

Then G is a sum of a grid function which has a unique equilibrium state and a function in the Bowen class. We do not know whether for $\tau \in (0, 1)$, $\tau G \circ \pi$ has a unique equilibrium state or not.

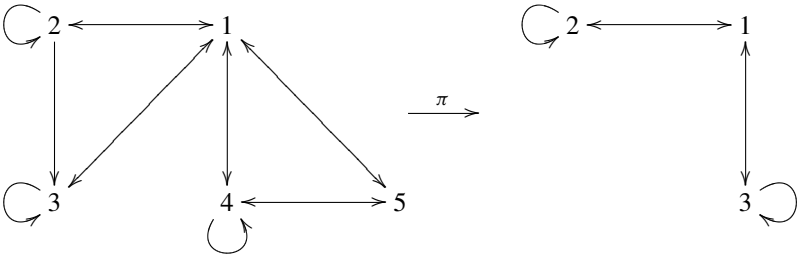


FIGURE 9. X, Y, and π in Example 7.8.

8. *Problems*

In this section, we list some questions and possible directions for future work. An overarching question is whether the geometrical properties of an SFT-NC carpet can be understood in a fundamentally different way, in particular without using saturated

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compensation functions. It would be interesting to determine completely the properties of an example for which the associated symbolic factor map does not admit a saturated compensation function. We note that the key formula (2) of Shin [**18**, p. 7], which connects the measures maximizing the weighted entropy functional ϕ_α with a saturated compensation function, was proved by using the relative variational principle with a saturated compensation function as a potential.

Question 1. Can an SFT-NC carpet (or an NC-carpet corresponding to another subshift not necessarily of finite type) have several measures of full Hausdorff dimension? Can a measure of full Hausdorff dimension fail to have a Bernoulli natural extension?

Question 2. Let T be an expanding map given by a non-diagonal matrix A over $\mathbb{Z}$. For a compact T-invariant subset of the torus, is there an ergodic measure of full Hausdorff dimension? If so, is it unique? What are the properties of such measure(s)? In this case the existence of measures of full Hausdorff dimension is not known. Bedford [**2**] and Ito and Ohtsuki [**8**] showed that there is a Markov partition consisting of $|\det A|$ elements. Ito and Ohtsuki also showed that if μ is Lebesgue measure on the torus, then the partition is a one-sided Bernoulli partition. Also, Stolot [**20**] gave a construction of 'extended Markov partitions' for the particular toral endomorphism given by $T(x, y) = ((3x + y) \bmod 1, (x + y) \bmod 1)$, and this may also be useful.

Question 3. What is the Hausdorff dimension of an SFT-NC 'sponge' (an NC carpet on an n-dimensional torus with $n \geq 3$)? Is there any measure of full Hausdorff dimension? If so, is it unique? What are the properties of the measure(s)? The Ledrappier–Young formula [**9**] extends to higher dimensions. If T is a toral endomorphism given by a diagonal matrix $\mathrm{Diag}(m_1, m_2, \ldots, m_r)$, where m_i are integers with $m_i < m_{i+1}$ for $i = 1, 2, \ldots, r-1$ and μ is an ergodic T-invariant probability measure on the r-torus,

$$\dim_{\mathrm{H}}(\mu) = \sum_{\nu=1}^{r} \frac{1}{\log m_\nu}[h(\pi_\nu\mu) - h(\pi_{\nu-1}\mu)],$$

where the entropy $h(\pi_\nu\mu)$ is with respect to the endomorphism $\mathrm{Diag}\,(m_1, \ldots, m_\nu)$ of the ν-torus, and $h(\pi_0\mu) = 0$ by convention. The existence of measures of full Hausdorff dimension of a compact T-invariant set is known [**9**].

If $r = 3$, we get

$$\dim_{\mathrm{H}}(\mu) = \frac{1}{\log m_3}[h(\pi_3\mu) + \alpha h(\pi_2\mu) + \beta h(\pi_1\mu)], \tag{27}$$

where $\alpha = -1 + \log_{m_2} m_3$ and $\beta = -\log_{m_2} m_3 + \log_{m_1} m_3$. Thus we need to find the measure(s) that maximize the right-hand side of (27). To do this, first, using the natural Markov partition for T, define X to be a symbolic representation of a compact T-invariant subset (an SFT-NC sponge) of the 3-torus, Y to be a symbolic representation of the projection of the compact T-invariant subset of the 3-torus to the 2-torus (the first two coordinates), and Z to be a symbolic representation of the projection of the compact T-invariant subset of the 3-torus to the 1-torus (the first coordinate). Define $\pi_1 : (X, \sigma_X) \to (Y, \sigma_Y)$ to be the projection to the first two coordinates and $\pi_2 : (Y, \sigma_Y) \to (Z, \sigma_Z)$ to be the projection to the first coordinate. Note that π_1 and π_2 are factor maps and so $\pi_2 \circ \pi_1$

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is also a factor map. Then we need to find an ergodic shift-invariant measure on X that maximizes

$$h_\mu(\sigma_X) + \alpha h_{\pi_1\mu}(\sigma_Y) + \beta h_{\pi_2\pi_1\mu}(\sigma_Z).$$

So we have a sequence of factor maps rather than just one, complicating the symbolic dynamics considerably.

Question 4. Let X, Y be topological dynamical systems (e.g. arbitrary subshifts) and let $\pi : X \to Y$ be a factor map. When can we find a continuous compensation function, that is, a function $f : X \to \mathbb{R}$ satisfying

$$P_X(f + \phi \circ \pi) = P_Y(\phi)$$

for all $\phi \in C(Y)$? If there is no such continuous f, can we always find a measurable one? To what extent can any such (measurable but not continuous) function substitute for a compensation function in the above arguments?

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