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An extremely sharp phase transition threshold for the slow growing hierarchy

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We investigate natural systems of fundamental sequences for ordinals below the Howard–Bachmann ordinal and study growth rates of the resulting slow growing hierarchies. We consider a specific assignment of fundamental sequences that depends on a non-negative real number ε. We show that the resulting slow growing hierarchy is eventually dominated by a fixed elementary recursive function if ε is equal to zero. We show further that the resulting slow growing hierarchy exhausts the provably recursive functions of ID_1 if ε is strictly greater than zero. Finally, we show that the resulting fast growing hierarchies exhaust the provably recursive functions of ID_1 for all non-negative values of ε.

Our result is somewhat surprising since usually the slow growing hierarchy along the Howard–Bachmann ordinal exhausts precisely the provably recursive functions of PA. Note that the elementary functions are a very small subclass of the provably recursive functions of PA, and the provably recursive functions of PA are a very small subclass of the provably recursive functions of ID_1. Thus the jump from ε equal to zero to ε greater than zero is one of the biggest jumps in growth rates for subrecursive hierarchies one might think of.

1. Introduction

This article is part of our general research program on phase transitions in logic and combinatorics. Phase transition phenomena are ubiquitous in a wide variety of branches of mathematics and neighbouring sciences, in particular, physics (see, for example, Chandler (1987)). An informal description of a 'phase transition effect' is the behaviour seen when 'small' changes in certain parameters of a system occasion dramatic shifts in some globally observed behaviour of the system, such shifts being marked by a 'sharp threshold point'. An everyday life example of this is the change from one material state to a different one as temperature is increased, with the 'threshold' being given by the melting or boiling point. Similar phenomena occur in mathematical and computational

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contexts such as evolutionary graph theory (see, for example, Bollobás (2001) and Janson *et al.* (2000)), percolation theory (see, for example, Grimmett (1999)), computational complexity theory and artificial intelligence (see, for example, Clote and Kranakis (2002) and Monasson *et al.* (1999)).

The purpose of PTLC is to study Phase Transitions in Logic and Combinatorics. We are particularly interested in the transition from provability to unprovability of a given assertion by varying a threshold parameter. From the perspective of hierarchies of recursive functions, this reduces to classifing the phase transition for the growth rates of the functions involved. The work reported in this paper is concerned with phase transitions for the slow growing hierarchy and is a continuation of the investigations already described in Weiermann (1995; 2001; 1999).

From the purely logical perspective, this article is motivated by the classical classification problem for the recursive functions and the resulting problem of comparing the slow and fast growing hierarchies. It has been claimed, for example in Girard (1987, page 439, line -5), that for sufficiently big proof-theoretic ordinals the slow and fast growing hierarchies will match up. Our results in this paper suggest that this claim might not be true in general.

To formulate the results precisely, we introduce some notation. For an ordinal α less than the Howard–Bachmann ordinal, let $N\alpha$ be the number of symbols in α that are different from 0 and $+$. The idea is essentially that $N\alpha$ is the number of edges in the tree that represents the term for α.

For a limit ordinal λ let $\lambda[x] := \max\{\beta < \lambda : N\beta \leqslant N\lambda + x\}$. This assignment of fundamental sequences is natural and, as we will show in the appendix, does not change the growth rate of the induced fast growing hierarchy. But, as our first main theorem shows, the induced slow growing hierarchy (along the Howard–Bachmann ordinal) consists of elementary functions only. This generalises results from Weiermann (2004) where the resulting slow growing hierarchy along Γ_0 was shown to consist of elementary recursive functions only.

At first sight the resulting slow growing hierarchies always seem to collapse under this assignment of fundamental sequences, and one may wonder how robust this phenomenon is. We prove therefore in a separate section a very surprising and extremely sharp phase transition threshold for the slow growing hierarchy. The upshot is that small changes prevent the hierarchies from collapsing. For a given real number $\varepsilon \geqslant 0$, let $\lambda[x]_\varepsilon := \max\{\beta < \lambda : N\beta \leqslant (1+\varepsilon)\cdot N\lambda + x\}$. Then, as we have just said, for $\varepsilon = 0$ the resulting slow growing hierarchy is very slow growing, but for any $\varepsilon > 0$ the resulting slow growing hierarchy becomes fast growing and matches up with the fast growing hierarchy at all ϵ-numbers below the Howard–Bachmann ordinal. We conjecture that within the phase transition, that is, when in the definition of $\lambda[x]_\varepsilon$ the number ε is a function of λ and x, we may arrange other behaviours of the resulting slow growing hierarchy.

The paper is not fully self-contained. The proof of the first main theorem requires basic familiarity with Buchholz-style notation systems for the Howard–Bachmann ordinal (knowledge of Buchholz (1986) should be more then sufficient). The proof of the second main result should be generally accessible (at least when one restricts consideration to ordinals below ε_0).

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2. Proof of the first main result

2.1. *Basic concepts*

We recall some basic definitions and facts from Buchholz's papers on ordinal notations. Missing proofs can be found in Buchholz (1986).

Definition 2.1. We inductively define a set of terms T and a set $P \subseteq T$ as follows:

1. $0 \in T$.
2. If $a \in T$, $i \in \{0,1\}$, then $D_i a \in P$.
3. If $a_0, \ldots, a_n \in P$, and if $n \geqslant 1$, then $(a_0, \ldots, a_n) \in T$.

We assume the following notational conventions for the current section. The symbols a, b, c, d, e will always denote elements of T. If $a \in P$, we identify the one element sequence (a) with the term a. The empty sequence $( )$ is identified with the term 0. The letters x, y, z, i, l, m, n will range over non-negative integers (unless stated otherwise).

Definition 2.2. We define recursively an ordering relation $\prec$ on T as follows. $a \prec b$ holds, if and only if one of the following cases holds:

1. $a = 0$ and $b \neq 0$.
2. $(a = D_0 a_0$ and $b = D_1 b_0)$.
3. $(a = D_i a_0$ and $b = D_i b_0$ and $a_0 \prec b_0)$.
4. $a = (a_0, \ldots, a_m)$, $b = (b_0, \ldots, b_n)$, $1 \leqslant m + n$ and $[(m < n \,\&\, \forall i \leqslant m(a_i = b_i))$ or $(\exists k \leqslant \min\{m,n\}(\forall i < k(a_i = b_i) \,\&\, a_k \prec b_k))]$.

Lemma 2.3. The structure $(T, \prec)$ is a linear order.

Proof. This is a special case of results from Buchholz (1986). □

It is easy to see that $(T, \prec)$ is not a well-order. To see this, let $a_0 := D_1 0$ and $a_{n+1} := D_0 a_n$. Then $a_{i+1} \prec a_i$ for all i. Later in the paper we will thin out T to a smaller set OT that is well-ordered by $\prec$.

We assume the following additional notational conventions for the current section. Suppose that $M, N \subseteq T$. We write $M \preceq N$ if and only if for all $x \in M$ there is a $y \in N$ such that $x \preceq y$. We write $a \preceq N$ for $\{a\} \preceq N$. Moreover, $M \prec a$ means that $x \prec a$ for all $x \in M$. Finally, we define a formal term addition on T as follows: $(a_1, \ldots, a_m) + (b_1, \ldots, b_n) := (a_1, \ldots, a_i, b_1, \ldots, b_n)$ where $i \in \{1, \ldots, m\}$ is maximal with $b_1 \preceq a_i$. This includes the cases $m = 0$ and $n = 0$ where $0 + (b_1, \ldots, b_n) = (b_1, \ldots, b_n)$ and $(a_1, \ldots, a_m) + 0 = (a_1, \ldots, a_m)$.

Definition 2.4. We define recursively T-subsets $K^* a$ and Ka for $a \in T$ as follows:

1. $K^* 0 := \varnothing$, $\quad K0 := \varnothing$.
2. $K^*(a_0, \ldots, a_n) := \bigcup_{i \leqslant n} K^* a_i$, $\quad K(a_0, \ldots, a_n) := \bigcup_{i \leqslant n} K a_i$.
3. $K^* D_1 a := K^* a$, $\quad K D_1 a := Ka$.
4. $K^* D_0 a := \{a\} \cup K^* a$, $\quad K D_0 a := \{D_0 a\}$.

Lemma 2.5. If $K^* c \prec a$, then $Kc \prec D_0 a$.

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Definition 2.6. We define inductively a set of terms $\mathrm{OT} \subseteq \mathrm{T}$ as follows:

1. $0 \in \mathrm{OT}$.
2. If $a_0, \ldots, a_n \in \mathrm{OT} \cap P$, $n \geqslant 1$ and $a_n \preceq \ldots \preceq a_0$, then $(a_0, \ldots, a_n) \in \mathrm{OT}$.
3. If $a \in \mathrm{OT}$, then $D_1 a \in \mathrm{OT}$.
4. If $a \in \mathrm{OT}$ & $\mathrm{K}^* a \prec a$, then $D_0 a \in \mathrm{OT}$.

We use the following notation. As expected, we set $1 := D_0 0$, $\omega := D_0 1$ and $\Omega := D_1 0$. Moreover, we define $\mathrm{T}_0 := \{x \in \mathrm{T} : x \prec \Omega\}$, $\mathrm{OT}_0 := \mathrm{OT} \cap \mathrm{T}_0$. Note that addition of terms in OT is already given via the restriction of the addition on T.

Theorem 2.7. $(\mathrm{OT}, \prec)$ is a well-order. The order type of $(\mathrm{OT}_0, \prec)$ is equal to the Howard–Bachmann ordinal.

Proof. A proof of this standard but fundamental fact is given, for example, in Buchholz (1986). □

Definition 2.8. We define a majorisation relation $\lhd_c$ on T as follows. $b \lhd_c a$ iff $b \prec a$ and for all $d \in \mathrm{OT}$ we have $(b \preceq d \preceq a \Rightarrow \mathrm{K}^* b \preceq \mathrm{K}^* d \cup \mathrm{K}^* c)$ $[a, b, c \in \mathrm{T}]$.

Lemma 2.9.

1. $a \prec D_1 a$.
2. If $a, b \in \mathrm{OT}_0$ and $a \prec b$, then $\mathrm{K}^* a \preceq \mathrm{K}^* b$.
3. If $b \lhd_c a$, $\mathrm{K}^* a \prec a$, $\mathrm{K}^* c \prec b$ and $a, b \in \mathrm{OT}$, then $\mathrm{K}^* b \prec b$.
4. If $b \lhd_c a$, then $d + b \lhd_c d + a$ and $D_i b \lhd_c D_i a$ $(i \in \{0, 1\})$.

Proof. We will just prove assertion 3. Assume that $b \preceq \mathrm{K}^* b$. Choose a subterm d of b with $b \preceq \mathrm{K}^* d$ such that the length of d is the minimal possible. Then $d = D_0 e$ with $\mathrm{K}^* e \prec b \preceq e \preceq a$, since $\mathrm{K}^* b \preceq \mathrm{K}^* c \cup \mathrm{K}^* a \prec a$. Then we obtain $\mathrm{K}^* b \preceq \mathrm{K}^* c \cup \mathrm{K}^* e \prec b$, which gives a contradiction. □

Definition 2.10. For $a \in \mathrm{T}$ we define recursively the cofinality type $\mathsf{tp}(a)$ to be an element of $\{0, 1, \omega, \Omega\}$ as follows:

1. $\mathsf{tp}(0) := 0$.
2. $\mathsf{tp}(1) := 1$.
3. $\mathsf{tp}(\Omega) := \Omega$.
4. If $\mathsf{tp}(a) = 1$, then $\mathsf{tp}(D_i a) := \omega$.
5. If $\mathsf{tp}(a) = \omega$, then $\mathsf{tp}(D_i a) := \omega$
6. If $\mathsf{tp}(a) = \Omega$, then $\mathsf{tp}(D_0 a) := \omega$.
7. If $\mathsf{tp}(a) = \Omega$, then $\mathsf{tp}(D_1 a) := \Omega$.
8. $\mathsf{tp}((a_0, \ldots, a_n)) := \mathsf{tp}(a_n)$.

Definition 2.11. For $c \in T_0$ and $a \in \mathrm{T}$ with $tp(a) = \Omega$ we define $a\{c\} \in \mathrm{T}$ recursively as follows (informally, $a\{c\}$ is the c'th member of the fundamental sequence for a):

1. $\Omega\{c\} := c$.
2. $(D_1 a)\{c\} := D_1 a\{c\}$.
3. If $a = (a_0, \ldots, a_n)$, then $a\{c\} := (a_0, \ldots, a_{n-1}) + a_n\{c\}$.

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Lemma 2.12.

1. If $\mathsf{tp}(a) = \Omega$ and $c \in T_0$, then $a\{c\} \lhd_c a$.
2. If $\mathsf{tp}(a) = \Omega$ and $c, d \in T_0$ and $c \prec d$, then $a\{c\} \prec a\{d\}$.
3. If $\mathsf{tp}(a) = \Omega$ and $c \in T_0$, then $\mathrm{K}^* a\{c\} \preceq \mathrm{K}^* a\{0\} \cup \mathrm{K}^* c$.
4. If $a \in \mathrm{OT}, \mathsf{tp}(a) = \Omega$ and $c \in \mathrm{OT}_0$, then $a\{c\} \in \mathrm{OT}$.

2.2. *Refined concepts*

Here we collect some technical material that will be needed during the proof of the first main result.

Definition 2.13. We define recursively the norm of a, Na, for $a \in T$ as follows:

1. $N0 := 0$.
2. $N(a_0, \ldots, a_n) := Na_0 + \ldots + Na_n$.
3. $ND_i a := 1 + Na$.

Definition 2.14 (fundamental sequences). For $a \in \mathrm{OT} \setminus \{0\}$ we define $a[x]$ and $a[\![x]\!]$ explicitly as follows:

1. $a[x] := \max\{b \in \mathrm{OT} : b \prec a \text{ and } Nb \leqslant Na + x\}$.
2. $a[\![x]\!] := \max\{b \in \mathrm{OT} : b \prec a \text{ and } Nb \leqslant x\}$.

Lemma 2.15.

1. $(a_0, \ldots, a_{n-1}, a_n)[x] = (a_0, \ldots, a_{n-1}) + a_n[x]$.
2. If $a = (a_0, \ldots, a_n)$, $b = (a_1, \ldots, a_n)$, $x \geqslant Na_0$, then $a[\![x]\!] = a_0 + b[\![x - Na_0]\!]$.
3. If $a = (a_0, \ldots, a_n)$, $x < Na_0$, then $a[\![x]\!] = a_0[\![x]\!]$.
4. $a[\![0]\!] = 0$.
5. $(D_1 a)[x] = D_1 a[x]$.
6. If $x > 0$, then $(D_1 a)[\![x]\!] = D_1 a[\![x - 1]\!]$.

Definition 2.16. For $a \in \mathrm{OT}_0$ we define recursively a natural number $G_a(x)$ as follows:

1. $G_0(x) := 0$.
2. $G_{a+1}(x) := G_a(x) + 1$.
3. $G_a(x) := G_{a[x]}(x)$ if $\mathsf{tp}(a) = \omega$.

Lemma 2.17. Let $a, b \in \mathrm{OT}_0$.

1. If $a = (a_0, \ldots, a_n)$, then $G_a(x) = G_{a_0}(x) + \cdots + G_{a_n}(x)$.
2. If $a = D_0(b + 1)$, then $G_a(x) = G_{D_0 b}(x) + G_{a[\![x+1]\!]}(x)$.
3. If $a \preceq b$ & $Na \leqslant x + 1$, then $G_a(x) \leqslant G_b(x)$.

Assertion 2 motivates the definition of the assignment $\cdot[\![\cdot]\!]$. Moreover, it gives a first indication of why the resulting slow growing hierarchy will collapse since the second term $G_{a[\![x+1]\!]}(x)$ refers to $a[\![x + 1]\!]$, which is in general very small when compared to a. Later in the paper we will verify that this phenomenon also holds true in more complicated situations.

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Definition 2.18. For $a \in T$ we define by transfinite recursion a natural number $T_a(x)$ as follows:

1. $T_0(x) := 1$.
2. $T_a(x) := T_{a[\![x]\!]}(x) + 2$.

Remark. $T_a(x)$ is defined by recursion on the cardinality of the set $\{b \prec a : Nb \leqslant x\}$. The asymptotic of T_a is very interesting from the analytic number theory point of view. We conjecture that sharp bounds on T_a will prove useful for obtaining good upper bounds on G_a, but in this paper we will just prove that each G_a is elementary for $a \in \mathrm{OT}$.

Lemma 2.19.

1. If $a \preceq b$ and $Na \leqslant x$, then $T_a(x) \leqslant T_b(x)$.
2. If $a \preceq b$, then $T_a(x) \leqslant T_b(x)$.

Proof.

1. We use induction on the cardinality of the set $\{c \prec b : Nc \leqslant x\}$. Assume that $T_b(x) = T_{b[\![x]\!]}(x) + 2$ and $a \neq b$. Then $a \preceq b[\![x]\!]$ and the induction hypothesis yield $T_a(x) \leqslant T_{b[\![x]\!]}(x)$ and the assertion follows.
2. If $a = 0$, the assertion is clear. Assume that $T_a(x) = T_{a[\![x]\!]}(x) + 2$ and $T_b(x) = T_{b[\![x]\!]}(x) + 2$. Then $a[\![x]\!] \preceq b[\![x]\!]$, and the assertion follows from 1. □

Definition 2.20. For $a \in \mathrm{OT}$ and $g, x < \omega$ we define recursively a natural number $C_x(a, g)$, the pointwise collapse of a (at arguments x and g), as follows:

1. $C_x(0, g) := 0$.
2. $C_x((a_0, \ldots, a_n), g) := C_x(a_0, g) + \cdots + C_x(a_n, g)$.
3. $C_x(D_0 a, g) := g \cdot G_{D_0 a}(x)$.
4. $C_x(D_1 a, g) := g^{2^{T_{D_1 a}(x+1)}} \cdot (C_x(a, g) + 1)$.

Lemma 2.21. If $a \in \mathrm{OT}_0$, then $C_x(a, g) = g \cdot G_a(x)$.

Proof. We use induction on Na:

1. $a = 0$.
   The assertion is obvious in this case.
2. $a = (a_0, \ldots, a_n)$.
   The induction hypothesis yields

$$\begin{aligned} C_x(a, g) &= C_x(a_0, g) + \cdots + C_x(a_n, g) \\ &= g \cdot G_{a_0}(x) + \cdots + g \cdot G_{a_n}(x) \\ &= g \cdot G_{(a_0, \ldots, a_n)}(x). \end{aligned}$$

3. $a = D_0 b$.
   In this case $C_x(a, g) = g \cdot G_a(x)$. □

The following Lemma is a crucial tool in proving the hierarchy collapse.

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Lemma 2.22 (Main lemma). If we have $a, b \in \mathrm{OT}$, $\mathrm{K}^* b \prec a$, $Nb \leqslant x+1$ and $g = G_{(D_0 a)[\![x+1]\!]}(x)$, then

$$C_x(b, g) \leqslant g^{2^{T_b(x+1)+1}}.$$

Proof. We use induction on Nb:

1. $b = 0$.
   The assertion is obvious in this case.
2. $b = (b_0, \ldots, b_n)$.
   We have $n+1 \leqslant x+1$, $Nb_i \leqslant x+1$ and $\mathrm{K}^* b_i \subseteq \mathrm{K}^* b \prec a$ for $i = 0, \ldots, n$. The induction hypothesis yields $C_x(b_i, g) \leqslant g^{2^{T_{b_i}(x+1)+1}}$ for $i = 0, \ldots, n$. Thus

$$\begin{aligned} C_x(b, g) &= C_x(b_0, g) + \cdots + C_x(b_n, g) \\ &\leqslant g^{2^{T_{b_0}(x+1)+1}} + \cdots + g^{2^{T_{b_n}(x+1)+1}} \\ &\leqslant (x+1) \cdot g^{2^{T_b(x+1)-1}} \\ &\leqslant g^{2^{T_b(x+1)+1}} \end{aligned}$$

   since $x+1 \leqslant g$ and $T_{b_i}(x+1) < T_b(x+1)$ by Lemma 2.19.
3. $b = D_0 c$.
   We have $C_x(b, g) = g \cdot G_{D_0 c}(x)$. $\mathrm{K}^* b \prec a$ yields $b \prec D_0 a$. Thus $Nb \leqslant x+1$ yields $b \preceq (D_0 a)[\![x+1]\!]$ and $G_b(x) \leqslant G_{(D_0 a)[\![x+1]\!]}(x)$, so $C_x(b, g) \leqslant g^2$.
4. $b = D_1 c$.
   In this case the induction hypothesis yields $C_x(c, g) \leqslant g^{2^{T_c(x+1)+1}}$, so

$$\begin{aligned} C_x(b, g) &= g^{2^{T_{D_1 c}(x+1)}} \cdot (C_x(c, g) + 1) \\ &\leqslant g^{2^{T_{D_1 c}(x+1)}} \cdot (g^{2^{T_c(x+1)+1}} + 1) \\ &\leqslant g^{2^{T_{D_1 c}(x+1)+1}} \\ &= g^{2^{T_b(x+1)+1}} \end{aligned}$$

   since $T_c(x+1) + 1 < T_{D_1 c}(x+1)$ because $c \prec D_1 c$ and $Nc \leqslant x$. □

2.3. *Collapsing ordinals with countable cofinalities*

Lemma 2.23. If $x < Na$, then $\mathrm{K}^* a[\![x]\!] \preceq \mathrm{K}^* a$.

Proof. We use induction on Na using Lemma 2.15:

1. $a = 0$.
   The assertion is obvious in this case.
2. $a = (a_0, \ldots, a_n)$.
   Let $b := (a_1, \ldots, a_n)$.
   2.1. $x < Na_0$.
   We have $a[\![x]\!] = a_0[\![x]\!]$ and the induction hypothesis yields

$$\mathrm{K}^* a[\![x]\!] = \mathrm{K}^* a_0[\![x]\!] \preceq \mathrm{K}^* a_0 \subseteq \mathrm{K}^* a$$

   by assertion 2 of Lemma 2.9.

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2.2. $x = Na_0$.
We have $a[\![x]\!] = a_0$ and $\mathrm{K}^* a_0 \subseteq \mathrm{K}^* a$.

2.3. $x > Na_0$.
We have $a[\![x]\!] = a_0 + b[\![x - Na_0]\!]$. Then $x < Na$ gives $x - Na_0 < Nb$, and the induction hypothesis yields $\mathrm{K}^* b[\![x - Na_0]\!] \preceq \mathrm{K}^* b$. Thus

$$\mathrm{K}^* a[\![x]\!] = \mathrm{K}^* a_0 \cup \mathrm{K}^* b[\![x - Na_0]\!] \preceq \mathrm{K}^* a_0 \cup \mathrm{K}^* b = \mathrm{K}^* a.$$

3. $a = D_0 b$.
We have $a[\![x]\!] \prec a \prec \Omega$, so $\mathrm{K}^* a[\![x]\!] \preceq \mathrm{K}^* a$ by assertion 2 of Lemma 2.9.

4. $a = D_1 b$.
Therefore $Na = 1 + Nb$.

4.1. $b = 0$.
We have $Na = 1$, hence $x = 0$, and the assertion is obvious.

4.2. $\mathsf{tp}(b) \in \{\omega, \Omega\}$.
We may assume that $x > 0$. So $(D_1 b)[\![x]\!] = D_1 b[\![x - 1]\!]$. Then $x < Na$ yields $x - 1 < Nb$. The induction hypothesis yields $\mathrm{K}^* b[\![x-1]\!] \preceq \mathrm{K}^* b$, so $\mathrm{K}^* a[\![x]\!] \preceq \mathrm{K}^* a$.

4.3. $b = c + 1$.
We have $a[\![x]\!] = D_1 c \cdot y + (D_1 c)[\![z]\!]$ where $ND_1 c > z$. The induction hypothesis yields $\mathrm{K}^*(D_1 c)[\![z]\!] \preceq \mathrm{K}^* D_1 c$, so $\mathrm{K}^* a[\![x]\!] \preceq \mathrm{K}^* a$. □

Lemma 2.24. If $a, b \in \mathrm{OT}$, $\mathsf{tp}(b) = \omega$ and $b[x] \preceq a \preceq b$, we have $\mathrm{K}^* b[x] \leqslant \mathrm{K}^* a$.

Proof. We use induction on Nb:

1. $b = (b_0, \ldots, b_n)$.
We have $b[x] = (b_0, \ldots, b_{n-1}) + b_n[x]$ by assertion 1 of Lemma 2.15. Then $b[x] \preceq a \preceq b$ yields $a = (b_0, \ldots, b_{n-1}) + c$ for some c with $b_n[x] \preceq c \preceq b_n$. The induction hypothesis yields $\mathrm{K}^* b_n[x] \preceq \mathrm{K}^* c$, and hence $\mathrm{K}^* b[x] \preceq \mathrm{K}^* a$.
2. $b = D_0 c$.
We have $b[x] \preceq a \preceq b \prec \Omega$ and $\mathrm{K}^* b[x] \preceq \mathrm{K}^* a$.
3. $b = D_1 c$.

3.1. $\mathsf{tp}(c) = \omega$.
We have $b[x] = D_1 c[x] \preceq a \preceq D_1 c$. Thus $a = D_1 d + e$ for some d with $c[x] \preceq d \preceq c$. The induction hypothesis yields $\mathrm{K}^* c[x] \preceq \mathrm{K}^* d$, so $\mathrm{K}^* b[x] \preceq \mathrm{K}^* a$.

3.2. $c = d + 1$.
We have $(D_1 c)[x] = D_1 d \cdot y + (D_1 d)[\![z]\!]$ with $z < ND_1 d$ and $y > 0$. Lemma 2.23 yields $\mathrm{K}^*(D_1 d)[\![z]\!] \leqslant \mathrm{K}^* D_1 d$. Then $D_1 d \cdot y + (D_1 d)[\![z]\!] \preceq a \preceq D_1 c$ yields $a = D_1 d + e$ for some $e \prec D_1 c$, so $\mathrm{K}^* b[x] \preceq \mathrm{K}^* a$.

Lemma 2.25. Assume that $b \in \mathrm{OT}$ and $\mathsf{tp}(b) = \omega$. The following assertions hold:

1. $b[x] \lhd_x b$.
2. $\mathrm{K}^* b[x] \prec b[x]$.
3. $D_0 b[x] \in \mathrm{OT}$.

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Proof. Assertions 2 and 3 follow from assertion 1. Assertion 1 itself follows from Lemma 2.24. □

Lemma 2.26. Assume $D_0 b \in \mathrm{OT}$. If $x < \omega$ and $\mathsf{tp}(b) = \omega$, we have $D_0 b[x] \in \mathrm{OT}$ and $(D_0 b)[x] = D_0 b[x]$.

Proof. Lemma 2.25 yields $\mathrm{K}^* b[x] \prec b[x]$, so $D_0 b[x] \in \mathrm{OT}$. By definition, we have $D_0 b[x] \preceq (D_0 b)[x]$. Assume now that $(D_0 b)[x] = D_0 c + d$ for some $d \prec D_0(c+1)$. If $d \neq 0$, then $D_0(c+1)$ would be a better choice for $(D_0 b)[x]$ than $D_0 c + d$. Hence $d = 0$. We have $N(D_0 b)[x] = 1 + Nb + x = 1 + Nc$, so $Nc = Nb + x$. $c \prec b$ yields $c \preceq b[x]$, and hence $D_0 c \leqslant D_0 b[x]$. Therefore $D_0 b[x] = (D_0 b)[x]$. □

Lemma 2.27. Assume that $a, b \in \mathrm{OT}$, $\mathsf{tp}(b) = \omega$, $\mathrm{K}^* b \prec a$ and $g = G_{(D_0 a)[\![x+1]\!]}(x)$. Then $C_x(b[x], g) \leqslant C_x(b, g)$.

Proof. We use induction on b:

1. $b = (b_0, \ldots, b_n)$ with $\mathsf{tp}(b_n) = \omega$.
   We have $\mathrm{K}^* b_n \prec a$, and the induction hypothesis yields $C_x(b_n[x], g) \leqslant C_x(b_n, g)$. Thus
   $$\begin{aligned} C_x(b[x], g) &= C_x(b_0, g) + \cdots + C_x(b_n[x], g) \\ &\leqslant C_x(b_0, g) + \cdots + C_x(b_n, g) \\ &= C_x(b, g). \end{aligned}$$
2. $b = D_0 c$.
   We have $b[x] \prec \Omega$ and Lemma 2.21 yields $C_x(b[x], g) = g \cdot G_{b[x]}(x) = g \cdot G_b(x) = C_x(b, g)$.
3. $b = D_1 c$ where $\mathsf{tp}(c) = \omega$.
   We have $\mathrm{K}^* c \subseteq \mathrm{K}^* b \prec a$ and the induction hypothesis yields $C_x(c[x], g) \leqslant C_x(c, g)$. Therefore Lemma 2.19 yields
   $$\begin{aligned} C_x(b[x], g) &= C_x(D_1 c[x], g) \\ &= g^{2^{T_{D_1 c[x]}(x+1)}} \cdot (C_x(c[x], g) + 1) \\ &\leqslant g^{2^{T_{D_1 c}(x+1)}} \cdot (C_x(c, g) + 1) \\ &= C_x(b, g). \end{aligned}$$
4. $b = D_1 c$ where $c = d + 1$.
   In this critical case we have $b[x] = D_1 d + (D_1 c)[\![x+1]\!] = D_1 d \cdot y + (D_1 c)[\![z]\!]$ where $z < N D_1 c$ and $y > 0$. Lemma 2.23 yields $\mathrm{K}^*(D_1 c)[\![z]\!] \preceq \mathrm{K}^* D_1 c \preceq \mathrm{K}^* b \prec a$. Thus $\mathrm{K}^*(D_1 c)[\![x+1]\!] \prec a$. Hence Lemma 2.22 yields
   $$\begin{aligned} C_x(b[x], g) &= C_x(D_1 d, g) + C_x((D_1 c)[\![x+1]\!], g) \\ &\leqslant g^{2^{T_{D_1 d}(x+1)}} \cdot (C_x(d, g) + 1) + g^{2^{T_{(D_1 c)[\![x+1]\!]}(x+1)+1}} \\ &\leqslant g^{2^{T_{D_1 c}(x+1)}} \cdot (C_x(c, g) + 1) \\ &= C_x(b, g) \end{aligned}$$
   since $T_{D_1 c}(x+1) > T_{(D_1 c)[\![x+1]\!]}(x+1) + 1$ and $T_{D_1 c}(x+1) \geqslant T_{D_1 d}(x+1)$. □

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2.4. *Collapsing ordinals with uncountable cofinalities*

Lemma 2.28. If $\mathsf{tp}(a) = \Omega$ and $a\{0\} \prec b \prec a$, then $Na\{0\} < Nb$.

Proof. We use induction on Na:

1. $a = \Omega$.
   We have $a\{0\} = 0$ and the assertion is obvious.
2. $a = (a_0, \ldots, a_n)$.
   We have $a\{0\} \prec b \prec a$ yields $b = (a_0, \ldots, a_{n-1}, c)$ for some c with $a_n\{0\} \prec c \prec a_n$. The induction hypothesis yields $Na_n\{0\} < Nc$, so $Na\{0\} < Nb$.
3. $a = D_1 c$.
   We have $a\{0\} = D_1 c\{0\} \prec b \prec D_1 c$ yields $b = D_1 d + e$ for some $e \prec D_1(d+1)$ and $c\{0\} \preceq d \prec c$. The induction hypothesis yields $Nc\{0\} \leqslant Nd$. If $e \neq 0$, we have $Nb \geqslant Nd + Ne + 1 > Na$. If $e = 0$, we have $c\{0\} \prec d \prec c$. The induction hypothesis yields $Nc\{0\} < Nd$ and hence the assertion. □

Lemma 2.29. Assume that $D_0 a, b, c \in \mathrm{OT}$, $\mathsf{tp}(a) = \mathsf{tp}(c) = \Omega$, $b \preceq c \preceq a$, $\mathrm{K}^* b \prec a$ and $Nb \leqslant Nc + x$. Then $b \preceq c\{(D_0 a)[\![x+1]\!]\}$.

Proof. We use induction on Nb:

1. $b = 0$.
   The assertion is obvious in this case.
2. $b = (b_0, \ldots, b_m)$.
   Assume that $c = (c_0, \ldots, c_n)$.
   2.1. $b_0 = c_0, \ldots, b_m = c_m$ and $m < n$.
   We have $b \preceq (c_0, \ldots, c_m) \preceq c\{(D_0 a)[\![x+1]\!]\}$.
   2.2. $\exists i \leqslant \min\{m, n\}[b_0 = c_0, \ldots, b_{i-1} = c_{i-1}, b_i \prec c_i]$.
   If $i < n$, we have $b \prec (c_0, \ldots, c_i) \preceq c\{(D_0 a)[\![x+1]\!]\}$. Assume $i = n$ and $b_n, \ldots, b_m \prec c_n]$.
   2.2.1. $m = n$.
   We have $Nb \leqslant Nc + x$ yields $Nb_n \leqslant Nc_n + x$. The induction hypothesis yields $b_n \preceq c_n\{(D_0 a)[\![x+1]\!]\}$, so $b \preceq c\{(D_0 a)[\![x+1]\!]\}$.
   2.2.2. $m > n$.
   We have $Nb \leqslant Nc + x$ yields $Nb_n, \ldots, Nb_m \leqslant Nc_n + x - 1$. For $x > 0$ we get $b_n, \ldots, b_m \preceq c_n\{(D_0 a)[\![x]\!]\} \prec c_n\{(D_0 a)[\![x+1]\!]\}$ from the induction hypothesis. For $x = 0$ we get $b_n, \ldots, b_m \preceq c_n\{0\} = c_n\{(D_0 a)[\![x]\!]\}$ by Lemma 2.28. Since $c_n\{(D_0 a)[\![x+1]\!]\} \in P$, we have $(b_n, \ldots, b_m) \prec c_n\{(D_0 a)[\![x+1]\!]\}$, so $b \prec c\{(D_0 a)[\![x+1]\!]\}$ holds for $x \geqslant 0$.
3. $b = D_0 c$.
   We have $\mathrm{K}^* b \prec a$ yields $\mathrm{K}^* c \cup \{c\} \prec a$, so $b \prec D_0 a$, and hence $b \preceq (D_0 a)[\![Nb]\!]$.
   3.1. $c = \Omega$.
   We have $Nc = 1$ and $b \preceq (D_0 a)[\![x+1]\!] = c\{(D_0 a)[\![x+1]\!]\}$.
   3.2. $\Omega \prec c$.
   We have $b \preceq \Omega \preceq c[0] \preceq c\{(D_0 a)[\![x+1]\!]\}$.

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4. $b = D_1 d$.

   4.1. $c = (c_0, \ldots, c_n)$ with $n \geqslant 1$.
   We have $b = D_1 d \leq c_0 \leq c\{0\} \leqslant c\{(D_0 a)[\![x+1]\!]\}$.

   4.2. $c = D_1 e$.
   We have $Nb \leqslant Nc + x$ yields $Nd \leqslant Ne + x$. The induction hypothesis yields $d \leq e\{(D_0 a)[\![x+1]\!]\}$ since $e \prec D_1 e \leq a$. Thus $b = D_1 d \leq D_1 e\{(D_0 a)[\![x+1]\!]\} = c\{(D_0 a)[\![x+1]\!]\}$. □

Corollary 2.30. Assume that $\mathsf{tp}(a) = \Omega$, $b \leq a$, $\mathrm{K}^* b \prec b$ and $Nb \leqslant Na + x$. Then $b \leqslant a\{(D_0 a)[\![x+1]\!]\}$.

Proof. Put $c = a$ in Lemma 2.29. □

Lemma 2.31. Assume that $D_0 a \in \mathrm{OT}$ and $\mathsf{tp}(a) = \Omega$. Let $z \prec ND_0 a$. Then $(D_0 a)[\![z]\!] = D_0 a\{0\}$ if $z = ND_0 a[0]$ and $(D_0 a)[\![z]\!] = (D_0 a\{0\})[\![z]\!]$ otherwise.

Proof. It suffices to show $(D_0 a)[\![z]\!] \leq D_0 a\{0\}$. To show a contradiction, we assume that $D_0 a\{0\} \prec (D_0 a)[\![z]\!] \prec D_0 a$. Then $(D_0 a)[\![z]\!] = D_0 b + d$ for some b with $a\{0\} \leq b \prec a$. Lemma 2.28 yields $Nb \geqslant Na\{0\}$. If $d \neq 0$, we have $N(D_0 b + d) \geqslant ND_0 a\{0\} + 1 = ND_0 a$. This contradicts $z \prec ND_0 a$. Hence $d = 0$ and $a\{0\} \prec b \prec a$. Lemma 2.28 yields $Nb > Na\{0\}$, so $ND_0 b \geqslant ND_0 a$, which contradicts $z \prec ND_0 a$. □

Lemma 2.32. Assume that $D_0 a \in \mathrm{OT}$ and $\mathsf{tp}(a) = \Omega$. Let $z \prec N(D_0 a)$. Let $d_0 a(0, z) := (D_0 a\{0\})[\![z]\!]$ and $d_0 a(y+1, z) := D_0 a\{d_0 a(y, z)\}$. Then $d_0 a(y, z) \prec d_0 a(y+1, z)$ and $d_0 a(y, z) \in \mathrm{OT}$. Moreover, $(D_0 a)[\![x]\!] = d_0 a(y, z')$ where y and z' are chosen such that $(Na + 1) \cdot y + z' = x$ and $z' \prec Na + 1$.

Proof. We use induction on y to show $d_0 a(y, z) \prec d_0 a(y+1, z)$:

1. $y = 0$.
   We have $d_0 a(0, z) = (D_0 a\{0\})[\![z]\!]$ and $d_0 a(1, z) = D_0 a\{(D_0 a\{0\})[\![z]\!]\}$. If $z = 0$, it is obvious that $d_0 a(0, z) \prec d_0 a(1, z)$. Assume that $z \neq 0$. Lemma 2.23 yields $\mathrm{K}^*(D_0 a\{0\})[\![z]\!] \leq \mathrm{K}^*(D_0 a\{0\}) \leq \mathrm{K}^* a\{0\} \cup \{a\{0\}\} \prec a\{(D_0 a\{0\})[\![z]\!]\}$. Lemma 2.5 yields $\mathrm{K}(D_0 a\{0\})[\![z]\!] \prec D_0 a\{(D_0 a\{0\})[\![z]\!]\}$, so
   $$(D_0 a\{0\})[\![z]\!] \prec D_0 a\{(D_0 a\{0\})[\![z]\!]\}.$$
2. $y = y' + 1$.
   The induction hypothesis yields
   $$d_0 a(y', z) \prec d_0 a(y'+1, z),$$
   so
   $$a\{d_0 a(y', z)\} \prec a\{d_0 a(y'+1, z)\},$$
   and thus
   $$d_0 a(y, z) \prec d_0 a(y+1, z).$$

We now show $d_0 a(y, z) \in \mathrm{OT}$ by induction on y:

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1. $y = 0$.
   Then $d_0a(y,z) = (D_0a\{0\})[\![z]\!] \in \mathrm{OT}$.
2. $y = y' + 1$.
   Then $d_0a(y,z) := D_0a\{d_0a(y',z)\}$. The induction hypothesis yields $d_0a(y',z) \in \mathrm{OT}$, so $a\{d_0a(y',z)\} \in \mathrm{OT}$.
   We have to show $\mathrm{K}^*a\{d_0a(y',z)\} \prec a\{d_0a(y',z)\}$ and compute $\mathrm{K}^*a\{d_0a(y',z)\} \preceq \mathrm{K}^*a\{0\} \cup \mathrm{K}^*d_0a(y',z)$. Lemma 2.12 yields $a\{0\} \lhd_0 a$, so $\mathrm{K}^*a\{0\} \prec a\{0\}$ since $\mathrm{K}^*a \prec a$. We first consider the case $y' = 0$. If $z = 0$, we have $d_0a(y',z) = 0$, so

$$\mathrm{K}^*a\{d_0a(y',z)\} = \mathrm{K}^*a\{0\} \prec a\{0\} = a\{d_0a(y',z)\}.$$

   If $z > 0$, we have $(D_0a\{0\})[\![z]\!] \neq 0$, and Lemma 2.23 yields

$$\begin{aligned} \mathrm{K}^*d_0a(y',z) &\preceq \mathrm{K}^*D_0a\{0\} \\ &\preceq \mathrm{K}^*a\{0\} \cup \{a\{0\}\} \\ &\preceq a\{0\} \\ &\prec a\{(D_0a\{0\})[\![z]\!]\}. \end{aligned}$$

   Now assume that $y' > 0$. We have already shown that $\{d_0a(y'-1,z)\} \prec d_0a(y',z)$. The induction hypothesis yields $d_0a(y',z) \in \mathrm{OT}$, so $\mathrm{K}^*a\{d_0a(y'-1,z)\} \prec a\{d_0a(y'-1,z)\}$, hence

$$\begin{aligned} \mathrm{K}^*a\{d_0a(y',z)\} &= \mathrm{K}^*D_0a\{d_0a(y'-1,z)\} \\ &\preceq \mathrm{K}^*a\{0\} \cup \mathrm{K}^*\{d_0a(y'-1,z)\} \cup \{d_0a(y'-1,z)\} \\ &\preceq \{d_0a(y'-1,z)\} \prec d_0a(y',z). \end{aligned}$$

   Now we prove $(D_0a)[\![x]\!] = d_0a(y,z')$ by induction on x, where $(Na+1)\cdot y + z' = x$ and $z' < Na+1$. If $x < N(D_0a)$, the assertion follows from Lemma 2.31. Now assume that $x \geqslant N(D_0a)$. The choice of y and z' and the definition of $(D_0a)[\![x]\!]$ yield $(D_0a)[\![x]\!] \geqslant d_0a(y,z')$ since $Nd_0a(y,z') = x$.
   Now assume that $(D_0a)[\![x]\!] = D_0b + c$ with $b \prec a$ and $c \prec D_0(b+1)$. Then $c = 0$ since otherwise $D_0(b+1)$ would be a better choice than $D_0b + c$ for $(D_0a)[x]$. We have $b \preceq a$, $\mathsf{tp}(a) = \Omega$, $\mathrm{K}^*b \prec b$ and $Nb \leqslant x = Na + 1 + x - Na - 1$. The induction hypothesis yields $(D_0a)[\![x - Na - 1]\!] = d_0a(y-1,z)$. Lemma 2.30 yields $b \preceq a\{(D_0a)[\![x - Na]\!]\}$, so $D_0b \preceq D_0a\{(D_0a)[\![x - Na - 1]\!]\} = d_0a(y,z) \in \mathrm{OT}$. □

Corollary 2.33. Assume $\mathsf{tp}(a) = \Omega$. Then $D_0a\{(D_0a)[\![x+1]\!]\} \in \mathrm{OT}$ and $(D_0a)[x] = D_0a\{(D_0a)[\![x+1]\!]\}$.

Proof. Lemma 2.32 yields $(D_0a)[x] = (D_0a)[\![Na + 1 + x]\!] = D_0a\{(D_0a)[\![x+1]\!]\} \in \mathrm{OT}$. □

Definition 2.34. For $a \in \mathrm{OT}$ with $\mathsf{tp}(a) = \Omega$ and $g < \omega$ we define recursively a nominal form $\mathscr{C}_x(a,g)$ as follows:

1. $\mathscr{C}_x(\Omega, g) := \star$.
2. $\mathscr{C}_x((a_0,\ldots,a_{n-1},a_n),g) := C_x(a_0,g) + \cdots + C_x(a_{n-1},g) + \mathscr{C}_x(a_n,g)$.
3. $\mathscr{C}_x(D_1a,g) := g^{2^{T_{D_1a}(x+1)}} \cdot (\mathscr{C}_x(a,g)+1)$.

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If $\mathscr{C}$ is a nominal form, $\mathscr{C}[\star := c]$ denotes the result of replacing every occurrence of $\star$ in $\mathscr{C}$ by c.

Lemma 2.35. If $a \in \mathrm{OT}$, $\mathsf{tp}(a) = \Omega$ and $c \in \mathrm{OT}_0$, then $C_x(a[c], g) \leqslant \mathscr{C}_x(a, g)[\star := g \cdot G_c(x)]$.

Proof. We use induction on Na:

1. $a = \Omega$.
   Lemma 2.21 yields $C_x(a[c], g) = C_x(c, g) = g \cdot G_c(x) = \star[\star := g \cdot G_c(x)]$.
2. $a = (a_0, \ldots, a_n)$.
   The induction hypothesis yields

$$\begin{aligned} C_x(a[c], g) &= C_x(a_0, g) + \cdots + C_x(a_{n-1}, g) + C_x(a_n[c], g) \\ &\leqslant C_x(a_0, g) + \cdots + C_x(a_{n-1}, g) + \mathscr{C}_x(a_n, g)[\star := g \cdot G_c(x)] \\ &= \mathscr{C}_x(a, g)[\star := g \cdot G_c(x)]. \end{aligned}$$

3. $a = D_1 b$.
   Assertion 2 of Lemma 2.19 and the induction hypothesis yield

$$\begin{aligned} C_x(a[c], g) &= C_x(D_1(b[c]), g) \\ &= g^{2^{T_{D_1 b[c]}(x+1)}} \cdot (C_x(b[c], g) + 1) \\ &\leqslant g^{2^{T_{D_1 b}(x+1)}} \cdot (\mathscr{C}_x(b, g)[\star := g \cdot G_c(x)] + 1) \\ &= \mathscr{C}_x(a, g)[\star := g \cdot G_c(x)]. \end{aligned}$$

□

Lemma 2.36. If $a \in \mathrm{OT}$ and $\mathsf{tp}(a) = \Omega$, then $\mathscr{C}_x(a, g)[\star := g^2] \leqslant C_x(a, g)$.

Proof. We use induction on Na:

1. $a = \Omega$.
   We have $\mathscr{C}_x(a, g)[\star := g^2] = g^2$ and $C_x(D_1 0, g) = g^{2^{T_{D_1 0}(x+1)}} \cdot (0 + 1) \geqslant g^2$.
2. $a = (a_0, \ldots, a_n)$
   The induction hypothesis yields

$$\begin{aligned} \mathscr{C}_x(a, g)[\star := g^2] &= C_x(a_0, g) + \cdots + C_x(a_{n-1}, g) + \mathscr{C}_x(a_n, g)[\star := g^2] \\ &\leqslant C_x(a_0, g) + \cdots + C_x(a_{n-1}, g) + C_x(a_n, g) \\ &= C_x(a, g). \end{aligned}$$

3. $a = D_1 b$.
   The induction hypothesis yields

$$\begin{aligned} \mathscr{C}_x(a, g)[\star := g^2] &= g^{2^{T_{D_1 b}(x+1)}} \cdot (\mathscr{C}_x(b, g)[\star := g^2] + 1) \\ &\leqslant g^{2^{T_{D_1 b}(x+1)}} \cdot (C_x(b, g) + 1) \\ &= C_x(a, g). \end{aligned}$$

□

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2.5. *Putting things together*

Theorem 2.37. Let $D_0a \in \mathrm{OT}_0$ and $g := G_{(D_0a)[\![x+1]\!]}(x)$. Let $D_0b \in \mathrm{OT}$ and assume that $\mathrm{K}^* D_0b \prec a$. Then

$$G_{D_0b}(x) \leqslant 1 + C_x(b, g).$$

Proof. We use induction on D_0b:

1. $G_{D_00}(x) = 1 \leqslant 1 + C_x(0, g)$.
2. $b = c + 1$.
   We have
$$\begin{aligned} G_{D_0b}(x) &= G_{D_0c+(D_0b)[\![x+1]\!]}(x) \\ &= G_{D_0c}(x) + G_{(D_0b)[\![x+1]\!]}(x) \\ &\leqslant 1 + C_x(c, g) + g \\ &= 1 + C_x(c+1, g) \end{aligned}$$
   since $C_x(1, g) = g$ and $G_{(D_0b)[\![x+1]\!]}(x) \leqslant G_{(D_0a)[\![x+1]\!]}(x)$ by assertion 3 of Lemma 2.17.
3. $tp(b) = \omega$.
   The induction hypothesis and Lemmas 2.26 and 2.27 yield
   $G_{D_0b}(x) = G_{(D_0b)[x]}(x) = G_{D_0(b[x])}(x) \leqslant 1 + C_x(b[x], g) \leqslant 1 + C_x(b, g)$.
4. $tp(b) = \Omega$.
   The induction hypothesis and Lemmas 2.33, 2.35 and 2.36 yield
$$\begin{aligned} G_{D_0b}(x) &= G_{D_0b[(D_0b)[\![x+1]\!]]}(x) \\ &\leqslant 1 + C_x(b\{(D_0b)[\![x+1]\!]\}, g) \\ &\leqslant 1 + \mathscr{C}_x(b, g)[\star := g \cdot G_{(D_0b)[\![x+1]\!]}(x)](x) \\ &\leqslant 1 + \mathscr{C}_x(b, g)[\star := g^2] \\ &\leqslant 1 + C_x(b, g). \end{aligned}$$

□

Lemma 2.38. Let $U_x := \{a \in T : Na \leqslant x\}$ and $\#U_x$ be the cardinality of U_x. Then $\#U_x \leqslant 4^{4^{4^x}}$.

Proof. We use induction on x. Obviously, $\#U_0 = 1$ and $\#U_{x+1}$ is less than or equal to one plus the cardinality of the Cartesian product $\{0, 1, 2\} \times U_x \times \cdots U_x \times U_x$ with $x + 2$ factors since if $a \in T$, we know a is either of the form $(a_0, \ldots, a_n)$ with $n \leqslant x$ and $a_i \in T_x$ or a is of the form D_0b or D_1b for $b \in T_x$. Hence, arguing by induction,

$$\#U_{x+1} \leqslant 3 \cdot (\#U_x)^x \leqslant 3 \cdot (4^{4^{4^x}})^x \leqslant 4^{4^{4^{x+1}}}.$$

□

Theorem 2.39. Let $p(x) := 4^{4^{4^x}}$. If $a \in \mathrm{OT}_0$ and $Na \leqslant x$, then $G_a(x) \leqslant p(p(T_a(x+1)))$.

Proof. We use induction on Na:

1. $a = 0$.
   The assertion is obvious in this case.

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2. $a = (a_0, \ldots, a_n)$.
   The induction hypothesis yields
   $$G_a(x) = G_{a_0}(x) + \cdots + G_{a_n}(x) \leqslant p^2(T_{a_0}(x+1)) + \cdots + p^2(T_{a_n}(x+1)) \leqslant p^2(T_a(x+1)).$$
3. $a = D_0 b$.
   Let $g := (D_0 a)[\![x+1]\!]$. Then the induction hypothesis and Lemma 2.22 yield
   $$\begin{aligned} G_a(x) &\leqslant 1 + C_x(b, g) \\ &\leqslant g^{2^{T_b(x+1)+1}} \\ &\leqslant (p^2(T_{a[\![x+1]\!]}(x+1)))^{2^{4^{4^{4^x}}}} \\ &\leqslant p^2(T_a(x+1)). \end{aligned}$$ □

Corollary 2.40. Let $4_0(x) := x$ and $4_{n+1}(x) := 4^{4_n(x)}$. If $a \in \mathrm{OT}_0$ and $Na \leqslant x$, then $G_a(x) \leqslant 4_9(x)$.

Proof. The corollary follows from Lemma 2.38 and Theorem 2.39. □

We find it an interesting open question to decide whether G_a can be majorised eventually by a double (or triple) exponential function. Another interesting open question is whether our first main result extends to the proof-theoretic ordinal of the theory $\mathsf{ID}_{<\omega}$. Then the usual first subrecursively inaccessible ordinal would not be subrecursively inaccessible for the assignment of fundamental sequences considered in this section. The corresponding phase transition would then be even sharper than the one obtained in this paper.

3. The second main result

3.1. *Preliminaries*

We collect here some folklore material that will be useful in proofs later on. In this section we use small Greek letters (except $\varepsilon, \varepsilon'$) to denote ordinals. The idea is to indicate that the results of this section are to a large extent independent of the notation system OT. In particular, for $\varepsilon > 0$ the resulting slow growing hierarchy, when restricted to the segment of ordinals below ε_0, will exactly exhaust all provably recursive functions of PA. Therefore, this section can be read by readers without knowledge of higher ordinal notations.

Definition 3.1. For a given real number $\varepsilon \geqslant 0$ let $\lambda[x]_\varepsilon := \max\{\beta < \lambda : N\beta \leqslant (1+\varepsilon)\cdot N\lambda + x\}$. Any such system will be called a norm-based assignment of fundamental sequences. If $\varepsilon = 0$ we call $\cdot[\cdot]_\varepsilon$ the standard norm-based assignment.

Definition 3.2.
Let $\cdot[\cdot]$ be an assignment of fundamental sequences. With respect to $\cdot[\cdot]$, we define certain ordinal relations as follows:

1. We write $\alpha \succ_x \beta$ iff there are an $n > 0$ and $\gamma_0, \ldots, \gamma_n$ such that $\alpha = \gamma_0$ and $\beta = \gamma_n$, and $\gamma_{i+1} = \gamma_i[x]$ for all $i < n$.
2. We write $\alpha \succeq_x \beta$ iff $\alpha \succ_x \beta \vee \alpha = \beta$.
3. We write $\beta \sqsupseteq_x m$ iff there is an α such that $\beta \succeq_x \alpha$ and $N\alpha \geqslant m$.

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The following lemma provides some useful properties for investigating the growth rate of pointwise hierarchies.

Lemma 3.3. Let $\cdot[\cdot]$ be a norm-based assignment of fundamental sequences and let the slow growing hierarchy (G_α) be defined with respect to $\cdot[\cdot]$. Then we have:

1. If $\alpha \succeq_x \beta$, then $G_\alpha(x) \geqslant G_\beta(x)$.
2. $G_\beta(x) \geqslant N\beta$.
3. If $\beta \sqsupseteq_x m$, then $G_\beta(x) \geqslant m$.

Proof. The proof is straightforward. □

Lemma 3.4. Assume that $\cdot[\cdot]$ is a norm-based assignment. Let $\lambda \in Lim$. Then $N\lambda + x \leqslant N\lambda[x]$. If $\lambda[x] + 1 < \alpha < \lambda$ also, then $\lambda[x] + 1 \leqslant \alpha[0]$.

Proof. The first claim is obvious. Assume that $\alpha[0] < \lambda[x]+1 < \alpha < \lambda$. Then $N\lambda[x]+1 > N\alpha[0] \geqslant N\alpha > N\lambda[x] + 1$, which is a contradiction. □

Corollary 3.5. Assume that $\cdot[\cdot]$ is a norm-based assignment. Let $\succ_y$ be defined with respect to $\cdot[\cdot]$.

1. Let $\lambda \in Lim$. Then $\lambda[x + 1] \succeq_y \lambda[x] + 1$.
2. Assume that $\alpha \succ_x \beta$ and $\delta = \omega^\gamma + \alpha$ where $\alpha < \omega^{\gamma+1}$. Then $\delta \succ_x \omega^\gamma + \beta$.
3. Assume that $\alpha \succ_x$. Then $\omega^\alpha \succ_x \omega^\beta$.

Proof. Assertion 1 follows from Lemma 3.4. Assertions 2 are 3 are proved by induction on β with the use of 1. □

The following lemma shows that monotonicity for the indices of the assignment of fundamental sequences yields the expected monotonicity for the induced assignments and pointwise hierarchies.

Lemma 3.6. Let $\varepsilon, \varepsilon'$ be real numbers with $1 \leqslant \varepsilon \leqslant \varepsilon'$. Let (G_α) be defined with respect to $\cdot[\cdot]_\varepsilon$ and (G'_α) be defined with respect to $\cdot[\cdot]_\varepsilon$. Let $\succeq_y$ be defined with respect to $\cdot[\cdot]$. Then the following hold:

1. If $\lambda \in Lim$, then $\lambda[x]'_\varepsilon \succeq_y \lambda[x]_\varepsilon$.
2. $G_\alpha(x) \leqslant G'_\alpha(x)$ for any $\alpha \in \mathrm{OT}$ and $x < \omega$.

Proof. The proof is straightforward. □

We are going to show that for $\varepsilon > 0$ the pointwise hierarchies in fact consist of fast growing functions. To this end, we recall some basic facts from hierarchy theory.

Definition 3.7 (The Hardy-hierarchy). With regard to a given norm-based system $\cdot[\cdot]$ of fundamental sequences we define recursively number-theoretic functions H_α as follows:

1. $H_0(x) := x$.
2. $H_{\alpha+1}(x) := H_\alpha(x + 1)$.
3. $H_\lambda(x) := H_{\lambda[x]}(x)$ if λ is a limit.

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Lemma 3.8. Let $\succeq_x$ be defined with respect to a given norm-based assignment $\cdot[\cdot]$. Then $\alpha \succeq_x \beta$ yields $H_\alpha(y) \geqslant H_\beta(y)$ for all $y \geqslant x$. Furthermore, each function H_α is strictly monotonic increasing.

Lemma 3.9. Let $\varepsilon, \varepsilon'$ be real numbers with $1 \leqslant \varepsilon \leqslant \varepsilon'$. Let (H_α) be defined with respect to $\cdot[\cdot]_\varepsilon$ and let (H'_α) be defined with respect to $\cdot[\cdot]_{\varepsilon'}$. Let $\succeq_y$ be defined with respect to $\cdot[\cdot]$. Then $H_\alpha(x) \leqslant H'_\alpha(x)$ for any α and $x < \omega$.

Proof. The proof is straightforward. □

Lemma 3.10. Let (H_α) be defined with respect to a norm-based assignment. Then (H_α) is a fast growing hierarchy.

Proof. The proof is straightforward using techniques from Weiermann (1999), but is postponed to the appendix. □

3.2. *Putting things together*

If f is an operation on natural numbers, we write $f(m/n)$ for $f(l)$ where l is the largest integer less than or equal to $m/n := m \cdot n^{-1}$. Later in the paper we show the fast growingness of (G_α) when defined with respect to $\cdot[\cdot]$ for $\varepsilon > 0$ by a straightforward, but tedious, calculation.

Theorem 3.11. Assume $k \geqslant 4$. Let $\cdot[\cdot] := \cdot[\cdot]_{1/k}$, and $\sqsupseteq_0, \succeq_0$ and (H_α) be defined with respect to $\cdot[\cdot]$. If $\gamma =_{NF} \delta + \omega^{\omega^\alpha} \cdot k^k$, then $\gamma \sqsupseteq_0 H_\alpha(N\delta/k)$.

Proof. We use induction on α. In the following calculations we frequently make use of assertions 1, 2 and 3 of Corollary 3.5.

We may assume that $\alpha > 0$.

Case 1. $\alpha = \beta + 1$.

We have $\delta + \omega^{\omega^{\beta+1}} \cdot k^k \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + \gamma}$ for some $\gamma < \omega^{\beta+1}$ with $N\gamma \geqslant (N\beta + 1) \cdot k^{k-1} + 1$. Let $\xi_0 := \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + \gamma}$.

If $\gamma < \omega$, then

$$\xi_0 \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1} + 1} =: \xi_1.$$

If $\gamma \geqslant \omega$, then

$$\xi_0 \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + \omega} \succeq_0 \xi_1.$$

In both cases $\xi_0 \succeq_0 \xi_1$. We have

$$\xi_1 \succeq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} \cdot 2 =: \xi_2$$

since

$$\begin{aligned} N\xi_1 \cdot 1/k &\geqslant (3 + N\beta) \cdot (k^k - 1)/k + (2 + N\beta + 1 + N\beta \cdot k^{k-1}) \cdot k \\ &\geqslant (3 + N\beta) \cdot k^{k-1} + N\beta \cdot k^{k-2} \\ &\geqslant N\omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}}. \end{aligned}$$

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Similarly, we obtain

$$\begin{aligned}\xi_2 &= \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} \cdot 2 \\ &\geq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1} - 1} \cdot 2 \\ &\geq_0 \cdots \\ &\geq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1} - 1} + \cdots \\ &\quad + \omega^{\omega^\beta + 2} + \omega^{\omega^\beta + 1} =: \xi_3.\end{aligned}$$

We have

$$\begin{aligned}N\xi_3 \cdot (1/k) &\geqslant (1/k) \cdot (N\delta + (3 + N\beta) \cdot (k^{k-1} - 1)) + (1/k) \cdot \sum_{i=1}^{2+N\beta+(1+N\beta)\cdot k^{k-1}} i \\ &\geqslant (2 + N\beta) \cdot k^k,\end{aligned}$$

so the induction hypothesis yields

$$\begin{aligned}\xi_3 &= \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1} - 1} + \cdots \\ &\quad + \omega^{\omega^\beta + 2} + \omega^{\omega^\beta + 1} \\ &\geq_0 \delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1}} + \omega^{\omega^\beta + (1+N\beta)\cdot k^{k-1} - 1} + \cdots \\ &\quad + \omega^{\omega^\beta + 2} + \omega^{\omega^\beta} \cdot k^k \\ &\sqsupseteq_0 H_\beta((N(\delta + \omega^{\omega^{\beta+1}} \cdot (k^k - 1) + \ldots))/k) \geqslant H_\beta(N\delta/k + 1) = H_\alpha(N\delta/k).\end{aligned}$$

Case 2. $\alpha \in Lim$.

We have

$$\begin{aligned}\delta + \omega^{\omega^\alpha} \cdot k^k &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (2+N\alpha)\cdot k^{k-1}]}} \\ &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} + 1]}} \\ &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}] + 1}} =: \eta_0 \\ &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}] \cdot 2}} =: \eta_1\end{aligned}$$

since

$$N\eta_0 \cdot 1/k \geqslant N\delta/k + (2 + N\alpha)k^k + (1 + N\alpha) \cdot k^{k-2} \geqslant N[\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}$$

. Furthermore,

$$\begin{aligned}\eta_1 &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1] + 1}} \\ &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1] \cdot 2}} \\ &\geq_0 \cdots \\ &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} \\ &\quad + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1] + \ldots + \omega^{\alpha[1]} + \omega}} \\ &\geq_0 \delta + \omega^{\omega^\alpha} \cdot (k^k - 1) + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1}]}} \\ &\quad + \omega^{\omega^{\alpha[N\delta/k + (1+N\alpha)\cdot k^{k-1} - 1] + \ldots + \omega^{\alpha[1]} + N\delta/k \cdot k^{k-1}}}.\end{aligned}$$

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Let $\eta_2 := \omega^{\alpha[N\delta/k+(1+N\alpha)\cdot k^{k-1}-1]} + \ldots + \omega^{\alpha[1]}$. Then

$$\begin{aligned}
\eta_2 &= \delta + \omega^{\omega^\alpha}\cdot(k^k-1) + \omega^{\eta_2+N\delta/k\cdot k^{k-1}} \\
&\geq_0 \delta + \omega^{\omega^\alpha}\cdot(k^k-1) + \omega^{\eta_2+N\delta/k\cdot k^{k-1}-1} + \omega^{\eta_2+N\delta/k\cdot k^{k-1}-1} \\
&\geq_0 \delta + \omega^{\omega^\alpha}\cdot(k^k-1) + \omega^{\eta_2+N\delta/k\cdot k^{k-1}-1} + \omega^{\omega^{\alpha[N\delta/k+(1+N\alpha)\cdot k^{k-1}]}} \\
&\geq_0 \delta + \omega^{\omega^\alpha}\cdot(k^k-1) + \omega^{\eta_2+N\delta/k\cdot k^{k-1}-1} \\
&\quad + \omega^{\omega^{\alpha[N\delta/k+(1+N\alpha)\cdot k^{k-1}-1]}+\omega^{\alpha[N\delta/k+(1+N\alpha)\cdot k^{k-1}-2]}+\cdots\omega^{\alpha[1]}+N\delta/k\cdot k^{k-1}-1} \\
&\quad + \cdots + \\
&\quad + \omega^{\omega^{\alpha[N\delta/k]}+\cdots+N\delta/k\cdot k^{k-1}-1} \\
&\geq_0 \delta + \omega^{\omega^\alpha}\cdot(k^k-1) + \cdots + \omega^{\omega^{\alpha[N\delta/k]}+1} =: \eta_3.
\end{aligned}$$

The induction hypothesis yields

$$\begin{aligned}
\eta_3 &= \cdots + \omega^{\omega^{\alpha[N\delta/k]}+1} \\
&\geq_0 \omega^{\omega^{\alpha[N\delta/k]}}\cdot k^k \\
&\sqsupseteq_0 H_{\alpha[N\delta/k]}(N\delta/k) = H_\alpha(k)
\end{aligned}$$

since

$$\begin{aligned}
N\eta_3 &\geqslant N\delta/k\cdot k^{k-1}\cdot k^{k-1} + 1/2\cdot(N\alpha+1)^2\cdot(k^{k-1})^2 \\
&\geqslant (2+N\alpha(k+1)/k + N\delta/k)k^k.
\end{aligned}$$

□

Lemma 3.12. For any $k, x \in \mathbb{N}$ we have

$$\omega^{\omega^{\omega^\alpha+1}}\cdot k^k + \omega^{\omega^{\omega^\alpha+\omega}} \geq_x \omega^{\omega^{\omega^\alpha+1}}\cdot k^k + \omega^{\omega^{\omega^\alpha+x}}\cdot k^k.$$

Proof. We obtain

$$\begin{aligned}
\omega^{\omega^{\omega^\alpha+1}}\cdot k^k + \omega^{\omega^{\omega^\alpha+\omega}} &\geq_x \omega^{\omega^{\omega^\alpha+1}}\cdot k^k + \omega^{\omega^{\omega^\alpha+x+k^{k-1}}} \\
&\geq_0 \omega^{\omega^{\omega^\alpha+1}}\cdot k^k + \omega^{\omega^{\omega^\alpha+x+k^{k-1}-1}}\cdot 2 \\
&\geq_0 \omega^{\omega^{\omega^\alpha+1}}\cdot k^k + \ldots + \omega^{\omega^{\omega^\alpha+x+1}} \\
&\geq_0 \omega^{\omega^{\omega^\alpha+1}}\cdot k^k + k^k + \omega^{\omega^{\omega^\alpha+x+k^{k-1}-1}} + \ldots + \omega^{\omega^{\omega^\alpha+x}}\cdot k^k.
\end{aligned}$$

□

Theorem 3.13. Assume $k \geqslant 4$. Let $\cdot[\cdot] := \cdot[\cdot]_{1/k}$. Assume that (G_α) is defined with respect to $\cdot[\cdot]$ and that (H_α) is defined with respect to the standard norm-based assignment. Then $G_{\omega^{\omega^{\omega^\alpha+1}}\cdot k^k+\omega^{\omega^{\omega^\alpha+\omega}}}(x) \geqslant H_{\omega^\alpha}(x)$.

Proof. The theorem follows from assertion 2 of Lemma 3.3 and Lemma 3.12 □

Corollary 3.14. Let $\varepsilon > 0$ and assume that the hierarchy (G_α) is defined with respect to $\cdot[\cdot]_\varepsilon$. Then (G_α) is fast growing.

Proof. The corollary follows from Theorem 3.13. □

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Appendix

We stick to the notational conventions of Section 1. In this appendix we first describe the standard system of fundamental sequences in terms of the norms function, and show that it gives rise to a normed Bachmann system. Then we define the standard Hardy hierarchy (H^*_a) along OT and compare it with (H_a). For an intermediate calculation we also introduce a fast growing (as shown in Arai (1998)) hierarchy (A_a) (which looks slow growing at first sight).

Definition 3.15. For $a \in \mathrm{OT}$ with $\mathsf{tp}(a) = \omega$ and $x < \omega$ we define a non-negative integer $p(a + x)$ as follows:

1. If $a = (a_0, \ldots, a_{n-1}, a_n)$, then $p(a + x) := Na_0 + \ldots + Na_{n-1} + p(a_n + x)$.
2. If $a = D_i(b + 1)$, then $p(a + x) := (Nb + 1) \cdot (x + 1)$.
3. If $a = D_i b$ and $\mathsf{tp}(b) = \omega$, then $p(a + x) := 1 + p(b + x)$.
4. If $a = D_0 b$ and $\mathsf{tp}(b) = \Omega$, then $p(a + x) := (Nb + 1) \cdot (x + 1)$.

Definition 3.16. The definition of $a\{\{x\}\}$ for $a \in \mathrm{OT}_0$ with $\mathsf{tp}(a) = \omega$ and $x < \omega$ is given by $a\{\{x\}\} := \max\{b \in \mathrm{OT} : b \prec a \text{ and } Nb \leqslant p(a + x)\}$.

Lemma 3.17. The structure $\langle \mathrm{OT}_0, \cdot\{\{\cdot\}\}, N\rangle$ is a normed Bachmann system.

Proof. The lemma follows from Buchholz *et al.* (1994, Theorem 5). □

Lemma 3.18. The characterisation of $a\{\{x\}\}$ for $a \in \mathrm{OT}_0$ with $\mathsf{tp}(a) = \omega$ and $x < \omega$ is given by:

1. If $a = (a_0, \ldots, a_{n-1}, a_n)$, then $a\{\{x\}\} = (a_0, \ldots, a_{n-1}) + a_n\{\{x\}\}$.
2. If $a = D_0(b + 1)$, then $a\{\{x\}\} = D_0 b \cdot (x + 1)$.
3. If $a = D_0 b$ and $\mathsf{tp}(b) = \omega$, then $a\{\{x\}\} = D_0 b\{\{x\}\}$.
4. If $a = D_0 b$ and $\mathsf{tp}(b) = \Omega$, then $a\{\{x\}\} = D_0 b_x$ where $b_0 := b\{0\}$ and $b_{y+1} := b\{D_0 b_y\}$.

Lemma 3.18 shows that $\cdot\{\{\cdot\}\}$ coincides with the usual Buchholz definition of fundamental sequences for the limits below the Howard–Bachmann ordinal.

From now on a, b, c, d range over OT_0.

Lemma 3.19. The following properties hold for the norm function N:

1. $N(a\{\{0\}\}) = N(a) + 1$.
2. $N(\omega \cdot a\{0\}) \leqslant N(\omega \cdot a) + 1$.
3. $N(a\{\{x\}\}) \leqslant Na \cdot (x + 1)$.
4. $N(\omega^i \cdot a) \leqslant Na \cdot (i + 1)$.

Definition 3.20. We define H^*_a and A_a for $a \in \mathrm{OT}_0$ by transfinite recursion functions as follows:

1. (a) $H^*_0(x) := x$.
   (b) $H^*_{a+1}(x) := H^*_a(x + 1)$.
   (c) $H^*_a(x) := H^*_{a\{\{x\}\}}(x)$ if $\mathsf{tp}(a) = \omega$.
2. (a) $A_0(x) := x$.
   (b) $A_a(x) := \max\{A_b(x) + 1 : b \prec a \text{ and } Nb \leqslant Na + x\}$.

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Lemma 3.21. We have the following elementary properties for the functions H_a, H^*_a and A_a:

1. If $NF(a,b)$, then $H_{a+b}(x) = H_a(H_b(x))$.
2. $H_{\omega\cdot 10}(x) \geqslant 10 \cdot x$.
3. $H_{\omega^2\cdot a+\omega\cdot a+\omega\cdot(k+11)}(x) \geqslant H_{\omega^2\cdot a+\omega\cdot a+\omega\cdot(k+1)}(10\cdot x)$.
4. If $a \prec b$ and $Na \leqslant Nb + x$, then $H_a(x) < H_b(x)$.
5. $A_a(x) \leqslant H_a(x) \leqslant H^*_a(x)$.

Definition 3.22.

1. Put $a \trianglerighteq^1_x b$ iff $a \succeq b$ and $Na \geqslant Nb + x$.
2. Put $a \trianglerighteq_x b$ iff there exist $a_0, \ldots, a_n$ with $a_0 = a$ and $a_n = b$ and $a_i \trianglerighteq_x a_{i+1}$ for all $i < n$.
3. $a \geqslant_x k$ iff there is a b such that $a \trianglerighteq_x b$ and $Nb \geqslant k$.

Lemma 3.23. We have the following elementary properties for the functions A_a and the relation $\trianglerighteq_x$:

1. If $a \trianglerighteq_x b$, then $A_a(x) \geqslant A_b(x)$.
2. $A_a(x) \geqslant Na$.
3. If $a \geqslant_x k$, then $A_a(x) \geqslant k$.
4. $\omega \cdot a \geqslant_{10\cdot x} Na \cdot (10 \cdot x - 1)$.

The last assertion of the following Lemma shows that the hierarchy (H_a) catches up with the hierarchy (H^*_a), as expected.

Lemma 3.24.

1. If $x \geqslant 2$, then $\omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega \geqslant_{10x} H^*_a(k + x)$.
2. If $x \geqslant 2$, then $H_{\omega^2\cdot a+\omega\cdot a+\omega\cdot k+\omega}(10x) \geqslant H^*_a(k + x)$.
3. If $c = \omega^{\omega+d}$, then for all c there exists an ordinal term $b \prec c$ such that $H_b(x) \geqslant H^*_a(x)$ for all $x \in \mathbb{N}$

Proof. We prove the first assertion by induction on a:

1. $a = 0$.
   In this case,
$$\begin{aligned} \omega \cdot k + \omega &\trianglerighteq_{10\cdot x} \omega \cdot k + 10 \cdot x + 1 \\ &\trianglerighteq_{10\cdot x} \cdots \\ &\trianglerighteq_{10\cdot x} k \cdot (10 \cdot x + 1) + 10 \cdot x + 1 \\ &\geqslant_{10\cdot x} k + x = H^*_0(k + x). \end{aligned}$$
2. $a = b + 1$.
   The induction hypothesis yields
$$\begin{aligned} \omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega &\trianglerighteq_{10\cdot x} \omega^2 \cdot b + \omega^2 + \omega \cdot b + \omega \cdot (k + 1) + \omega \\ &\geqslant_{10\cdot x} H^*_b(k + x + 1) = H^*_a(k + x). \end{aligned}$$
3. $\mathsf{tp}(a) = \omega$.

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The induction hypothesis yields

$$\begin{aligned}\omega^2 \cdot a + \omega \cdot a + \omega \cdot k + \omega &\trianglerighteq_{10\cdot x} \omega^2 \cdot a + \omega \cdot k + \omega + Na \cdot (10 \cdot x - 1)\\ &\trianglerighteq_{10\cdot x} \omega^2 \cdot a + k \cdot Na \cdot (10 \cdot x - 1)\\ &\trianglerighteq_{10\cdot x} \omega^2 \cdot a\{\{x\}\} + \omega \cdot a\{\{x\}\} + \omega \cdot k + \omega\\ &\geqslant_{10\cdot x} H^*_{a\{\{k+x\}\}}(k+x) = H^*_a(k+x).\end{aligned}$$

The second assertion follows from assertion 3 of Lemma 3.23 and assertion 5 of Lemma 3.21, and the last assertion follows from the second assertion and assertion 2 of Lemma 3.21. □

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