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THE DIFFERENCE BETWEEN CONSECUTIVE PRIME NUMBERS. II

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1. In a previous paper*, under the same title, I considered the problem of how far apart two consecutive primes can be. The present paper is concerned with the opposite question. How near together can large primes lie? The published literature on this subject is scanty and, though interesting, is mainly negative in character. It appears to be very difficult to give any answer that is not trivial, or that is at all illuminating.

It is immediately obvious from the Prime Number Theorem that there exists an infinity of prime pairs p, p' satisfying

$$0 < p - p' < (1+\epsilon)\log p$$

for every fixed $\epsilon > 0$. But no one has yet succeeded in replacing the upper bound $(1+\epsilon)\log p$ by any function $\lambda(p)$ of less order than $\log p$.

Hardy and Littlewood, in their unpublished paper *Partitio Numerorum VII*†, were the first to make any contribution to this problem. I have had the privilege of reading their manuscript, and the argument of the present paper is the same in principle, but contains several minor improvements.

Let Θ be the lower bound of all positive numbers σ_0 such that no Dirichlet L-function $L(s, \chi)$ has a zero at $s = \sigma + it$, where $\sigma > \sigma_0$. Then Hardy and Littlewood show‡ that

$$0 < p - p' < \left(\frac{1+2\Theta}{3} + \epsilon\right)\log p \qquad (1{\cdot}1)$$

for infinitely many prime pairs p, p', and every fixed $\epsilon > 0$, provided that $\Theta < \frac{3}{4}$. This result is obtained by an extension of the methods of the authors' paper§ *Partitio Numerorum III*. My method differs from theirs mainly in the use of trigonometrical sums in place of infinite series. By the help of a lemma on sums involving characters, I prove the following

THEOREM. *Given any* $\epsilon > 0$, *there exists a positive number* $N_0 = N_0(\epsilon)$, *such that,*

* R. A. Rankin(4). † Hereinafter referred to as *P.N. VII*.

‡ They restrict themselves to the hypothesis $\Theta = \frac{1}{2}$, but their method leads to (1·1) if $\Theta < \frac{3}{4}$. § G. H. Hardy and J. E. Littlewood(1).

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if $N \geqslant N_0$, there always exists at least one pair of primes p, p' satisfying the two relations

$$\left.\begin{aligned} &N < p' < p \leqslant 2N, \\ &p - p' < \left(\frac{1+4\Theta}{5} + \epsilon\right) \log p. \end{aligned}\right\} \tag{1·2}$$

This theorem, like the Hardy-Littlewood result (1·1), gives no significant information except under the hypothesis $\Theta < 1$. If this assumption is made, (1·2) is a slight improvement on (1·1), since the coefficient of $\log p$ in the former is the smaller. In particular, if the "grand Riemann hypothesis" is true, i.e. if $\Theta = \frac{1}{2}$, the theorem gives an upper bound $(\frac{3}{5} + \epsilon) \log p$ as against the $(\frac{2}{3} + \epsilon) \log p$ of *P.N. VII*. We do not require the restriction $\Theta < \frac{3}{4}$ for the proof.

I must mention that Dr P. Erdös has communicated to me an interesting proof, independent of any hypothesis, that

$$0 < p - p' < c \log p$$

for infinitely many prime pairs p, p', where c is some positive constant less than unity. His demonstration, which has not yet been published, depends on Brun's method. He has not informed me what is the best value he can assign to c.

2. In this section I give a sketch of the idea behind the method employed in *P.N. VII* and in the present paper.

Let $P(n, x)$ denote the number of prime pairs less than x which are a distance n apart. Then, if p denotes a prime,

$$P(n, x) = \int_0^1 \left| \sum_{p \leqslant x} e^{2\pi i \alpha p} \right|^2 e^{2\pi i \alpha n} \, d\alpha. \tag{2·1}$$

The problem of evaluating $P(n, x)$ is of the same depth as the Goldbach problem for two primes. If we split up the range of integration into Farey arcs, we find that, in the absence of more precise knowledge about the sums $\Sigma e^{2\pi i \alpha p}$, we cannot prove that the contribution from the "major arcs" predominates, and so obtain an asymptotic formula. Hardy and Littlewood's idea is to replace the $e^{2\pi i \alpha n}$ in the integrand by some trigonometrical polynomial which is positive for all α in the range of integration. The left-hand side of (2·1) will then become some function of $\pi(x)$, and the $P(n, x)$'s for different n's. On the other hand, the integral can be estimated fairly accurately on the major arcs, and if we confine ourselves to these arcs we shall have that the left-hand side is greater than the contribution from these arcs. Our aim now is to show that this would be impossible if all the $P(n, x)$'s (with $n > 0$) on the left-hand side were zero or $O(1)$.

The actual integral considered contains the sum $\Sigma e^{2\pi i \alpha p} \log p$ instead of $\Sigma e^{2\pi i \alpha p}$ since the former is easier to manipulate. Unfortunately, it is possible to prove that no matter what kind of positive polynomial we choose to substitute

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for $e^{2\pi i a n}$ in the integrand, we cannot prove more than the theorem, if we use the same methods. For this reason we use the simple polynomial

$$\left| \sum_{\nu=1}^{k} e^{2\pi i a \nu} \right|^2.$$

The analysis of the succeeding sections is abbreviated as much as possible, since the complication of the methods employed is somewhat out of proportion to the importance of the result obtained.

3. We need the following eight lemmas:

LEMMA 1. $$\sum_{(\nu, q)=1} e^{2\pi i \nu a/q} = \sum_{d \mid (a, q)} d\mu\left(\frac{q}{d}\right).$$

See E. Landau(3), 1, 188.

LEMMA 2. *If $\chi(n)$ is a primitive character modulo q, and if $\chi(n)$ is not the principal character* (i.e. $q > 2$), *then*

$$\sum_{n=1}^{q} \chi(n)\, e^{2\pi i m n/q} = \epsilon(\chi)\, q^{\frac{1}{2}} \overline{\chi}(m),$$

where $\epsilon(\chi)$ is a number of unit modulus depending only on χ, and where

$$\epsilon(\chi)\, \epsilon(\overline{\chi}) = \chi(-1).$$

See E. Landau(2), 1, 483–94.

Let $\chi_0(n)$ be the principal character modulo q.

LEMMA 3. *If $\chi(n)$ is any character modulo q, and if $X(n)$ is the associated primitive character modulo a, where $q = ab$, then*

$$\sum_{n=1}^{q} \chi(n)\, e^{2\pi i m n/q} = \begin{cases} a^{\frac{1}{2}} k \epsilon(X)\, X(b')\, \overline{X}(m') \sum_{d \mid (m', b')} d\mu\left(\frac{b'}{d}\right) & (k \mid m), \\ 0 \quad (k \nmid m), \end{cases}$$

where b' is the largest factor of b such that $(a, b') = 1$, and where

$$k = b/b', \quad m' = m/k.$$

This lemma will only be required for the particular case $(a, b) = 1$, $\mu(q) \neq 0$, but it has been stated for the general case since it is interesting in itself, and, so far as I know, is a new result.

Both Lemmas 1 and 2 are included in Lemma 3. Hence we need only prove it for an imprimitive non-principal character $\chi(n)$.

(i) Suppose that $(a, b) = 1$. Then the congruence

$$n_1 + \lambda a \equiv c \pmod{b}$$

has exactly one solution for λ modulo b. Let $c_1, c_2, \ldots, c_{b-h}$ be the $b - h = b - \phi(b)$ residue classes modulo b which are not prime to b, and let

$$n_1 + \lambda_\nu a \equiv c_\nu \pmod{b}.$$

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Then $\lambda_\nu = \lambda_\nu(n_1)$, and, putting $n = n_1 + \lambda a$, $\xi = e^{2\pi i m/q}$, $\xi_1 = e^{2\pi i m/a}$, $\xi_2 = e^{2\pi i m/b}$, we have

$$\Omega = \sum_{n=1}^{q} \chi(n)\,\xi^n = \sum_{n_1=1}^{a} X(n_1)\,\xi^{n_1} \sum_{\lambda=1}^{b} \xi_2^{\lambda} - \sum_{\nu=1}^{b-h} \sum_{n_1=1}^{a} X(n_1)\,\xi^{n_1}\xi_2^{\lambda_\nu}$$

$$= \rho a^{\frac{1}{2}} b\epsilon(X)\,\overline{X}(m/b) - \sum_{\nu=1}^{b-h} \sum_{n_1=1}^{a} X(n_1)\,\xi^{n_1}\xi_2^{\lambda_\nu}, \qquad (3\cdot1)$$

by Lemma 2, where

$$\rho = 0 \quad (b \nmid m), \qquad \rho = 1 \quad (b \mid m),$$

and $\overline{X}(m/b)$ is to be taken as zero when $b \nmid m$. Now choose a residue class t modulo b so that

$$at \equiv 1 \pmod{b},$$

and let $at = 1 + \mu b$. This is possible since $(a, b) = 1$. Then

$$n_1 t + \lambda_\nu \equiv c_\nu t \pmod{b},$$

and therefore

$$\xi_2^{\lambda_\nu} = \xi_2^{c_\nu t - n_1 t} = \xi^{-n_1 a t}\,\xi_2^{c_\nu t}.$$

Hence, writing n for n_1, we have

$$\sum_{\nu=1}^{b-h} \sum_{n=1}^{a} X(n)\,\xi^n\xi_2^{\lambda_\nu} = \sum_{\nu=1}^{b-h} \sum_{n=1}^{a} X(n)\,\xi^{-nb\mu}\,\xi_2^{c_\nu t}$$

$$= \sum_{\nu=1}^{b-h} \xi_2^{c_\nu t} \sum_{n=1}^{a} X(n)\,\xi_1^{-n\mu} = a^{\frac{1}{2}}\epsilon(X)\,\overline{X}(-m\mu) \sum_{\nu=1}^{b-h} \xi_2^{c_\nu t}$$

$$= \rho a^{\frac{1}{2}} b\epsilon(X)\,\overline{X}(-m\mu) - a^{\frac{1}{2}}\epsilon(X)\,\overline{X}(-m\mu) \sum_{\substack{\nu=1 \\ (\nu, b)=1}}^{b} \xi_2^{\nu t} \qquad (3\cdot2)$$

by Lemma 2. If $b \mid m$, the right-hand side of (3·2) becomes

$$a^{\frac{1}{2}}(b-h)\,\epsilon(X)\,\overline{X}(-m\mu) = a^{\frac{1}{2}}(b-h)\,\epsilon(X)\,\overline{X}(m/b).$$

Therefore, by (3·1),

$$\Omega = a^{\frac{1}{2}}\epsilon(X)\,\overline{X}(m/b)\,\phi(b) = a^{\frac{1}{2}}\epsilon(X)\,\overline{X}(m/b) \sum_{d \mid (m, b)} d\mu\left(\frac{b}{d}\right).$$

Now suppose that $b \nmid m$. Then, by (3·1), (3·2) and Lemma 1,

$$\Omega = a^{\frac{1}{2}}\epsilon(X)\,\overline{X}(-m\mu) \sum_{\substack{\nu=1 \\ (\nu, b)=1}}^{b} \xi_2^{\nu t} = a^{\frac{1}{2}}\epsilon(X)\,\overline{X}(-m\mu) \sum_{d \mid (b, m)} d\mu\left(\frac{b}{d}\right)$$

$$= a^{\frac{1}{2}}\epsilon(X)\,X(b)\,\overline{X}(m) \sum_{d \mid (m, b)} d\mu\left(\frac{b}{d}\right).$$

This completes the proof of the case (i).

(ii) Now suppose that $(a, b) = u > 1$, and that $u \nmid m$. The congruence

$$n + \lambda a \equiv c \pmod{b}$$

has u solutions if $u \mid n - c$, and none otherwise. Take $c_1, c_2, \ldots, c_{b-h}$ as before. If $u \mid n - c$, let

$$\lambda_\nu,\ \lambda_\nu + b/u,\ \ldots,\ \lambda_\nu + (u-1)\,b/u$$

be the u solutions of

$$n + \lambda a \equiv c_\nu \pmod{b}.$$

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Then
$$\Omega = \sum_{n=1}^{q} X(n)\,\xi^n \sum_{\lambda=1}^{b} \xi_2^{\lambda} - \sum_{\substack{\nu=1 \\ u \mid n-c_\nu}}^{b-h} \sum_{n=1}^{a} X(n)\,\xi^n \xi_2^{\lambda_\nu} \sum_{r=0}^{u-1} e^{2\pi i m r/u}$$
$$= 0,$$

since $u \nmid m$, and therefore $b \nmid m$.

(iii) Suppose that $(a,b) > 1$, and that $(a,b) \mid m$. Put $b = b'k$, where b' is the largest divisor of b such that $(a,b') = 1$. Also let

$$l = (m,k), \quad q = q_1 l, \quad b = b_1 l, \quad m = m_1 l.$$

Then $\xi = e^{2\pi i m_1/q_1}$, and

$$\Omega = \sum_{\lambda=1}^{l} \sum_{\mu=1}^{q_1} \chi(\lambda q_1 + \mu)\,\xi^{\lambda q_1 + \mu}.$$

Now $$\chi(\lambda q_1 + \mu) = X(\lambda q_1 + \mu)\,\chi_0(\lambda q_1 + \mu) = X(\mu)\,\chi_0(\mu) = \chi(\mu),$$

since $(\lambda q_1 + \mu, q) \geqslant 1$ according as $(\mu, q) \geqslant 1$. Therefore

$$\Omega = \sum_{\lambda=1}^{l} \sum_{\mu=1}^{q_1} \chi(\mu)\,\xi^{\mu} = l \sum_{\mu=1}^{q_1} \chi(\mu)\,\xi^{\mu}. \tag{3·3}$$

We now consider the two cases (α) $l < k$, (β) $l = k$.

(α) If $l < k$, then $k \nmid m$ and $(a, b_1) \nmid m_1$, since

$$(a, b_1) = (a, b'k/l) = (a, k/l) > 1,$$

and $(m_1, k/l) = (m/l, k/l) = 1$. Hence, applying (ii) for q_1 instead of q, we have

$$\sum_{\mu=1}^{q_1} \chi(\mu)\,\xi^{\mu} = 0,$$

since $\chi(\mu)$ is a character modulo q_1, and therefore, by (3·3), $\Omega = 0$.

(β) $l = k$. Then $b_1 = b'$, $m_1 = m'$, and

$$(a, b_1) = (a, b') = 1, \quad k \mid m.$$

Hence, applying (i) to q_1 instead of q, we have

$$\Omega = a^{\frac{1}{2}} k \epsilon(X)\,X(b')\,\overline{X}(m') \sum_{d \mid (m', b')} d\mu(b'/d),$$

by (3·3), since $\chi(n)$ is a character modulo q_1.

This completes the proof of the lemma.

LEMMA 4.* *If X is large*
$$\sum_{1 \leqslant q \leqslant X} \frac{\mu^2(q)}{\phi^2(q)} \sim \log X.$$

The proof is omitted. See, for instance, E. C. Titchmarsh (5), 415–16.

LEMMA 5. *If $\chi(n)$ is a character modulo q and $q < x$, and if*

$$\vartheta(x, \chi) = \sum_{p \leqslant x} \chi(p) \log p,$$

then
$$\vartheta(x, \chi) = xE(\chi) + O(x^{\theta} \log^2 x),$$

where $E(\chi) = 1$ if $\chi = \chi_0$, and $E(\chi) = 0$ otherwise.

* This lemma occurs in *P.N. VII*.

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See E. C. Titchmarsh (5), 425–8, for the case $\Theta = \frac{1}{2}$. If $\frac{1}{2} < \Theta < 1 - \delta$, the lemma follows by similar methods*. If $\Theta = 1$, the lemma is trivial.

LEMMA 6. *If*

$$\Psi(\alpha) = \sum_{\nu=1}^{k} e^{2\pi i \nu \alpha},$$

where

$$\alpha = \frac{a}{q} + z, \quad (a, q) = 1, \quad 0 \leqslant |z| \leqslant \frac{1}{T},$$

and T is large compared with k, then

$$|\Psi(\alpha)|^2 = k + 2\Re \sum_{\nu=1}^{k} (k-\nu) e^{2\pi i \nu \alpha}, \tag{3·4}$$

and

$$|\Psi(\alpha)|^2 = \left|\Psi\left(\frac{a}{q}\right)\right|^2 + O(k^3|z|).$$

(3·4) is obtained by multiplying $\Psi(\alpha)$ by its conjugate. For there are $k-\nu$ solutions of the equation $\nu_1 - \nu_2 = \nu$, when $1 \leqslant \nu_2 \leqslant \nu_1 \leqslant k$. For the second part of the lemma, we have, by (3·4),

$$|\Psi(\alpha)|^2 = k + 2\Re \sum_{\nu=1}^{k} (k-\nu) e^{2\pi i \nu a/q} e^{2\pi i \nu z}$$

$$= \left|\Psi\left(\frac{a}{q}\right)\right|^2 + O\left(\sum_{\nu=1}^{k} |z| \nu(k-\nu)\right) = \left|\Psi\left(\frac{a}{q}\right)\right|^2 + O(k^3|z|).$$

LEMMA 7. *If $\chi(a)$ is a character modulo q, and if $\mu(q) \neq 0$, then*

$$\sum_{a=1}^{q} \chi(a) \left|\Psi\left(\frac{a}{q}\right)\right|^2 = \begin{cases} O(k^4 q^{\frac{1}{2}}), & \text{if } \chi \neq \chi_0. \\ O(k^2 \phi(q)), & \text{if } \chi = \chi_0. \end{cases} \tag{3·5}$$

Also, if $\chi_1(a)$ and $\chi_2(a)$ are characters modulo q, then

$$\sum_{a=1}^{q} \chi_1(a) \overline{\chi}_2(a) \left|\Psi\left(\frac{a}{q}\right)\right|^2 = \begin{cases} O(k^4 q^{\frac{1}{2}}), & \text{if } \chi_1 \neq \chi_2. \\ O(k^2 \phi(q)), & \text{if } \chi_1 = \chi_2. \end{cases} \tag{3·6}$$

It is sufficient to prove (3·5), since $\chi_1(a)\,\overline{\chi}_2(a)$ is a character modulo q, and is the principal character only when $\chi_1 = \chi_2$.

(i) Suppose that $\chi \neq \chi_0$. Then, by Lemma 3, since $\mu(q) \neq 0$,

$$\sum_{a=1}^{q} \chi(a) e^{2\pi i \nu a/q} = q_1^{\frac{1}{2}} \epsilon(X) X(q/q_1) \overline{X}(\nu) \sum_{d | (\nu, q/q_1)} d \mu\left(\frac{q}{d q_1}\right),$$

where $X(a)$ is the primitive character modulo q_1 associated with $\chi(a)$. If $\nu = 0$, each side is zero. Hence, for $0 \leqslant |\nu| \leqslant k$,

$$\sum_{a=1}^{q} \chi(a) e^{2\pi i \nu a/q} = O(q^{\frac{1}{2}} \sum_{d | |\nu|} d) = O(q^{\frac{1}{2}} \sum_{d \leqslant k} d) = O(q^{\frac{1}{2}} k^2).$$

Therefore

$$\sum_{a=1}^{q} \chi(a) \left|\Psi\left(\frac{a}{q}\right)\right|^2 = O(q^{\frac{1}{2}} k^2 . k^2) = O(k^4 q^{\frac{1}{2}}).$$

* Part of the contour of integration will have to be taken more to the left. Instead of integrating along the line joining $\frac{1}{2} - Ti$, $\frac{1}{2} + Ti$, take the line joining $\frac{1}{2}\delta - Ti$, $\frac{1}{2}\delta + Ti$. Other minor modifications are necessary.

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(ii) If $\chi = \chi_0$,

$$\sum_{a=1}^{q} \chi(a) \left| \Psi\left(\frac{a}{q}\right) \right|^2 = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \Psi\left(\frac{a}{q}\right) \right|^2 = O(k^2 \phi(q)).$$

LEMMA 8. *If*

$$\mathfrak{S}(k, X) = \sum_{\nu=1}^{k} (k-\nu) \sum_{1 \leqslant q \leqslant X} \sum_{\substack{0 \leqslant a < q \\ (a,q)=1}} \frac{\mu^2(q)}{\phi^2(q)} e^{2\pi i \nu a/q},$$

then $\mathfrak{S}(k, X)$ *is real, and*

$$\mathfrak{S}(k, X) = \tfrac{1}{2}k^2 + O(k \log k) + O\{k^3 X^{-1} \log^2 (\log X)\}.$$

This lemma may be proved as in G. H. Hardy and J. E. Littlewood(1), §§ 3·21–3·23. We first show that

$$\mathfrak{S}(k, X) = 2 \prod_{p>2} \left\{1 - \frac{1}{(p-1)^2}\right\} \sum_{\nu \leqslant \frac{1}{2}k} (k-2\nu) \prod_{\substack{p \mid \nu \\ p>2}} \left(\frac{p-1}{p-2}\right) + O\{k^3 X^{-1} \log^2 (\log X)\},$$

and the lemma follows fairly easily from this.

4. *Proof of the theorem.* Since the theorem is trivial if $\Theta = 1$, we assume from now on that $\Theta < 1$. Let

$$S(\alpha) = \sum_{N < p \leqslant 2N} e^{2\pi i \alpha p} \log p,$$

where p runs through all the prime numbers in the range $N < p \leqslant 2N$. We consider the integral

$$I(N, k) = \int_0^1 |S(\alpha) \Psi(\alpha)|^2 d\alpha,$$

where

$$\Psi(\alpha) = \sum_{\nu=1}^{k} e^{2\pi i \nu \alpha},$$

and k is an integer depending on N, which is less than $\log N$. Then

$$\begin{aligned} I(N, k) &= \int_0^1 \sum_{p, p'} \sum_{\nu, \nu'} e^{2\pi i \alpha (p - p' + \nu - \nu')} \log p \log p' \, d\alpha \\ &= k \sum_{N < p \leqslant 2N} \log^2 p + 2 \sum_{\nu=1}^{k-1} (k-\nu) \sum_{\substack{N < p' < p \leqslant 2N \\ p - p' = \nu}} \log p \log p' \\ &= kN \log N (1 + o(1)) + 2(1 + o(1)) \log^2 N \sum_{\nu=1}^{k-1} (k-\nu) \varpi(\nu), \qquad (4{\cdot}1) \end{aligned}$$

by the Prime Number Theorem, where $\varpi(\nu)$ is the number of prime pairs p, p' which satisfy $N < p' < p \leqslant 2N$, and $p - p' = \nu$.

Since the integrand is periodic with period 1, we may substitute for the range of integration the range

$$-\frac{1}{T} \leqslant \alpha \leqslant 1 - \frac{1}{T}.$$

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Divide this range into Farey arcs with centre points a/q, where $(a, q) = 1$, and take arcs for which

$$q \leqslant X \leqslant \sqrt{T}/\sqrt{2}. \tag{4·2}$$

We shall consider only a part of each arc, namely the part $\mathfrak{M}_{a,q}$, where

$$\alpha = \frac{a}{q} + z, \quad -\frac{1}{T} \leqslant z \leqslant \frac{1}{T}, \tag{4·3}$$

and take

$$T = Nm^{-\gamma}, \quad m = \log N, \quad \gamma > 1. \tag{4·4}$$

This is possible without overlapping by (4·2). In addition, we consider only those arcs $\mathfrak{M}_{a,q}$ for which q is square-free, i.e. $\mu(q) \neq 0$. Let $\sigma_{a,q}$ be the contribution from the arc $\mathfrak{M}_{a,q}$ to $I(N, k)$. Then

$$I(N, k) > \sum_{1 \leqslant q \leqslant X} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \sigma_{a,q}. \tag{4·5}$$

5. *Estimation of $S(\alpha)$ on $\mathfrak{M}_{a,q}$.* Put

$$S_l(\alpha) = \sum_{\substack{N < p \leqslant 2N \\ p \equiv l \pmod q}} e^{2\pi i a p} \log p,$$

where $(l, q) = 1$. Then

$$S(\alpha) = \sum_{\substack{l=1 \\ (l,q)=1}}^{q} S_l(\alpha), \tag{5·1}$$

and

$$S_l(\alpha) = e^{2\pi i l a/q} \sum_{\substack{N < p \leqslant 2N \\ p \equiv l \pmod q}} e^{2\pi i z p} \log p.$$

If

$$\vartheta_l(x) = \sum_{\substack{p \leqslant x \\ p \equiv l \pmod q}} \log p,$$

then

$$\sum_{\substack{N < p \leqslant 2N \\ p \equiv l \pmod q}} e^{2\pi i z p} \log p = \int_N^{2N} e^{2\pi i z x} \, d\vartheta_l(x)$$

$$= \vartheta_l(2N)\, e^{4\pi i z N} - \vartheta_l(N)\, e^{2\pi i z N} - 2\pi i z \int_N^{2N} \vartheta_l(x)\, e^{2\pi i z x} \, dx.$$

Now

$$\vartheta_l(x) = \frac{1}{\phi(q)} \sum_{p \leqslant x} \log p \sum_{\chi} \overline{\chi}(l)\, \chi(p)$$

$$= \frac{1}{\phi(q)} \sum_{\chi} \overline{\chi}(l)\, \vartheta(x, \chi), \tag{5·2}$$

in the notation of Lemma 5. Put

$$\vartheta(x, \chi) = x E(\chi) + \xi(x, \chi).$$

Then, by Lemma 5,

$$\xi(x, \chi) \ll O(x^{\theta} \log^2 x). \tag{5·3}$$

By (5·2),

$$\vartheta_l(x) = \frac{x}{\phi(q)} + \frac{1}{\phi(q)} \sum_{\chi} \overline{\chi}(l)\, \xi(x, \chi).$$

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Therefore $$\phi(q)\sum_{\substack{N<p\leqslant 2N\\ p\equiv l(\mathrm{mod}\,q)}} e^{2\pi izp}\log p = e^{3\pi izn}\frac{\sin\pi zN}{\pi z}+\sum_{\chi}\bar{\chi}(l)\,\Xi(N,\chi),$$

where

$$\begin{aligned}\Xi(N,\chi) &= \xi(2N,\chi)\,e^{4\pi izN}-\xi(N,\chi)\,e^{2\pi izN}-2\pi iz\int_N^{2N}\xi(x,\chi)\,e^{2\pi izx}\,dx\\ &= O\{(1+N|z|)\max_{N\leqslant x\leqslant 2N}|\xi(x,\chi)|\} = O(N^{\theta}m^{2+\gamma}),\end{aligned}\tag{5·31}$$

by (4·3), (4·4), and (5·3). Hence,

$$\sum_{\substack{l=1\\(l,q)=1}}^{q} S_l(\alpha) = e^{3\pi izN}\frac{\sin\pi zN}{\pi z}\sum_{\substack{l=1\\(l,q)=1}}^{q}\frac{e^{2\pi ial/q}}{\phi(q)}+\frac{1}{\phi(q)}\sum_{\chi\,\mathrm{mod}\,q}\Xi(N,\chi)\sum_{\substack{l=1\\(l,q)=1}}^{q}\bar{\chi}(l)\,e^{2\pi ial/q}.$$

Let $X(n)$ be the primitive character associated with $\chi(n)$, and let its modulus be q_1. Then, by Lemmas 1, 2 and 3, since $(a,q)=1$,

$$\begin{aligned}S(\alpha) &= \frac{\mu(q)}{\phi(q)}e^{3\pi izN}\frac{\sin\pi zN}{\pi z}+\frac{q_1^{\frac{1}{2}}}{\phi(q)}\sum_{\chi\neq\chi_0}\Xi(N,\chi)\,\epsilon(\bar{X})\,\bar{X}\left(\frac{q}{q_1}\right)X(a)\,\mu\left(\frac{q}{q_1}\right)\\ &\qquad+\frac{\mu(q)}{\phi(q)}\Xi(N,\chi_0)\\ &= \frac{\mu(q)}{\phi(q)}e^{3\pi izN}\frac{\sin\pi zN}{\pi z}+\frac{1}{\phi(q)}\sum_{\chi}\chi(a)f(\chi),\end{aligned}$$

say, where

$$f(\chi)=\begin{cases}\Xi(N,\chi)\,\epsilon(\bar{X})\,q_1^{\frac{1}{2}}\bar{X}\left(\dfrac{q}{q_1}\right)\mu\left(\dfrac{q}{q_1}\right) & (\chi\neq\chi_0),\\ \Xi(N,\chi)\,\mu(q) & (\chi=\chi_0),\end{cases}\tag{5·4}$$

since $X(a)=\chi(a)$. $f(\chi)$ is independent of a, and

$$f(\chi) = O(q^{\frac{1}{2}}N^{\theta}m^{2+\gamma}),\tag{5·41}$$

by (5·31). Hence

$$\begin{aligned}|S(\alpha)|^2 &= \frac{\mu^2(q)}{\phi^2(q)}\frac{\sin^2\pi zN}{\pi^2z^2}+\frac{\mu(q)}{\phi^2(q)}\frac{\sin\pi zN}{\pi z}\left\{e^{-3\pi izN}\sum_{\chi}\chi(a)f(\chi)+e^{3\pi izN}\sum_{\chi}\bar{\chi}(a)\overline{f(\chi)}\right\}\\ &\qquad+\frac{1}{\phi^2(q)}\sum_{\chi_1}\sum_{\chi_2}\chi_1(a)\,\bar{\chi}_2(a)f(\chi_1)\overline{f(\chi_2)}\\ &= \frac{\mu^2(q)}{\phi^2(q)}\frac{\sin^2\pi zN}{\pi^2z^2}+\eta_1+\eta_2,\end{aligned}\tag{5·5}$$

say. Also, by (5·41),

$$|S(\alpha)|^2 = O\left(\frac{1}{\phi^2(q)}\frac{\sin^2\pi zN}{z^2}\right)+O(qN^{2\theta}m^{4+2\gamma}).\tag{5·6}$$

6. *Contributions from the arcs $\mathfrak{M}_{a,q}$ to the integral $I(N,k)$.* By Lemma 6,

$$|S(\alpha)\,\Psi(\alpha)|^2 = |S(\alpha)|^2\left|\Psi\left(\frac{a}{q}\right)\right|^2+O(k^3|z|\,|S(\alpha)|^2).$$

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Therefore

$$\sigma_{a,q} = \int_{-1/T}^{1/T} \left| S\left(\frac{a}{q}+z\right) \Psi\left(\frac{a}{q}\right) \right|^2 dz + O\left(k^3 \int_{-1/T}^{1/T} |z| \left| S\left(\frac{a}{q}+z\right) \right|^2 dz\right)$$
$$= \tau_{a,q} + E(a,q), \tag{6·1}$$

say. By (5·6), $E(a,q) = O\{k^3(m^\gamma \phi^{-2}(q) + qN^{2\theta-2}m^{4+4\gamma})\}$,

since $$\int_{-1/T}^{1/T} \frac{\sin^2 \pi z N}{\pi^2 z^2} dz = N\{1 + O(m^{-\gamma})\}.$$

Hence, by (6·1), since $k < m$,

$$\sigma_{a,q} = \tau_{a,q} + O(m^{3+\gamma}\phi^{-2}(q)) + O(qm^{7+4\gamma}N^{2\theta-2}).$$

Therefore

$$\sum_{1 \leqslant q \leqslant X} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \sigma_{a,q} = \sum_{1 \leqslant q \leqslant X} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \tau_{a,q} + O(m^{3+\gamma} \log X) + O(N^{2\theta-2} m^{7+4\gamma} X^3), \tag{6·2}$$

by Lemma 4. By (5·5) and (6·1),

$$\tau_{a,q} = \int_{-1/T}^{1/T} \left| S\left(\frac{a}{q}+z\right) \Psi\left(\frac{a}{q}\right) \right|^2 dz$$
$$= \frac{\mu^2(q)}{\phi^2(q)} \left| \Psi\left(\frac{a}{q}\right) \right|^2 \int_{-1/T}^{1/T} \frac{\sin^2 \pi z N}{\pi^2 z^2} dz + \left| \Psi\left(\frac{a}{q}\right) \right|^2 \int_{-1/T}^{1/T} (\eta_1 + \eta_2) dz$$
$$= \frac{\mu^2(q)}{\phi^2(q)} \left| \Psi\left(\frac{a}{q}\right) \right|^2 N\{1 + O(m^{-\gamma})\} + \left| \Psi\left(\frac{a}{q}\right) \right|^2 \int_{-1/T}^{1/T} (\eta_1 + \eta_2) dz. \tag{6·3}$$

Also

$$\sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \eta_1 \left| \Psi\left(\frac{a}{q}\right) \right|^2 = O\left\{ \frac{1}{\phi^2(q)} \left| \frac{\sin \pi z N}{z} \right| \sum_{\chi \neq \chi_0} |f(\chi)| \, k^4 q^{\frac{1}{2}} \right\}$$
$$+ O\left\{ \frac{1}{\phi^2(q)} \left| \frac{\sin \pi z N}{z} \right| k^2 \phi(q) \, |f(\chi_0)| \right\}$$
$$= O\left(\left| \frac{\sin \pi z N}{z} \right| \frac{q}{\phi(q)} N^\theta m^{6+\gamma} \right),$$

by Lemma 6, (5·5), and (5·41), since $\mu(q) \neq 0$. Therefore

$$\sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \left| \Psi\left(\frac{a}{q}\right) \right|^2 \int_{-1/T}^{1/T} \eta_1 dz = O\left\{ \frac{q}{\phi(q)} N^\theta m^{6+\gamma} \int_0^{1/T} \left| \frac{\sin \pi z N}{z} \right| dz \right\}$$
$$= O\left(\frac{q}{\phi(q)} N^\theta m^{6+\gamma} \log m \right). \tag{6·4}$$

By (5·5), $$\eta_2 = \frac{1}{\phi^2(q)} \sum_{\chi_1} \sum_{\chi_2} \chi_1(a) \overline{\chi}_2(a) f(\chi_1) \overline{f(\chi_2)},$$

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and therefore

$$\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\eta_2\left|\Psi\left(\frac{a}{q}\right)\right|^2 = O\left\{\frac{1}{\phi^2(q)}\sum_{\chi_1\neq\chi_2}|f(\chi_1)f(\chi_2)|\,q^{\frac{1}{2}}k^4\right\} + O\left\{\frac{1}{\phi^2(q)}\sum_{\chi_1=\chi_2}|f(\chi_1)|^2k^2\phi(q)\right\}$$
$$= O(q^{\frac{1}{2}}N^{2\Theta}m^{8+2\gamma}) + O(qN^{2\Theta}m^{6+2\gamma})$$
$$= O(q^{\frac{1}{2}}N^{2\Theta}m^{8+2\gamma}),$$

by Lemma 6, since $\mu(q)\neq 0$. Hence

$$\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\left|\Psi\left(\frac{a}{q}\right)\right|^2\int_{-1/T}^{1/T}\eta_2\,dz = O(q^{\frac{1}{2}}N^{2\Theta-1}m^{8+3\gamma}). \tag{6·5}$$

Combining (6·3), (6·4), and (6·5), we have

$$\sum_{1\leqslant q\leqslant X}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\tau_{a,q} = \sum_{1\leqslant q\leqslant X}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\left|\Psi\left(\frac{a}{q}\right)\right|^2\frac{\mu^2(q)}{\phi^2(q)}N\{1+O(m^{-\gamma})\}$$
$$+ O\left(N^{\Theta}m^{6+\gamma}\log m\sum_{1\leqslant q\leqslant X}\frac{q}{\phi(q)}\right) + O\left(N^{2\Theta-1}m^{8+3\gamma}\sum_{1\leqslant q\leqslant X}q^{\frac{1}{2}}\right)$$
$$= N\{1+O(m^{-\gamma})\}\sum_{1\leqslant q\leqslant X}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\left|\Psi\left(\frac{a}{q}\right)\right|^2\frac{\mu^2(q)}{\phi^2(q)}$$
$$+ O(N^{\Theta}m^{6+\gamma}X\log m\log\log X) + O(N^{2\Theta-1}m^{8+3\gamma}X^{\frac{1}{2}}). \tag{6·6}$$

The first term on the right-hand side is of order N at least; we therefore choose X to satisfy (4·2), and to make the two error terms $o(N)$. This will be so if we take

$$X = N^{\frac{1}{2}(1-\Theta)}m^{-8(1+\gamma)}. \tag{6·7}$$

Hence, by (6·2), (6·6), and (6·7),

$$\sum_{1\leqslant q\leqslant X}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\sigma_{a,q} = N\{1+O(m^{-\gamma})\}\sum_{1\leqslant q\leqslant X}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\left|\Psi\left(\frac{a}{q}\right)\right|^2\frac{\mu^2(q)}{\phi^2(q)} + o(N)$$
$$= N\{1+o(1)\}\sum_{1\leqslant q\leqslant X}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}k\frac{\mu^2(q)}{\phi^2(q)}$$
$$+ 2N\{1+o(1)\}\,\Re\sum_{1\leqslant q\leqslant X}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\frac{\mu^2(q)}{\phi^2(q)}\sum_{\nu=1}^{k-1}(k-\nu)\,e^{2\pi i\nu a/q} + o(N)$$
$$= kN\{1+o(1)\}\sum_{1\leqslant q\leqslant X}\frac{\mu^2(q)}{\phi(q)} + 2N\{1+o(1)\}\,\mathfrak{S}(k,X) + o(N)$$
$$= kN\{1+o(1)\}\log X + 2N\{1+o(1)\}\,\mathfrak{S}(k,X)$$
$$= \tfrac{1}{5}(1-\Theta)\,kNm\{1+o(1)\} + k^2N\{1+o(1)\}, \tag{6·8}$$

by Lemmas 4 and 8.

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7. Finally, using (4·1), (4·5), and (6·8), we have

$$Nm\{1+o(1)\}+2m^2k^{-1}\{1+o(1)\}\sum_{\nu=1}^{k-1}(k-\nu)\,\varpi(\nu)>N\{\tfrac{4}{5}(1-\Theta)\,m+k\}\{1+o(1)\}. \tag{7·1}$$

We now take

$$k=\left[\left(\frac{1+4\Theta}{5}+\epsilon\right)\log N\right]+1 \tag{7·2}$$

for any fixed $\epsilon>0$. From (7·1), we obtain

$$\tfrac{1}{5}(1+4\Theta)\,Nm\{1+o(1)\}>kN\{1+o(1)\}, \tag{7·3}$$

if we assume that $\varpi(\nu)=0$ for $\nu=1, 2, \ldots, k-1$. But if we take N sufficiently large (ϵ being fixed) (7·3) is false. Hence $\varpi(\nu)>0$ for some ν in the range

$$1\leqslant\nu\leqslant k-1.$$

The theorem now follows from this and (7·2).

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