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Completely stretched sets

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Introduction. All the sets X considered in this paper will be compact convex subsets of the Euclidean plane R. We shall require the following definitions. For $x, y, z \in R$ let $L(x, y, x)$ denote the minimum of all the lengths† of all connected sets containing x, y and z. Define the stretch of x with respect to X to be

$$L(X: x) = \sup_{y,z \in X} L(x, y, z)$$

and the stretch of X to be

$$L(X) = \sup_{x \in X} L(X; x).$$

The main purpose of this paper is to give some characterizations of those sets X with the property that $L(X; x) = \lambda$ for some constant λ for each x in the frontier ∂X of X. Denote this class of sets by $L(\lambda)$. If we necessarily require $y = z$ in the above definitions then $L(\lambda)$ is the class of sets of constant width λ.

We shall show that the sets in $L(\lambda)$ possess similar kinds of completeness properties to the sets of constant width and that this enables us to give a characterization of the inrevolvable curves of the regular hexagon.

For any two sets X, Y let (X, Y) denote the convex hull of X and Y. We shall write xy to denote both the line segment joining x and y and also the distance between x and y. For any three points x, y and z let $\phi(x, y, z)$ denote the connected set of minimal length containing x, y and z. We shall say that a set X of stretch λ is completely stretched if and only if $x \notin X$ implies $L(X; x) > \lambda$. We say that a support line l to X supports X regularly or that l is a regular support line to X if it meets X in a unique point.

LEMMA 1. *Let ABC be a triangle. On AB erect the triangle which is equilateral and such that D lies on the side of AB opposite to C.*

(*a*) *If each angle of ABC is less than $\frac{2}{3}\pi$ let K be the unique point such that*

$$A\hat{K}B = B\hat{K}C = C\hat{K}A = \tfrac{2}{3}\pi.$$

Then (i) *of all connected sets containing A, B and C each has length greater than or equal to $AK + BK + CK$*; (ii) *$AK + BK + CK = CD$.*

(*b*) *If one angle of ABC is greater than or equal to $\frac{2}{3}\pi$ then of all connected sets containing A, B and C each has length greater than or equal to the sum of the lengths of the two sides which contain that angle.*

Proof. The proof of (*a*) is given in (1) on page 63. The proof of (*b*) involves arguments similar to those used in (*a*), (i) and is omitted.

† For any connected set S define the length of S to be the linear measure of S.

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In view of Lemma 1 either:

(*a*) Each angle of the triangle x, y, z is less than $\frac{2}{3}\pi$ in which case $\phi(x, y, z)$ consists of the union of three segments wx, wy and wz where $x\hat{w}y = y\hat{w}z = x\hat{w}z = \frac{2}{3}\pi$.

Or (*b*) the angle at one vertex w of the triangle xyz is greater than or equal to $\frac{2}{3}\pi$ in which case $\phi(x, y, z)$ consists of the union of the two segments incident at that vertex w.

We shall say that $\phi(x, y, z)$ has three segments or two segments depending on whether case (*a*) or case (*b*) occurs respectively and that $\phi(x, y, z)$ has centre of connexion w.

LEMMA 2. *If for any set X, $L(X) = L(X, x) = L(x, y, z)$, where $x \in \partial X$, if $\phi(x, y, z)$ has centre of connexion w, and if v is any vertex of triangle x, y, z, then if $w \neq v$ the line through v perpendicular to wv supports X regularly and if $w = v$ (e.g. if $\phi(x, y, z)$ has two segments) then the external bisector of the obtuse angle between the two lines of the triangle incident at w supports X regularly.*

Proof. We consider two cases.

(i) If $\phi(x, y, z)$ has three segments:

Let u be the third vertex of the equilateral triangle uyz which lies on the side of yz opposite to x and suppose that the line l through x perpendicular to wx does not support X regularly. Then there is a point $x' \neq x$ in $l \cap X$ such that each angle of triangle $x'yz < \frac{2}{3}\pi$, and so by Lemma 1,

$$L(X) \geqslant L(x', y, z) = x'u > xu = L(x, y, z) = L(X).$$

This is impossible and so l supports X regularly.

(ii) If $\phi(x, y, z)$ has two segments:

Suppose $x \neq w$ and let l be the line through x perpendicular to wx. Then if the angle at w is greater than $\frac{2}{3}\pi$ and if l does not support X regularly, then there is a point $x' \neq x$ in $l \cap X$ such that the angle at w of triangle $x'yz$ is greater than $\frac{2}{3}\pi$. Then if without loss in generality $y = w$ we have

$$L(X) \geqslant L(X', y, z) = x'w + wz > xw + wz = L(x, y, z) = L(X).$$

This is impossible and so again l supports X regularly. If the angle at w is equal to $\frac{2}{3}\pi$ then a combination of the above methods again imply that l supports X regularly.

Next suppose $x = w$. Let l be the external bisector of the obtuse angle of triangle wyz at w. Then of all positions of the point r on l, $ry + rz$ takes a minimum when $r = w$. If the angle at w is greater than $\frac{2}{3}\pi$ and l does not support X regularly then there is a point $w' \neq w$ in $X \cap l$ such that $y\hat{w}'z$ is greater than $\frac{2}{3}\pi$. Hence

$$L(X) \geqslant L(w', y, z) = w'y + w'z > wy + wz = L(x, y, z) = L(X).$$

Again this is impossible and so l supports X regularly. If the angle at w is equal to $\frac{2}{3}\pi$, u is defined as in part (1) of this lemma and l does not support X regularly then there is a point $w' \neq w$ on l. We may assume w' is near w. Then if each angle of triangle $w'yz$ is less than $\frac{2}{3}\pi$, $L(w', y, z) = uw'$ and if $yw'z$ is greater than or equal to $\frac{2}{3}\pi$ then

$$L(w', y, z) = w'y + w'z.$$

Thus in all cases,

$$L(X) \geqslant L(w', y, z) > uw = L(x, y, z) = L(X).$$

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This is again impossible and so l supports X regularly. Since

$$L(X; x) = L(X; y) = L(X; z)$$

Lemma 2 is proved.

THEOREM 1. *$L(X; x) = \lambda$ for each $x \in \partial X$ if and only if X is completely stretched with stretch λ.*

Proof. We assume X is not a segment for in this case the result is trivial.

Suppose first the $L(X; x) = \lambda$ for each $x \in \partial X$. By the compactness of X,

$$L(X) = L(x, y, z)$$

for three points x, y, z in X and it is obvious that x, y and z each belong to ∂X. Hence X has stretch λ.

Choose $x_1 \notin X$ and let x be that point of X which is nearest to x_1. Then the line through x perpendicular to x_1, x supports X and this together with Lemma 2 implies that $L(X; x_1) > L(X)$. Thus the first part of the theorem is proved.

We suppose now that X is such that if $x \notin X$ then $L(X; x) > \lambda$, where $\lambda = L(X)$.

If the result is false then $\exists x_0 \in \partial X$ with $L(X; x_0) < L(X)$, where $L(X; x_0) = L(x_0, y_0, z_0)$ for some $y_0, z_0 \in \partial X$.

Let $D(x_0, \delta)$ denote the closed disc radius δ centre x_0.

Then if δ is sufficiently small $L(x, y, z) < L(X)$ for all x, y and z with $x \in D(x_0, \delta)$ and $y, z \in D(x_0, \delta) \cup X$.

For suppose that is not the case. Then $\exists$ a sequence $\delta_i (i = 1, 2, \ldots, \infty)$ tending to zero as i tends to infinity and points x_i, y_i and z_i $(i = 1, 2, \ldots, \infty)$ with $x_i \in D(x_0, \delta)$ and $y_i, z_i \in D(x_0, \delta) \cup X$ with $L(x_i, y_i, z_i) \geqslant L(X)$. Then x_i tends to x_0 as i tends to infinity and by extracting subsequences if necessary it follows that there exist points y' and z' in X such that y_i tends to y' and z_i tends to z' as i tends to infinity. Hence

$$\begin{aligned} L(X; x_0) \geqslant L(x_0, y', z') &= \lim_{i \to \infty} L(x_i, y_i, z_i) \\ &\geqslant L(X) \\ &> L(X; x_0). \end{aligned}$$

This is impossible and so $\exists \delta > 0$ with $L(x, y, z) < (L(X))$ for all x, y and z with $z \in D(x_0, \delta)$ and $y, z \in D(x_0, \delta) \cup X$.

We now choose any $x_1 \in D(x_0, \delta)$ and $x_1 \notin X$.

Then $L(X; x_1) \leqslant L(X)$.

This is impossible and so $L(X; x) = L(X) = \lambda$ for all $x \in \partial X$.

Thus the theorem is proved.

THEOREM 2. *If X has stretch λ then X is contained in a set Y of stretch λ which is completely stretched.*

Proof. Define

$$V(X) = \{x | L(X; x) = L(X)\},$$

$$\rho(X) = \sup_{x \in V(X)} \rho(X, x) \quad \text{where} \quad \rho(X, x) = \inf_{y \in X} xy,$$

$$W(x) = \{x | x \in V(X) \quad \text{and} \quad \rho(X, x) = \rho(X)\}.$$

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Select $x_1 \in W(X)$ and write $X_1 = (X, x_1)$.

Inductively define $X_{i+1} = (X_i, x_{i+1})$ with x_{i+1} selected from $W(X_i)$. We prove next that $L(X_i) = L((X_i, x_{i+1})) = L(X_i; x_{i+1})$.

Suppose $L((X_i, x_{i+1})) = L(u, v, w)$. Since by Lemma 2 there are support lines at u, v and w respectively which are regular, it follows that each of u, v and w is a member of $X_i \cup x_{i+1}$. The statement is proved. Thus for all i and j, $L(X_i) = L(X_j) = L(X)$.

The sequence of sets X_i $(i = 1, \ldots, \infty)$ converge to a set Y which is the closure of $\bigcup_{i=1}^{\infty} X_i$, under the usual Hausdorff metric and since also it is clear that $L(X)$ is a continuous function of X under this metric it follows that $L(X) = L(X_i) = L(Y)$ for each i. The remainder of this proof follows in exactly the same way as an analogous theorem on the sets of constant width which is given in (2) and so to save space is omitted.

THEOREM 3. *(a) If X is completely stretched with stretch λ then*

$$0 \leqslant d(X) \leqslant \tfrac{2}{3}\lambda \leqslant D(X) \leqslant \lambda,$$

where $d(X)$, $D(X)$ denote respectively the minimal width and diameter of X.

(b) X is a rotor of a regular hexagon of side $2\lambda/(3\sqrt{3})$ if and only if X is completely stretched with stretch λ and $d(X) = \frac{2}{3}\lambda$ or $D(X) = \frac{2}{3}\lambda$.

(We say that X is a rotor for a regular polygon P if and only if X can be rotated continuously inside P in such a way that each side of P is always a support line to X.)

Proof. We first show that if for each $x \in \partial X$, $\lambda = L(X) = L(X; x) = L(x, y, z)$ for some y and z (depending on x) where $\phi(x, y, z)$ has three segments then X is a rotor for an equilateral triangle of height λ. By Lemma 2 it follows that X is circumscribed by at least one equilateral triangle T of height λ. Suppose there exists an equilateral triangle T' of height $\mu > \lambda$ which circumscribes X. Then if x', y' and z' are three points of X, one on each side of T' we have $L(X) \geqslant L(x', y', z') \geqslant \mu > \lambda = L(X)$. But this is impossible and so we can assume that X can be rotated continuously inside T and is circumscribed by T in at least one orientation. We shall suppose that X is fixed and that T is moved around X so as always to contain X. Let T have vertices L, M, N and suppose that MN makes an angle θ, $0 \leqslant \theta < 2\pi$ with some fixed line.

We shall suppose L, M, N are labelled in anti-clockwise order around T with respect to an interior point of X and that θ increases as T is moved in a clockwise sense.

θ will be called the orientation of T.

We suppose the result false.

Then there exist orientations, θ_0, θ_1 of T such that T circumscribes X when $\theta = \theta_0$ and $\theta = \theta_1$, but does not circumscribe X for any θ in the interval $\theta_1 > \theta > \theta_0$. Let LM, LN and MN support X at points x_1, x_2 and x_3 respectively when $\theta = \theta_0$. It is clearly impossible that $x_1 = x_2 = x_3$ and so we can assume without any real loss in generality that $x_1 \neq x_2$.

Now X is not a segment and by Lemma 2 it follows that ∂X does not contain a segment.

Thus there exists a point $p \in \partial X$ strictly on the same side of the line $x_1 x_2$ as L, $p \neq x_1$,

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$p \neq x_2$ and moreover no support line to X through p can contain either x_1 or x_2. Let $\theta_2 > \theta_0$ be a value of θ for which LM is a support line to X at p.

Let $\theta_3 = \min(\theta_2, \theta_1)$. Then $0 < |\theta_3 - \theta_0| < \frac{2}{3}\pi$ since px_1 and px_2 are not support lines of X.

We suppose that LM, LN, MN support X at points y_1, y_2 and y_3 respectively when $\theta = \theta_1$. We consider two cases:

(I) $\theta_3 = \theta_1$. Thus $0 < |\theta_1 - \theta_0| < \frac{2}{3}\pi$.

Now it is not possible that $y_i = x_i$ for each $i = 1, 2$ and 3 for if this were the case we would have to have $|\theta_0 - \theta_1| = \frac{2}{3}\pi$, which is impossible.

We suppose then that $y_1 \neq x_1$. Thus we can choose $q \in \partial X$, $q \neq x_1$, $q \neq y_1$ and then any support line to X at q makes an angle θ with the given straight line, where

$$\theta_0 < \theta < \theta_1.$$

By Lemma 2 there exists an equilateral triangle of height λ which circumscribes X with one side supporting X at q. This contradicts the choice of θ_1 as no support line through q is a support line of X at x_1 or y_1. The cases $y_2 \neq x_2$ and $y_3 \neq x_3$ are similarly proved to be impossible. Thus we must have:

(II) $\theta_3 = \theta_2$.

This time we choose $q \in \partial X$ strictly between x_1 and p so that any support line to X at q makes an angle θ with the given fixed line where $\theta_0 < \theta < \theta_2 \leqslant \theta_1$.

We obtain a contradiction in the same way as in case (I) and so the statement is proved. We have shown that if for each $x \in \partial X$, $\lambda = L(X) = L(X; x) = L(x, y, z)$ for some y and z (depending on x) where $\phi(x, y, z)$ has three segments then X is a rotor for an equilateral triangle of height λ. Thus by (3), p. 106, the perimeter of X is equal to

$$\tfrac{2}{3}\pi\lambda = \frac{1}{2}\int_0^{2\pi} B(\phi)\, d\phi,$$

where $B(\phi)$ is the width of X in the direction ϕ. Hence $d(X) \leqslant \frac{2}{3}\lambda \leqslant D(X)$.

We suppose now that there exists $x \in \partial X$, $\lambda = L(X) = L(X; x) = L(x, y, z)$ for some y and z (depending on x), where $\phi(x, y, z)$ has two segments. We assume $y\hat{x}z \geqslant \frac{2}{3}\pi$.

Then $\lambda = yx + zx$ and $(D(X))^2 \geqslant yz^2 = (yx - \frac{1}{2}\lambda)^2 + \frac{3}{4}\lambda^2$ and so $D(X) \geqslant (\frac{1}{2}\sqrt{3})\,\lambda > \frac{2}{3}\lambda$.

This implies also that X contains a segment uv of length $[\sqrt{(3)}/2]\,\lambda$.

Let γ_1 be the small arc of a circle which contains u and v and is such that uv subtends an angle $\frac{2}{3}\pi$ on γ_1.

Let γ_2 be the reflexion of γ_1 in uv. Let l, m be support lines to γ_1 and γ_2 respectively which are parallel to uv and are on either side of uv. By Lemma 1 it follows that X is contained in the strip bounded by l and m.

Thus $d(X) \leqslant \frac{1}{2}\lambda$.

Thus in all cases $0 \leqslant d(X) \leqslant \frac{2}{3}\lambda \leqslant D(X) \leqslant \lambda$, and (a) is proved. We now prove (b).

Suppose now $d(X) = \frac{2}{3}\lambda$. Then X is necessarily a rotor of an equilateral triangle T of height λ and thus has diameter $D(X) = \frac{2}{3}\lambda$.

Hence X has constant width $\frac{2}{3}\lambda$ also and thus is a rotor for a regular hexagon of side $(2\lambda)/3\sqrt{3}$.

If $D(X) = \frac{2}{3}\lambda$ the result is similar. If on the other hand X is a rotor of a regular

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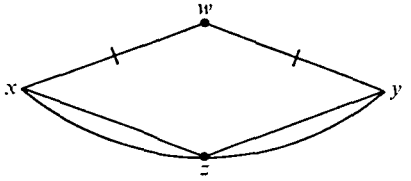
hexagon of side $(2\lambda)/3\sqrt{3}$, then X has constant width $\frac{2}{3}\lambda$ and this implies that for each $x \in \partial x$ the stretch of x, $L(X; x) = L(x, y, z)$ for some y, z (depending on x) is attained by some $\phi(x, y, z)$ which has three segments. Thus since X is also in particular a rotor of an equilateral triangle of height λ we must have by Lemma 2 that $L(X; x) = \lambda$ for each $x \in \partial X$. The theorem is proved.

Note. The completely stretched sets with stretch λ which have diameter $D(X)$, where $(\frac{1}{2}\sqrt{3})\lambda \geqslant D(X) \geqslant \frac{2}{3}\lambda$ are necessarily rotors of an equilateral triangle of height λ. This is a consequence of the last theorem. However, this is not necessarily true for the completely stretched sets of larger diameters since:

THEOREM 4. *Given any number μ, $(\frac{1}{2}\sqrt{3})\lambda < \mu \leqslant \lambda$ there exists a completely stretched set Y with stretch λ and diameter μ which is not a rotor for an equilateral triangle of height λ.*

Proof. If $\mu = \lambda$ we take Y to be a segment of length λ and the result is obvious. We suppose then that $(\frac{1}{2}\sqrt{3})\lambda < \mu < \pi$.

Let xy be a segment of length μ. Let w, z be points one on either side of xy with $xw = yw = yw = xz = yz = \frac{1}{2}\lambda$. Draw the arc γ of the circle xyz. Let X be the set bounded by γ and the segments xw and yw, we have that $z\hat{z}y = x\hat{w}y > \frac{2}{3}\pi$.



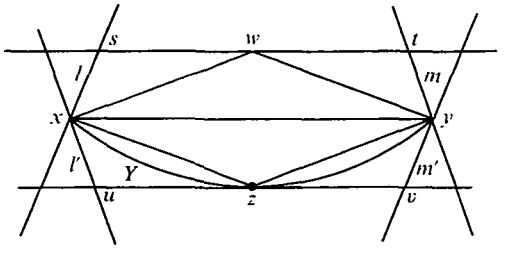
In view of Lemma 2 it follows that any three points which attain $L(X)$ must belong to $\gamma \cup \{w\}$. Thus it is not difficult to see that $L(X) = xz + yz = \lambda$. Now by Theorem 2 there exists a set Y which is completely stretched with stretch λ and which contains X.

Since $L(Y) = \lambda = L(x, y, z)$ it follows that x, y and z each belong to the frontier of Y and that there exist support lines l, m, n to Y with l passing through x perpendicular to xz, m passing through y perpendicular to zy and n passing through z parallel to xy. n is the unique support line to Y at z since n is the unique support line to X at z. Since also $L(Y) = L(x, y, w)$ we can define corresponding lines l', m', n' through x, y, w respectively, which are parallel to m, l and n respectively and which also support $L(Y)$.

Let n meet l' and m' in u and v respectively.

Let n' meet l and m in s and t respectively.

Then Y is contained in the hexagon $xstyvu$.



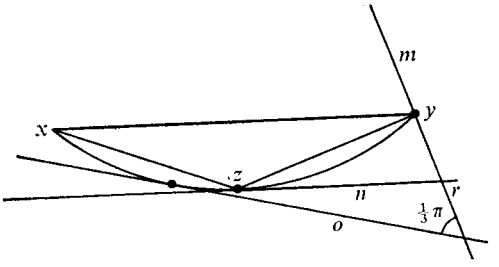
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By considering the parallelogram $xuyt$ it is obvious that the diameter of this hexagon is μ. Thus the diameter of Y is μ. We shall finally show that Y is not a rotor for an equilateral triangle T of height λ.

For suppose this was the case.

Consider the orientation of T when one side of T coincides with m. Let the lines m and n meet in r.

Since $x\hat{z}y > \frac{2}{3}\pi$ it follows that $y\hat{r}z > \frac{1}{3}\pi$.



Now let o be the line which supports Y and which makes an angle $\frac{1}{3}\pi$ with m. Then o contains one side of T when T has the above orientation. Also o does not contain z since n is the unique support line to Y at z. But as Y is a rotor of T the normals at the points of contact of Y with T are coincident by (3), p. 103. Thus in the above orientation of T the normals meet in some point q which lies on yz produced, and thus exterior to Y and thus exterior to T.

But this implies that $L(Y) > \lambda$ by considering the three points of contact of Y with T in this orientation. Thus the assumption that Y was a rotor of T was false and the theorem is proved.

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