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THE DICKEY–FULLER TEST FOR EXPONENTIAL RANDOM WALKS

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A common test in econometrics is the Dickey–Fuller test, which is based on the test statistic $DF(T) = T((\sum_1^T y_{t-1}y_t)/(\sum_1^T y_{t-1}^2) - 1)$. We investigate the behavior of the test statistic if the data y_t are given by an exponential random walk $\exp(Z_t)$ where $Z_t = Z_{t-1} + \sigma\epsilon_t$ and the ϵ_t are independent and identically distributed random variables. The test statistic $DF(T)$ is a nonlinear transformation of the partial sums of ϵ_t process. Under certain moment conditions on the ϵ_t we show that $\lim_{T\to\infty}\mathbb{P}(-1 \le DF(T)/T \le -\lambda)$ tends to one as $\lambda \to 0$. For the particular case that the ϵ_t define a simple random walk it is shown that $\operatorname{plim}_{T\to\infty} DF(T)/T$ exists and the limit is evaluated. The theoretical results are illustrated by some simulation experiments.

1. INTRODUCTION

A common test in econometrics is the Dickey–Fuller test, which for data y_t, $t = 0,\ldots,T$ is defined by the test statistic

$$DF(T) = T\left(\frac{\sum_1^T y_{t-1}y_t}{\sum_1^T y_{t-1}^2} - 1\right).$$

The purpose of the test is to investigate the adequacy of a random walk model for the data y_t. In this paper we investigate the behavior of the test statistic if the data are better described by an exponential random walk. To be precise we consider a stochastic process $Y_t, t = 0,\ldots,T$ given by

$$Y_t = \exp(Z_t),$$

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where

$$Z_t = Z_{t-1} + \sigma\epsilon_t \tag{1}$$

and the ϵ_t are independent and identically distributed (i.i.d.) random variables. We write the test statistic in the form

$$DF(T) = T(\hat{\rho}(T) - 1), \tag{2}$$

where

$$\hat{\rho}(T) = \frac{\sum_1^T Y_{t-1}Y_t}{\sum_1^T Y_{t-1}^2}.$$

This is a nonlinear transformation of the partial sums of ϵ_t process. Park and Phillips (1998) have developed a general method of treating nonlinear functionals of integrated time series, but unfortunately their results do not cover the asymptotic behavior of (2). There are two reasons for this. First, if we write the statistic in the form

$$DF(T) = T\frac{\sum_1^T Y_{t-1}^2(\exp(\sigma\epsilon_t) - 1)}{\sum_1^T Y_{t-1}^2} = T\frac{S_1(T)}{S_0(T)} \tag{3}$$

it is seen that the functional does not fit into the Park and Phillips framework because of the explicit appearance of the increment ϵ_t. The second reason is not quite so obvious. Park and Phillips develop a theory that is applicable to the the sum $S_0(T)$ of (3). Suppose that the ϵ_t have mean zero and finite nonzero variance. On writing

$$M_T = \max_{0\le t\le T} Z_t \tag{4}$$

Theorem 5.5 of Park and Phillips (1998) gives

$$\frac{S_0(T)}{\sqrt{T}\exp(2M_T)} \Rightarrow L(1, smax), \tag{5}$$

where $\Rightarrow$ denotes weak convergence, $L(x,t)$ is the local time of Brownian motion W on [0,1], and

$$smax = \max_{0\le t\le 1} W(t).$$

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The interpretation of $L(x,t)$ as an occupation time density and the continuity of $L(x,t)$ make it clear that $L(1,smax) = 0$. The consequence is that (5) reduces to

$$S_0(T) = o_{\mathbb{P}}(\sqrt{T}\exp(2M_T)), \tag{6}$$

so that it is of no help in analyzing the Dickey–Fuller statistic. For this we require the exact order of magnitude of $S_0(T)$. This is derived in the next section. In Section 3 we apply the result to the Dickey–Fuller statistic, which allows us to derive the correct order of magnitude of the statistic. In the particular case of the simple random walk the exact limiting behavior can be given. This is done in Section 4. Finally in Section 5 we give the results of some simulations.

2. THE ASYMPTOTIC BEHAVIOR OF $\sum_0^T \exp(\sigma Z_t)$

We prove the following theorem.

THEOREM 1. *Suppose the increments* $(\epsilon_i)_1^\infty$ *are i.i.d. and satisfy*

$$\mathbb{E}(\epsilon_i) = 0, \qquad \mathbb{V}(\epsilon_i) = 1, \qquad \mathbb{E}(|\epsilon_i|^3) < \infty. \tag{7}$$

Then for all $\gamma > 0$

$$\lim_{\lambda\to\infty} \inf_{T>0} \mathbb{P}\left(\exp(\gamma M_T) \le \sum_0^T \exp(\gamma Z_t) \le \lambda \exp(\gamma M_T)\right) = 1. \tag{8}$$

Proof. Clearly $\exp(\gamma M_T) \le \sum_0^T \exp(\gamma Z_t)$ so it is sufficient to prove

$$\lim_{\lambda\to\infty} \inf_{T>0} \mathbb{P}\left(\sum_0^T \exp(\gamma Z_t) \le \lambda \exp(\gamma M_T)\right) = 1,$$

which is equivalent to

$$\lim_{\lambda\to\infty} \inf_{T>0} \mathbb{P}\left(\sum_0^T \exp(-\gamma(M_T - Z_t)) \le \lambda\right) = 1,$$

which in turn follows from

$$\sup_{T>0} \mathbb{E}\left(\sum_0^T \exp(-\gamma(M_T - Z_t))\right) \le A \tag{9}$$

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for some constant A. Consider the term

$$\mathbb{E}(\exp(-\gamma(M_T - Z_t))) = \int_0^\infty \exp(-\gamma x)\, dF_{t,T}(x)$$

$$= F_{t,T}(\{0\}) + \gamma \int_0^\infty F_{t,T}(x)\exp(-\gamma x)\, dx, \tag{10}$$

where $F_{t,T}$ denotes the distribution function of $M_T - Z_t$. In the following discussion we make repeated use of the reversal procedure described on page 394 of Feller (1971). We have

$$\begin{aligned} F_{t,T}(x) &= \mathbb{P}(M_T - Z_t \le x) \\ &= \mathbb{P}\Big(\Big\{\max_{j=0,\ldots,t}\{Z_j - Z_t\} \le x\Big\} \cap \Big\{\max_{j=1,\ldots,T-t}\{Z_{t+j} - Z_t\} \le x\Big\}\Big) \\ &= \mathbb{P}(\widetilde{M}_t^* \le x)\mathbb{P}(\widetilde{M}_{T-t} \le x), \end{aligned} \tag{11}$$

where $\widetilde{M}_t = \max\{0, Z_1, Z_2, \ldots, Z_t\}$ and $\widetilde{M}_t^* = \max\{0, -Z_1, -Z_2, \ldots, -Z_t\}$. The proof of Theorem 1 relies entirely on the estimate

$$\mathbb{P}(\widetilde{M}_t^* \le x) \le (a + bx)/\sqrt{t}, \tag{12}$$

where now and in the discussion that follows a and b denote generic constants whose values may change at each appearance. As $t^{-1/2}\widetilde{M}_t^*$ converges weakly to the maximum of a Brownian motion on [0,1], (12) would follow immediately if a Berry–Esséen type result were available for the distribution of $t^{-1/2}\widetilde{M}_t^*$. We have not found such a result in the literature so we now give a proof that relies on properties of the random walk in $\mathbb{R}$ as given in Chapter XII of Feller (1971). For the sake of simplicity we assume that the random variables Z_t have a continuous distribution function, in which case the distinction between weak and strict variables is unnecessary (see Feller, 1971, pp. 392–393). We consider the case of strict variables only; the arguments for weak variables are to be found in Chapter 12 of Feller (1971). We have

$$\begin{aligned} \mathbb{P}(\widetilde{M}_t \le x) &= \sum_{j=1}^{t} \mathbb{P}(\{Z_1 < Z_j, Z_2 < Z_j, \ldots, Z_{j-1} < Z_j, 0 < Z_j \le x\} \\ &\qquad \cap \{Z_{j+1} < Z_j, Z_{j+2} < Z_j, \ldots, Z_t < Z_j\}) \\ &= \sum_{j=1}^{t} \mathbb{P}(Z_1 > 0, Z_2 > 0, \ldots, Z_{j-1} > 0, 0 < Z_j \le x) \\ &\quad \times \mathbb{P}(Z_1 < 0, Z_2 < 0, \ldots, Z_{t-j} < 0) \\ &= \sum_{j=1}^{t} H_j(x) q_{t-j}, \end{aligned} \tag{13}$$

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where

$$H_j(x) = \mathbb{P}(Z_1 > 0, Z_2 > 0, \ldots, Z_{j-1} > 0, 0 < Z_j \leq x), \tag{14}$$

$$q_j = \mathbb{P}(Z_1 < 0, Z_2 < 0, \ldots, Z_j < 0). \tag{15}$$

To estimate q_j in (15) we note that the assumptions of the theorem imply

$$\sum_1^\infty \frac{1}{n}\left(\mathbb{P}(Z_n > 0) - \frac{1}{2}\right) = c \tag{16}$$

for some finite constant c (see Feller, 1971, p. 612). From this we deduce

$$q_j \asymp \exp(-c)\binom{2j}{j}2^{-2j} \asymp \frac{\exp(-c)}{\sqrt{\pi j}} \tag{17}$$

(see Feller, 1971, pp. 418–419).

We turn to estimates for $H_j(x)$ of (14). We note first that

$$\begin{aligned} H_n(x) &= \mathbb{P}(Z_1 > 0, Z_2 > 0, \ldots, Z_{n-1} > 0, 0 < Z_n \leq x) \\ &\leq \mathbb{P}\left(Z_1 > 0, Z_2 > 0, \ldots, Z_{n/2} > 0, 0 < \sum_{n/2+1}^{n} \epsilon_i + Z_{n/2} \leq x\right) \\ &= \mathbb{E}(\{Z_1 > 0, Z_2 > 0, \ldots, Z_{n/2} > 0\} \\ &\qquad \times \mathbb{E}(\{0 \leq Z_n - Z_{n/2} + Z_{n/2} \leq x\} | Z_{n/2})). \end{aligned} \tag{18}$$

As $Z_n - Z_{n/2}$ is independent of $Z_{n/2}$ the Berry–Esséen theorem for $(Z_n - Z_{n/2})/\sqrt{n/2}$ gives

$$\mathbb{E}(\{0 \leq Z_n - Z_{n/2} + Z_{n/2} \leq x\} | Z_{n/2}) \leq (a + bx)/\sqrt{n},$$

and on substituting this into (18) we obtain

$$\begin{aligned} H_n(x) &\leq \mathbb{P}(Z_1 > 0, Z_2 > 0, \ldots, Z_{n/2} > 0)(a + bx)/\sqrt{n} \\ &\leq (a + bx)/n, \end{aligned} \tag{19}$$

where we have used the argument leading to (17). The second estimate for $H_n(x)$ that we require is

$$\sum_1^\infty H_n(x) = \psi(x) \leq a + bx, \tag{20}$$

where $\psi(x)$ is the renewal function of the first ladder height (see the duality lemma in Feller, 1971, p. 395). The inequality in (20) follows from the renewal theorem (Feller, 1971, pp. 358–366).

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Applying the inequalities (17), (19), and (20) in (13) we obtain

$$\begin{aligned}\mathbb{P}(\widetilde{M}_t \le x) &= \sum_{j=1}^{t} H_j(x) q_{t-j} \\ &= \sum_{j=1}^{t/2} H_j(x) q_{t-j} + \sum_{j=t/2+1}^{t} H_j(x) q_{t-j} \\ &\le (a+bx)/\sqrt{t} + ((a+bx)/t)\sqrt{t} \\ &= (a+bx)/\sqrt{t},\end{aligned}$$

where the first estimate follows from (17) and (20) and the second from (17) and (19). This finally proves (12), and on using this in (11) we obtain

$$\mathbb{E}(\exp(-\gamma(M_T - Z_t))) \le \frac{a}{\sqrt{(t+1)(T+1-t)}}. \tag{21}$$

From (21) we conclude

$$\begin{aligned}\mathbb{E}\left(\sum_{0}^{T} \exp(-\gamma(M_T - Z_t))\right) &\le B \sum_{0}^{T} \frac{1}{\sqrt{(t+1)(T+1-t)}} \\ &\le a.\end{aligned}$$

This proves (9) with $A = a$ and with it the theorem. ■

3. THE ASYMPTOTIC BEHAVIOR OF THE DICKEY–FULLER STATISTIC

We turn to the asymptotic behavior of the Dickey–Fuller statistic when applied to the Y_t.

THEOREM 2. *Suppose the increments* $(\epsilon_i)_1^\infty$ *are i.i.d. and satisfy*

$$\mathbb{E}(\epsilon_i) = 0, \qquad \mathbb{V}(\epsilon_i) = 1, \qquad \mathbb{E}(|\epsilon_i|^3) < \infty.$$

Then

$$\lim_{\lambda \downarrow 0} \lim_{T \to \infty} \mathbb{P}(-T \le DF(T) \le -\lambda T) = 1. \tag{22}$$

Proof. The lower bound of $-T$ follows directly from (2) on noting that $\hat{\rho}(T)$ is always positive. To obtain the upper bound we require an upper bound for the sum $S_1(T)$. This requires an analysis of the path $(t, Z_t)_0^T$ with $Z_0 = 0$. We denote the path by $B(T)$ and write $S_1(B(T)) = S_1(T)$. If part of the path is constant, that is, $Z_t = Z_{t+1} = \cdots = Z_{t+s}$ for some $s \ge 1$, then the points $(t+1, Z_{t+1}), \ldots, (t+s, Z_{t+s})$ can be removed without changing the value of S_1. Without loss of generality we can therefore suppose that the path has no intervals where it is constant. We consider the effect of removing the local maxima

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from the path $B(T)$. As the path contains no intervals of constancy there is a local maximum at the point t if

$$\max\{Z_{t-1}, Z_{t+1}\} < Z_t.$$

We suppose that $Z_{t-1} \le Z_{t+1} < Z_t$. A similar argument works if $Z_{t+1} \le Z_{t-1} < Z_t$. A new path of length $T-1$ is constructed by setting $\tilde{Z}_j = Z_j, 0 \le j \le t-1$, $\tilde{Z}_t = Z_{t+1}$, and $\tilde{Z}_j = Z_{j+1}, t+1 \le j \le T-1$. We denote this new path by $\tilde{B}(T-1)$ and set $\Delta = Z_t - Z_{t+1}$ and $\gamma + \Delta = Z_t - Z_{t-1}$ with γ and Δ both nonnegative. We have

$$\begin{aligned} S_1(B(T)) - S_1(\tilde{B}(T-1)) &= \exp(2Z_t)(\exp(-2(\gamma+\Delta))(\exp(\gamma+\Delta)-1) \\ &\quad + \exp(-\Delta) - 1 - \exp(-2\gamma - 2\Delta)(\exp(\gamma)-1)) \\ &= -\exp(2Z_t)(1-\exp(-\Delta))(1-\exp(-\Delta-\gamma)) \\ &\le -\exp(2Z_t)(1-\exp(-\Delta))^2. \end{aligned} \tag{23}$$

We note that the value of the final point of the path is not altered, i.e., $\tilde{Z}_{T-1} = Z_T$. Under the conditions of the theorem $\lim_{T\to\infty} \mathbb{P}(M_T > \max\{0, Z_T\} + a) = 1$ for all $a > 0$ so that the global maximum is an interior local maximum with probability tending to one as T tends to infinity. We reserve the Δ in (23) for the point corresponding to the global maximum. We now proceed to remove all local maxima and finally obtain a path $\tilde{B}(T-k)$ of length $T-k$ with no local maxima and that satisfies

$$S_1(B(T)) \le -\exp(2M_T)(1-\exp(-\Delta))^2 + S_1(\tilde{B}(T-k)).$$

As $\tilde{B}(T-k)$ has no local maxima it follows that the global maximum of this path is located either at 0 or $T-k$. As the removal of local maxima does not alter the value of the last point we have $S_1(\tilde{B}(T-k)) \le (T-k)\max\{0, Z_T\}$, and hence

$$S_1(B(T)) \le -\exp(2M_T)(1-\exp(-\Delta))^2 + T\max\{0, Z_T\}.$$

As $\lim_{T\to\infty} \mathbb{P}(M_T > Z_T + a_T) = 1$ for any sequence a_T with $\lim_{T\to\infty} a_T/\sqrt{T} = 0$ it follows that $T\max\{0, Z_T\} = o_{\mathbb{P}}(\exp(2M_T))$, which yields

$$S_1(B_T) \le -\exp(2M_T)((1-\exp(-\Delta))^2 + o_{\mathbb{P}}(1)). \tag{24}$$

This combined with the result of Theorem 1 implies

$$\lim_{\lambda\downarrow 0} \liminf_{T\to\infty} \mathbb{P}\left(T\,\frac{S_1(T)}{S_0(T)} \le -\lambda((1-\exp(-\Delta))^2 + o_{\mathbb{P}}(1))\right) = 1.$$

To complete the proof of the theorem it suffices to show that

$$\lim_{x\downarrow 0} \liminf_{T\to\infty} \mathbb{P}(\Delta^* \ge x) = 1, \tag{25}$$

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where $\Delta^* = \min\{\epsilon_t, -\epsilon_{t+1}\}$ on $M_T = Z_t$. Conditioning on $M_T = Z_t$ we have for ϵ_t

$$\epsilon_t > 0, \epsilon_t \geq -\sum_{j=s}^{t-1} \epsilon_j, \qquad s = t-1,\ldots,0 \tag{26}$$

and

$$\sum_{j=t+1}^{s} \epsilon_j \leq 0, \qquad s = t+1,\ldots,T. \tag{27}$$

As the ϵ_j are assumed to be independently distributed the inequalities (27) have no effect on the conditional distribution of ϵ_t. The inequalities (26) may be written in the form

$$\epsilon_t > \max\{0, f(\epsilon_1,\ldots,\epsilon_{t-1})\}.$$

On conditioning on $\epsilon_1,\ldots,\epsilon_{t-1}$ and using the fact that

$$\mathbb{P}(X \geq x | X \geq a) \geq \mathbb{P}(X \geq x | X \geq 0)$$

for any random variable X and for any x and $a \geq 0$ we deduce

$$\mathbb{P}(\epsilon_t \geq x | M_T = Z_t) \geq \mathbb{P}(\epsilon_t \geq x | \epsilon_t \geq 0).$$

Similarly we have

$$\mathbb{P}(-\epsilon_{t+1} \geq x | M_T = Z_t) \geq \mathbb{P}(-\epsilon_{t+1} \geq x | -\epsilon_{t+1} \geq 0).$$

These latter inequalities imply

$$\begin{aligned}\mathbb{P}(\Delta^* \geq x | M_T = Z_t) &= \mathbb{P}(\min\{\epsilon_t, -\epsilon_{t+1}\} \geq x | M_T = Z_t) \\ &\geq \mathbb{P}(\min\{\epsilon_t, -\epsilon_{t+1}\} \geq x | \min\{\epsilon_t, -\epsilon_{t+1}\} \geq 0),\end{aligned}$$

from which we may deduce (25). This completes the proof of the theorem. ■

4. THE SIMPLE RANDOM WALK

In the case of the simple random walk a more precise result is possible. In fact we state the theorem somewhat more generally as we need only the properties (28) and (29), which follows and they are satisfied for the simple random walk. In particular we do not require that the ϵ_i are independently or identically distributed.

THEOREM 3. *Suppose that the process Z_t satisfies*

$$\epsilon_i \in \{-1,1\} \quad \textit{for all } i \tag{28}$$

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and

$$\lim_{T\to\infty} \mathbb{P}(M_T \geq \max(0, Z_T) + a) = 1 \tag{29}$$

for all $a > 0$. *Then*

$$\text{plim}_{T\to\infty} \frac{DF(T)}{T} = -\frac{(\exp(\sigma) - 1)^2}{\exp(2\sigma) + 1}, \tag{30}$$

where σ *is as in (1).*

Proof. Without loss of generality we may set $Z_0 = 0$, and to ease the notation we consider only the case $\sigma = 1$. Given a level j we consider those times t, if any, for which $Z_t = j$ and $Z_{t+1} = j + 1$. These are the set of up-crossings from j to $j + 1$. Similarly we define the set of down-crossings from $j + 1$ to j as those points t, if any, for which $Z_t = j + 1$ and $Z_{t+1} = j$. Consider first a path with $Z_T \geq 1$. Then for all $j \geq Z_T$ and for all $j \leq -1$ the number of up-crossings from j to $j + 1$ is equal to the number of down-crossings from $j + 1$ to j. For $0 \leq j \leq Z_T - 1$ the number of up-crossings from j to $j + 1$ exceeds the number of down-crossings from $j + 1$ to j by 1. Similarly if $Z_T \leq 0$ then for $j \geq 0$ and $j \leq Z_T - 1$ the number of up-crossings from j to $j + 1$ is equal to the number of down-crossings from $j + 1$ to j. For $Z_T \leq j \leq -1$ the number of down-crossings from $j + 1$ to j exceeds the number of up-crossings from j to $j + 1$ by 1. For each level j let $N(j, T)$ denote the number of up-crossings from j to $j + 1$ that can be paired with a down-crossing from $j + 1$ to j. Further let $E(j, T)$ denote the excess of up-crossings from j to $j + 1$ over the down-crossings from $j + 1$ to j. The preceding argument shows that $E(j, T)$ is either -1, 0, or 1.

The contribution of an up-crossing from j to $j + 1$ to the sum $S_1(T)$ of (3) is

$$\exp(2j)(\exp(1) - 1).$$

Similarly the contribution to $S_1(T)$ of a down-crossing from $j + 1$ to j is

$$\exp(2(j + 1))(\exp(-1) - 1) = -\exp(2j)(\exp(2) - \exp(1)).$$

We may therefore write

$$S_1(T) = -(\exp(1) - 1)^2 \sum_{-\infty}^{\infty} N(j, T)\exp(2j) + (\exp(1) - 1) \sum_{-\infty}^{\infty} E(j, T)\exp(2j). \tag{31}$$

Similarly for the sum $S_0(T)$ we have

$$S_0(T) = (1 + \exp(2)) \sum_{-\infty}^{\infty} N(j, T)\exp(2j) + \sum_{-\infty}^{\infty} E(j, T)\exp(2j). \tag{32}$$

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Suppose now that $Z_T > 0$. Then the argument given previously shows that $E(j,T) = 0$ for $j \geq Z_T$ and consequently

$$\left| \sum_{-\infty}^{\infty} E(j,T)\exp(2j) \right| \leq \sum_{-\infty}^{Z_T} \exp(2j) \leq 2\exp(2Z_T).$$

Similarly if $Z_T \leq 0$ then

$$\left| \sum_{-\infty}^{\infty} E(j,T)\exp(2j) \right| \leq \sum_{-\infty}^{0} \exp(2j) \leq 2.$$

On combining these last two inequalities we obtain

$$\left| \sum_{-\infty}^{\infty} E(j,T)\exp(2j) \right| \leq 2\exp(2\max(0,Z_T)). \tag{33}$$

Consider a random walk satisfying

$$M_T = k > 0 \quad \text{and} \quad Z_T = k - a \quad \text{with } a \geq 1.$$

It follows from the arguments given previously that $N(j,T) \geq 1$ for $j = M_T - 1$. This implies

$$\begin{aligned} \sum_{-\infty}^{\infty} N(j,T)\exp(2j) &\geq \exp(2M_T - 2) \\ &= \exp(2\max(Z_T,0) + 2(M_T - \max(Z_T,0))) \\ &\geq \frac{1}{2}\exp(2(M_T - \max(Z_T,0)))\left| \sum_{-\infty}^{\infty} E(j,T)\exp(2j) \right| \end{aligned}$$

by (33) and consequently

$$\left| \sum_{-\infty}^{\infty} E(j,T)\exp(2j) \right| \leq 2\exp(2(\max(Z_T,0) - M_T)) \sum_{-\infty}^{\infty} N(j,T)\exp(2j). \tag{34}$$

From (29) it follows that there exists a sequence $(a_T)_1^{\infty}$ satisfying $\lim_{T\to\infty} a_T = \infty$ such that

$$\lim_{T\to\infty} \mathbb{P}(\max(0,Z_T) \leq M_T - a_T) = 1.$$

Using this in (34) gives

$$\left| \sum_{-\infty}^{\infty} E(j,T)\exp(2j) \right| = o_{\mathbb{P}}\left(\sum_{-\infty}^{\infty} N(j,T)\exp(2j) \right),$$

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from which we obtain with (31)

$$S_1(T) = -(\exp(1)-1)^2 \sum_{-\infty}^{\infty} N(j,T)\exp(2j)(1+o_{\mathbb{P}}(1))$$

and with (32)

$$S_0(T) = (1+\exp(2)) \sum_{-\infty}^{\infty} N(j,T)\exp(2j)(1+o_{\mathbb{P}}(1)).$$

On using these asymptotic equivalences in (3) we have

$$\text{plim}_{T\to\infty} \frac{S_1(T)}{S_0(T)} = -\frac{(\exp(1)-1)^2}{\exp(2)+1}, \tag{35}$$

which proves the theorem. ■

5. SIMULATIONS

Figure 1 shows the results of a simulation of a simple random walk with the ϵ_i i.i.d. and taking the values 1 and -1 with probability 0.5, $\sigma = 0.01$, and

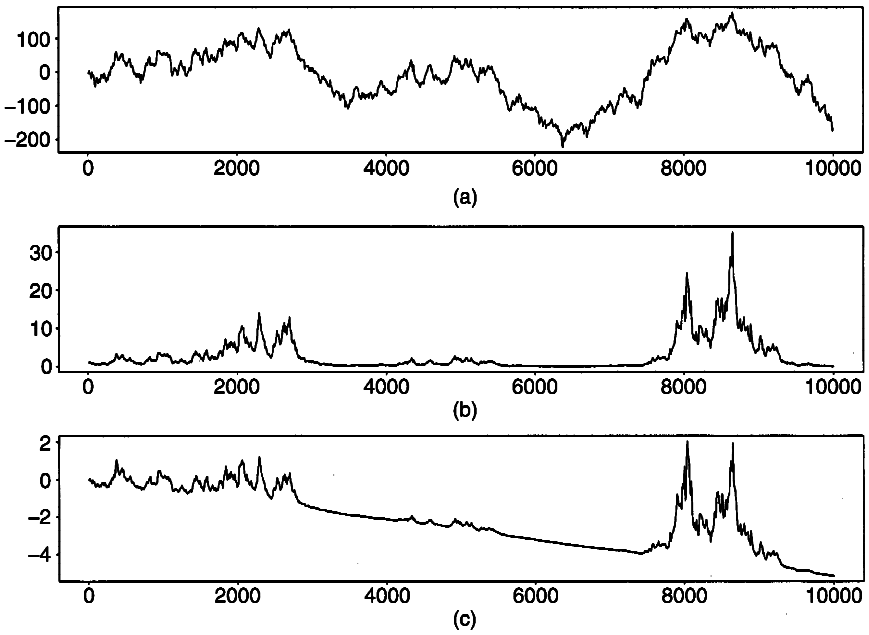


FIGURE 1. (a) The z_t process for the simple random walk with $\sigma = 0.01$ and $T = 100{,}000$. (b) The corresponding y_t process. (c) The path behavior of the Dickey–Fuller statistic.

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$T = 100{,}000$. Part (a) shows the path behavior of the Z_t-process, (b) shows the corresponding process Y_t, and (c) shows the path behavior of the Dickey–Fuller statistic. The final value of the statistic is $-5.163337e-05$, which compares well with the theoretical slope given by

$$-\frac{(\exp(0.01)-1)^2}{\exp(2\cdot 0.01)+1} = -4.999792e-05.$$

Figure 2 shows the corresponding results when the ϵ_i are i.i.d. standard Gaussian random variables. The variability of the individual paths is very high. To overcome this 500 paths of length 20,000 were simulated with $\sigma = 0.01$ and the ϵ_i being standard Gaussian white noise. The mean value of the Dickey–Fuller statistic was calculated at each of the time points 1,000, (1,000), 20,000. The results are shown in Figure 3, where a linear trend is apparent. This is consistent with the main theorem. It makes plausible the conjecture that the probability limit $\text{plim}_{T\to\infty} DF(T)/T$ exists and is negative and finite also in the Gaussian case

$$\text{plim}_{T\to\infty} \frac{DF(T)}{T} = -d \tag{36}$$

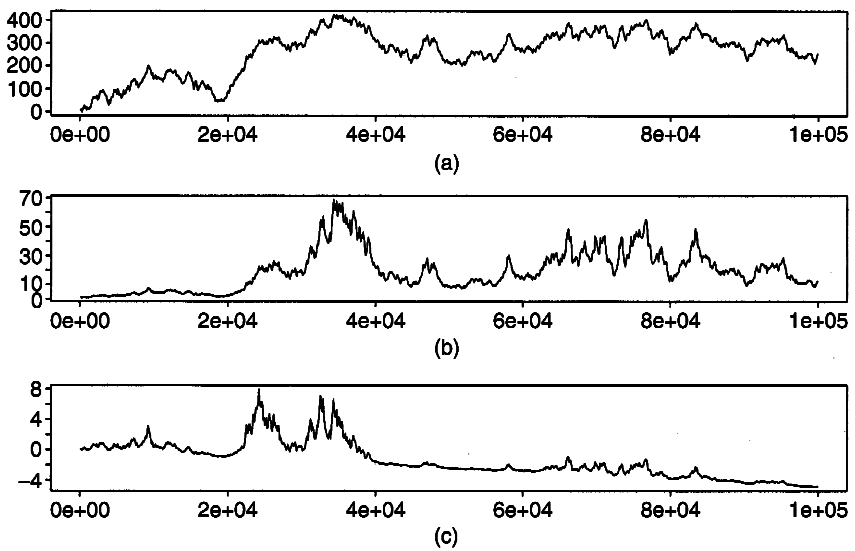


FIGURE 2. (a) The z_t process for standard Gaussian increments with $\sigma = 0.01$ and $T = 100{,}000$. (b) The corresponding y_t process. (c) The path behavior of the Dickey–Fuller statistic.

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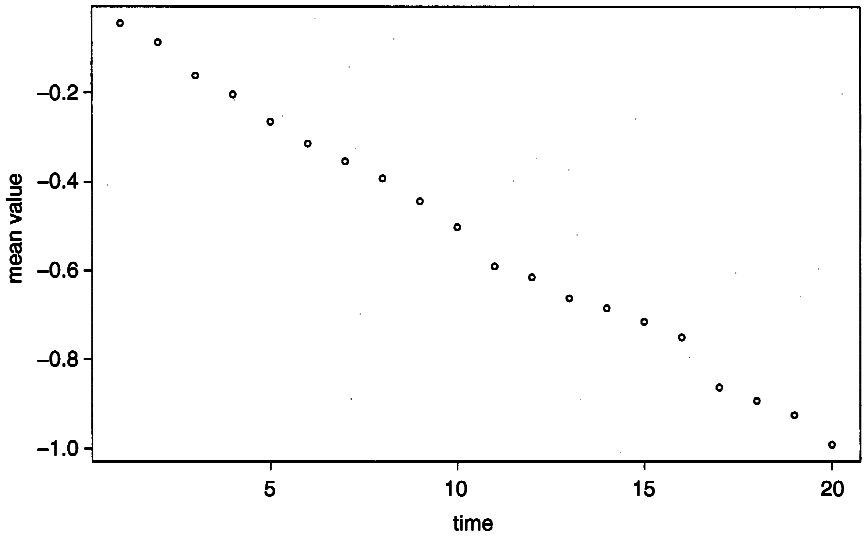


FIGURE 3. The value of the Dickey–Fuller statistic averaged over 500 simulated paths.

with $d > 0$ and finite. We offer no proof of this. There well may be other distributions for which (36) holds true, and in this case one can expect the constant d to be dependent on the underlying distribution of the ϵ_i.

REFERENCES

Feller, W. (1971) *An Introduction to Probability Theory and Its Applications*, 2nd ed., vol. 2. New York: Wiley.

Park, J.Y. and Phillips, P.C.B. (1998) Asymptotics for nonlinear transformations of integrated time series. *Econometric Theory* 15, 269–298.

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