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Nonlinear saturation of the Weibel instability in a dense Fermi plasma

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Abstract. We present an investigation for the generation of intense magnetic fields in dense plasmas with an anisotropic electron Fermi–Dirac distribution. For this purpose, we use a new linear dispersion relation for transverse waves in the Wigner–Maxwell dense quantum plasma system. Numerical analysis of the dispersion relation reveals the scaling of the growth rate as a function of the Fermi energy and the temperature anisotropy. The nonlinear saturation level of the magnetic fields is found through fully kinetic simulations, which indicates that the final amplitudes of the magnetic fields are proportional to the linear growth rate of the instability. The present results are important for understanding the origin of intense magnetic fields in dense Fermionic plasmas, such as those in the next-generation intense laser–solid density plasma experiments.

1. Introduction

The existence of feeble magnetic fields of several microgauss in our galaxies [1], as well as of those of gigagauss in intense laser–plasma interaction experiments [2] and those of billions of gauss in compact astrophysical objects [3] (e.g. superdense white dwarfs, neutron stars/magnetars, degenerate stars, supernovae) is well known. The generation mechanisms for seed magnetic fields in cosmic/astrophysical environments are still debated, while the spontaneous generation of magnetic fields in laser-produced plasmas is attributed to the Biermann battery [4] (also referred to as the baroclinic vector containing non-parallel electron density and electron temperature gradients) and to the return electron current from the solid target. Computer simulations of laser–fusion plasmas have shown evidence of localized anisotropic electron heating by resonant absorption, which in turn can drive a Weibel-like instability resulting in megagauss magnetic fields [5]. There have also been observations of the Weibel instability in high-intensity laser–solid interaction experiments [6]. Furthermore, a purely growing Weibel instability [7], arising from the electron temperature anisotropy (a bi-Maxwellian electron distribution function), is also capable of generating magnetic fields and associated shocks [8].

However, plasmas in the next-generation intense laser–solid density plasma experiments [9] would be very dense. Here the equilibrium electron distribution

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function may assume the form of a deformed Fermi–Dirac distribution due to the electron heating by intense laser beams. It may then turn out that, in such dense Fermi plasmas, quantum mechanical effects (e.g. electron tunneling and wave-packet spreading) would play a significant role [10]. The importance of quantum mechanical effects at nanometer scales has been recognized in the context of quantum diodes [11] and ultra-small semiconductor devices [12]. Also, recently there have been several developments on fermionic quantum plasmas, involving the addition of a dynamical spin force [13–16], turbulence or coherent structures in degenerate Fermi systems [17, 18], as well as the coupling between nonlinear Langmuir waves and electron holes in quantum plasmas [19]. The quantum Weibel or filamentational instability for non-degenerate systems has been treated in [20, 21].

In this work, we present an investigation of linear and nonlinear aspects of a novel instability that is driven by an equilibrium Fermi–Dirac electron temperature anisotropic distribution function in a non-relativistic dense Fermi plasma. Specifically, we show that the free energy stored in electron temperature anisotropy is coupled to purely growing electromagnetic modes. First, we take the Wigner–Maxwell system [22] with an anisotropic Fermi–Dirac distribution for the analysis of the linearly growing electromagnetic perturbations as a function of the physical parameters. Second, we use a fully kinetic simulation to assess the saturation level of the magnetic fields as a function of the growth rate. The treatment is restricted to transverse waves, since the latter are associated with the largest Weibel instability growth rates. The nonlinear saturation of the Weibel instability for classical, non-degenerate plasmas has been considered elsewhere [23].

2. Basic equations

It is well known [24] that a dense Fermi plasma with isotropic equilibrium distributions does not admit any purely growing linear modes. This can be verified, for instance, from the expression for the imaginary part of the transverse dielectric function, as derived by Lindhard [25], for a fully degenerate non-relativistic Fermi plasma. It can be proven (see [26, Equation (30)]) that the only exception would be for extremely small wavelengths, so that $k > 2k_{\mathrm{F}}$, where k is the wave number and k_{F} the characteristic Fermi wave number of the system. However, in this situation the wave would be super-luminal. On the other hand, in a classical Vlasov–Maxwell plasma containing an anisotropic electron distribution function, we have a purely growing Weibel instability [7], via which dc magnetic fields are created. The electron temperature anisotropy arises due to the heating of the plasma by laser beams [6], where there is also a signature of the Weibel instability. In the next-generation intense laser–solid density plasma experiments, it is likely that the electrons would be degenerate and that electron temperature anisotropy may develop due to anisotropic electron heating by intense laser beams via resonant absorption, similar to the classical laser plasma case [5].

In a dense laser-created plasma, quantum effects must play an important role in the context of the Weibel instability. In order to keep the closest analogy with the distribution function in phase space for the classical plasma, we use the Wigner–Maxwell formalism for a dense quantum plasma [27]. Here the distribution of the electrons is described by the Wigner pseudo-distribution function [28], which is related to the Fermi–Dirac distribution widely used in the random phase

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approximation [24]. Proceeding with the time evolution equation for the Wigner function (or the quantum Vlasov equation [27]), we derive a modified dispersion relation accounting for wave–particle duality and an anisotropic Wigner distribution function that is appropriate for the Fermi plasma. The results are consistent with those of the random phase approximation, in that they reproduce the well-known transverse density linear response function for a fully degenerate Fermi plasma [25].

Consider linear transverse waves in a dense quantum plasma composed of electrons and immobile ions, with $\mathbf{k}\cdot\mathbf{E} = 0$, where $\mathbf{k}$ is the wave vector and $\mathbf{E}$ is the wave electric field. Following standard procedure, we then obtain the general dispersion relation [21, 29] for the transverse waves of the Wigner–Maxwell system,

$$\omega^2 - \omega_{\mathrm{p}}^2 - c^2k^2 + \frac{m\omega_{\mathrm{p}}^2}{2n_0\hbar}\int d\mathbf{v}\left(\frac{v_x^2+v_y^2}{\omega - kv_z}\right) \times \left(f_0\left(v_x, v_y, v_z + \frac{\hbar k}{2m}\right) - f_0\left(v_x, v_y, v_z - \frac{\hbar k}{2m}\right)\right) = 0, \tag{1}$$

where ω is the frequency, c is the speed of light in vacuum, $\hbar$ is the Planck constant divided by 2π, m is the rest electron mass, n_0 is the unperturbed plasma number density, ω_{p} is the electron plasma frequency, $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity vector, and $f_0(v_x, v_y, v_z)$ is the equilibrium Wigner function associated to Fermi systems.

For spin-1/2 particles, the equilibrium pseudo-distribution function is in the form of a Fermi–Dirac function. Here we allow for velocity anisotropy and express

$$f_0 = \frac{\alpha}{\exp\left[\frac{1}{2}m\left(\left(v_x^2+v_y^2\right)/\kappa_{\mathrm{B}}T_\perp + v_z^2/\kappa_{\mathrm{B}}T_\parallel\right) - \beta\mu\right] + 1}, \tag{2}$$

where μ is the chemical potential, κ_{B} is the Boltzmann constant, and the normalization constant is

$$\alpha = -\frac{n_0}{\mathrm{Li}_{3/2}(-e^{\beta\mu})}\left(\frac{m\beta}{2\pi}\right)^{3/2} = 2\left(\frac{m}{2\pi\hbar}\right)^3. \tag{3}$$

Here $\mathrm{Li}_{3/2}$ is a polylogarithm function [30, 31]. Also, $\beta = 1/[\kappa_{\mathrm{B}}(T_\perp^2 T_\parallel)^{1/3}]$, where $T_\perp$ and $T_\parallel$ are related to the velocity dispersion in the direction perpendicular and parallel to the z-axis, respectively. In the special case when $T_\perp = T_\parallel$, the usual Fermi–Dirac equilibrium is recovered. The chemical potential is obtained by solving the normalization condition (3), yielding, in particular, $\mu = \mathscr{E}_{\mathrm{F}}$ in the limit of zero temperature, where $\mathscr{E}_{\mathrm{F}} = (3\pi^2 n_0)^{2/3}\hbar^2/(2m)$ is the Fermi energy. Also, the Fermi–Dirac distribution $\hat{f}(\mathbf{k})$, where $\mathbf{k}$ is the appropriated wave vector in momentum space, is related to the equilibrium Wigner function (2) by $\hat{f}(\mathbf{k}) = (1/2)(2\pi\hbar/m)^3 f_0(\mathbf{v})$, with the factor two coming from the spin [32, 33]. However, these previous works refer to the cases where there is no temperature anisotropy. Notice that it has been suggested [34] that in laser plasmas the Weibel instability is responsible for further increase of $T_\parallel$ with time.

Inserting (2) into (1) and integrating over the perpendicular velocity components, we obtain

$$\omega^2 - c^2k^2 - \omega_{\mathrm{p}}^2\left(1 + \frac{T_\perp}{T_\parallel}W_Q\right) = 0, \tag{4}$$

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where

$$W_Q = \frac{1}{2\sqrt{\pi}H\mathrm{Li}_{3/2}(-e^{\beta\mu})}\int\frac{d\nu}{\nu-\xi} \times\left(\mathrm{Li}_2\left\{-\exp\left[-\left(\nu+\frac{H}{2}\right)^2+\beta\mu\right]\right\}-\mathrm{Li}_2\left\{-\exp\left[-\left(\nu-\frac{H}{2}\right)^2+\beta\mu\right]\right\}\right). \tag{5}$$

In (5), Li_2 is the dilogarithm function [30, 31], $H = \hbar k/(mv_\parallel)$ is a characteristic parameter representing the quantum diffraction effect, $\xi = \omega/(kv_\parallel)$, and $\nu = v_z/v_\parallel$, with $v_\parallel = (2\kappa_\mathrm{B}T_\parallel/m)^{1/2}$. In the simultaneous limit of a small quantum diffraction effect ($H \ll 1$) and a dilute system ($e^{\beta\mu} \ll 1$), it can be shown that $W_Q \simeq -1-\xi Z(\xi)$, where Z is the standard plasma dispersion function [35]. It is important to notice that either (1) or (4) reproduces the transverse dielectric function calculated from the random phase approximation for a fully degenerate quantum plasma [25], in the case of an isotropic system. The simple way to verify this equivalence is to put $T_\perp = T_\parallel$ in (1) and then take the limit of zero temperature, so that $f_0 = 3n_0/(4\pi v_\mathrm{F}^3)$ for $|\mathbf{v}| < v_\mathrm{F}$, and $f_0 = 0$ otherwise, where $v_\mathrm{F} \equiv (2\mathscr{E}_\mathrm{F}/m)^{1/2}$ is the Fermi velocity. However, to the best of our knowledge, there is no corresponding calculation for an anisotropic Fermi equilibrium, as necessary in laser–solid interaction experiments with an anisotropic electron heating due to resonant absorption. Also notice that in this paper we are mainly interested in the real part of the transverse response function, since we are looking for purely growing instabilities ($\omega^2 < 0$), so that the contribution from the poles at (4) is not relevant.

3. Numerical results

We now solve our new dispersion relation (4) for a set of parameters that are representative of the next-generation laser–solid density plasma interaction experiments. The normalization condition (3) can also be written as $-\mathrm{Li}_{3/2}[-\exp(\beta\mu)] = (4/3\sqrt{\pi})(\beta\mathscr{E}_\mathrm{F})^{3/2}$, which is formally the same relation holding for isotropic Fermi–Dirac equilibria [36]. For a given value on the product $\beta\mu$ and the density, this relation yields the value β, from which the temperatures $T_\perp$ and $T_\parallel$ can be calculated, if we know $T_\perp/T_\parallel$. Consider only purely growing modes. From the definition (5), one can show that $W_Q \to -1$ when $\omega = i\gamma \to 0$ for a finite wave number k. From (4) we then obtain the maximum wave number for the instability as $k_\mathrm{max} = (\omega_\mathrm{p}/c)\sqrt{T_\perp/T_\parallel - 1}$. When $T_\perp/T_\parallel \to 1$, the range of unstable wave numbers shrinks to zero. In Figs 1 and 2, we have used the electron number density $n_0 = 10^{33}\ \mathrm{m}^{-3}$, which can be obtained in laser-driven compression schemes. The growth rates for different values of $T_\perp/T_\parallel$ are displayed in Fig. 1. We see that the maximum unstable wave number is $k_\mathrm{max} = (\omega_\mathrm{p}/c)\sqrt{T_\perp/T_\parallel - 1}$, as predicted, and that the maximum growth rate occurs at $k \approx k_\mathrm{max}/2$. Figure 1 also reveals that the maximum growth rate of the instability is almost linearly proportional to $T_\perp/T_\parallel - 1$. In Fig. 2, we have varied the product $\beta\mu$, which is a measure of the degeneracy of the quantum plasma. We see that for $\beta\mu > 5$, the instability reaches a limiting value, which is independent of the temperature, while thermal effects start to play an important role for $\beta\mu$ of the order of unity.

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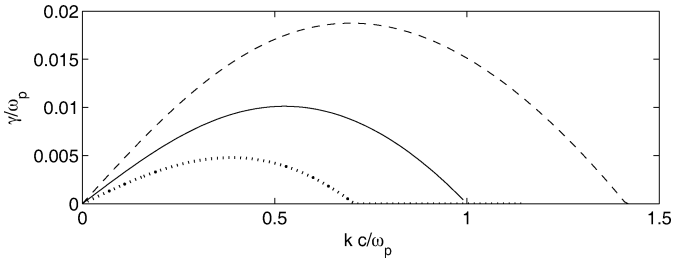


Figure 1. The growth rate for the Weibel instability of a dense Fermionic plasma with $n_0 = 10^{33}$ m^{-3} ($\omega_p = 1.8 \times 10^{18}$ s^{-1}) and $\beta\mu = 5$, relevant for the next generation inertially compressed material in intense laser–solid density plasma interaction experiments. The temperature anisotropies are $T_\perp/T_\| = 3$ (dashed line), $T_\perp/T_\| = 2$ (solid line) and $T_\perp/T_\| = 1.5$ (dotted line), yielding, respectively, $T_\| = 3.9 \times 10^6$ K, $T_\| = 5.2 \times 10^6$ K and $T_\| = 6.3 \times 10^6$ K.

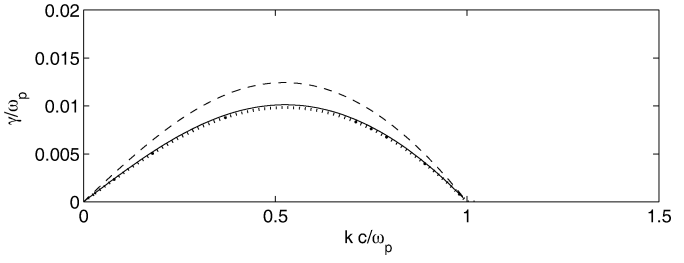


Figure 2. The growth rate for the Weibel instability of a dense Fermionic plasma with $n_0 = 10^{33}$ m^{-3} ($\omega_p = 1.8 \times 10^{18}$ s^{-1}). Here the temperature anisotropy is $T_\perp/T_\| = 2$. We used $\beta\mu = 1$ (dashed line), $\beta\mu = 5$ (solid line) and $\beta\mu = 10$ (dotted line), yielding $T_\| = 1.6 \times 10^7$ K, $T_\| = 5.2 \times 10^7$ K and $T_\| = 2.6 \times 10^6$ K, respectively.

From several numerical solutions of the linear dispersion relation, we have been able to deduce an approximate scaling law for the instability as $\gamma_{max}/\omega_p = \text{constant} \times n_0^{1/3}(T_\perp/T_\| - 1)$, where the constant is approximately 8.5×10^{-14} s^{-1} m. Using the fact that $n_0 = (2m\mathscr{E}_F/\hbar^2)^{3/2}/(3\pi^2) \approx 1.67 \times 10^{36}(\mathscr{E}_F/mc^2)^{3/2}$, we have

$$\frac{\gamma_{max}}{\omega_p} = 0.10\left(\frac{\mathscr{E}_F}{mc^2}\right)^{1/2}\left(\frac{T_\perp}{T_\|} - 1\right) \tag{6}$$

for the maximum growth rate of the Weibel instability in a degenerate Fermi plasma. This scaling law, where the growth rate depends on the Fermi energy and the temperature anisotropy, should be compared to that of a classical plasma [5, 37], where the growth rate depends on the thermal energy and the temperature anisotropy.

For a Maxwellian plasma, it has been found [38] that the Weibel instability saturates nonlinearly once the magnetic bounce frequency $\omega_c = eB/m$ has increased to a value comparable to the linear growth rate. In order to assess the nonlinear behavior of the Weibel instability for a degenerate plasma, we have carried out a kinetic simulation of the Wigner–Maxwell system. We have assumed that the quantum diffraction effect is small, so that the simulation of the Wigner equation can be approximated by simulations of the Vlasov equation by means of an electromagnetic Vlasov code [39]. As an initial condition for the simulation, we

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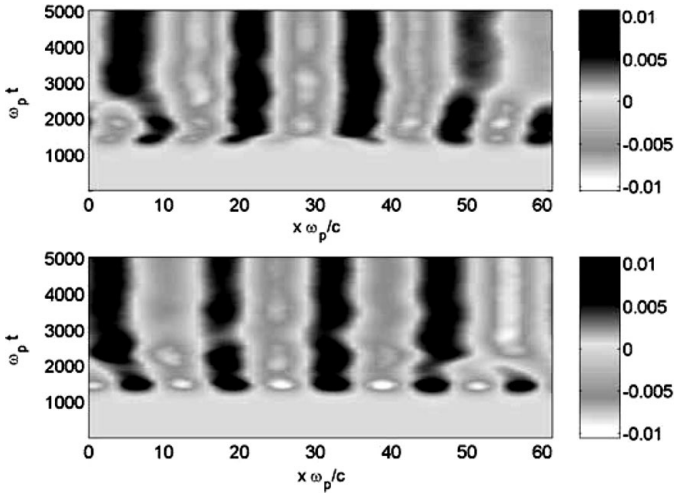


Figure 3. The magnetic field components B_y (top panel) and B_z (bottom panel) as a function of space and time, for $\beta\mu = 5$ and $T_\perp/T_{||} = 2$. The magnetic field has been normalized by $\omega_p m/e$. We see a nonlinear saturation of the magnetic field components at an amplitude of $\sim$0.01.

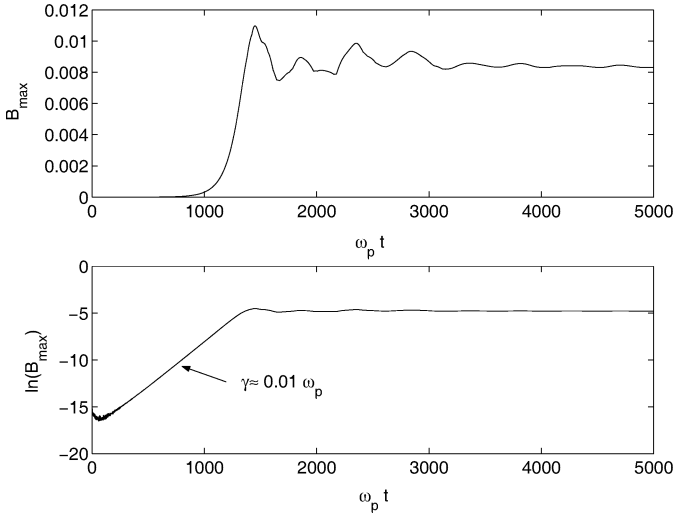


Figure 4. The maximum of the magnetic field amplitude, $B = (B_y^2 + B_z^2)^{1/2}$, over the simulation box (top panel), and the logarithm of the magnetic field maximum (bottom panel) as a function of time, for $T_\perp/T_{||} = 2$ and $\beta\mu = 5$. The magnetic field has been normalized by $\omega_p m/e$. From the logarithmic slope of the magnetic field in the linear regime we find $\gamma \approx \Delta \ln(B_{\max})/\Delta t \approx 0.01\,\omega_p$.

used the distribution function (2). In order to give a seed for any instability, the plasma density was perturbed with low-frequency fluctuations (random numbers). The results are displayed in Figs 3 and 4, for the parameters $\beta\mu = 5$ and $T_\perp/T_{||} = 2$, corresponding to the solid lines in Figs 1 and 2. Figure 3 shows the magnetic field components as a function of space and time. We see that the magnetic field

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initially grows, and saturates to steady-state magnetic field fluctuations with an amplitude of $eB/m\omega_{\rm p} \approx 0.008$. The maximum amplitude of the magnetic field over the simulation box as a function of time is shown in Fig. 4, where we see that the magnetic field saturates at $eB/m\omega_{\rm p} \approx 0.0082$, while the linear growth rate of the most unstable mode is $\gamma_{\max}/\omega_{\rm p} \approx 0.009$. Similar to the classical Maxwellian plasma case [38], we can thus estimate the magnetic field (in tesla) as

$$B = \frac{m\gamma_{\max}}{e}, \tag{7}$$

for a degenerate Fermi plasma. For our parameters that are relevant for intense laser–solid interaction experiments, we will thus have magnetic fields of the order of 10^5 T (1 gG).

4. Conclusion

In conclusion, we have demonstrated the existence of the Weibel instability for a Wigner–Maxwell dense quantum plasma, taking into account an anisotropic Fermi–Dirac equilibrium distribution function and the quantum diffraction effect. Numerically solving the dispersion relation for transverse waves, we found the dependence of the growth rate on the Fermi energy and the temperature anisotropy. The nonlinear saturation level of the magnetic field was found by means of kinetic simulations, which show a linear dependence between the growth rate and the saturated magnetic field. The present results may account for intense magnetic fields in dense quantum plasmas, such as those in the next generation of intense laser–solid density plasma interaction experiments.

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