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Non-singular Smale flows on three-dimensional manifolds and Whitehead torsion

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Abstract. This paper deals with non-singular Smale flows on oriented 3-manifolds. We shall show a relation between the properties of invariant sets of a Smale flow and a kind of Whitehead torsion of the underlying manifold.

1. *Introduction*

There is a question, with a long history, of how dynamical systems are constrained by the topology of underlying phase space. The Poincaré–Hopf index theorem for the singular points of vector fields may be the first significant result in this direction. It is natural to ask for its generalization when the given invariant set is a little more complicated than a singular point. A Smale flow is a structurally stable flow with one-dimensional invariant set. It is a pioneer work that Franks in [**5**] found out the relation between the linking matrices of basic sets of a non-singular Smale flow in S^3 and the Alexander polynomial of link consisting of some closed orbits. Some results for more restricted Smale flow, namely Morse–Smale flow, can be found in [**13**].

The main result in this paper shows that the Franks' relation still holds for all Smale flows on any oriented compact 3-manifold (see Theorem 5.1). The Alexander polynomial is replaced by its generalization—a kind of Whitehead torsion.

Milnor noticed in [**10**] that there is a close connection between Alexander polynomials of links and Whitehead torsion. This connection was discussed systematically by Turaev. The torsion we shall use in this paper is just a slight generalization of that in [**17**], which was called Milnor torsion. Such a generalization has its advantage: torsion of all CW complex pairs in a given manifold will take their values in the same field. This enables us to prove our conclusion by using the multiplicativity of torsion to the filtration related to basic sets. The method here is geometric, a little different from those in [**5**, §3]. Of most importance is the construction of isolated neighborhoods and Conley index pairs of basic sets having indices 1. This enables us to 'see' these saddle sets more explicitly, and to make Smale flow more visible (cf. [**16**]).

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This paper is organized as follows. We shall review the definition and some related properties of Whitehead torsion in §2. By modifying the Conley index pair, we define the RSC index pair (see Definition 2.3) of any isolated invariant of a flow in §3. Section 4 devotes to the computation of torsions and homologies of RSC index pairs for all kinds of one-dimensional basic sets in a 3-manifold. Our main result lies in §5, moreover we illustrate some simple applications by using two examples. In §6, we shall explain the relation between our results and existed one.

2. *Torsions of complex pairs*

In this section, we review the definition of general Whitehead torsion (see [**11, 17**] for more details), and give a special one for our purpose. Some basic properties are obtained.

Let $C = (0 \to C_m \to C_{m-1} \to \cdots \to C_1 \to C_0 \to 0)$ be a finitely generated chain complex over a field F with finite length. If we are given an ordered basis $c = \{(c_i^{(1)}, c_i^{(2)}, \ldots, c_i^{(r_i)})\}_{i=0}^m$. The torsion $\tau(C, c)$ of C with preferred basis c is defined as follows. If the homology $H_*(C)$ of the complex C is non-zero, the torsion $\tau(C, c)$ is defined as 0. If $H_*(C)$ is zero, then for $i = 1, 2, \ldots, m$, there is a sequence $b_i = (b_i^{(1)}, b_i^{(2)}, \ldots, b_i^{(r_i')})$ of elements in C_i such that its image $\partial(b_i)$ under $\partial\colon C_i \to C_{i-1}$ consists of a basis of the boundary group B_{i-1}, which is equal to the cycle Z_{i-1} because of the null-homology. Hence, $\partial(b_{i+1})b_i = (\partial(b_{i+1}^{(1)}), \ldots, \partial(b_{i+1}^{(r_{i+1}')}), b_i^{(1)}, \ldots, b_i^{(r_i')})$ is also a basis of C_i. The torsion $\tau(C, c)$ is defined as

$$\tau(C, c) := \prod_{i=0}^{m} \det[\partial(b_{i+1})b_i / c_i]^{(-1)^{i+1}},$$

where $[\partial(b_{i+1})b_i / c_i]$ is the transition matrix from c_i to $\partial(b_{i+1})b_i$.

In the definition above, $\tau(C, c)$ depends on the choice of the basis c. It is not hard to check that this torsion does not depend on the choice of the bases b_i of B_i (see [**11**, §4]).

Let (Y, Z) be a CW complex pair in a connected CW complex X, and $p_\rho\colon \tilde{X} \to X$ be the covering determined by a homomorphism $\rho\colon \pi_1(X) \to G$, where G is a free abelian group. Pick a lifting $\tilde{e}$ for each cell e in $Y - Z$; the relative cellular chain complex $C_*(p_\rho^{-1}(Y), p_\rho^{-1}(Z))$ will have a natural free $\mathbb{Z}G$-module structure in basis with these $\tilde{e}$. Then we have the following.

Definition 2.1. We define

$$\tau^\rho(Y, Z) := \tau(C_*(p_\rho^{-1}(Y), p_\rho^{-1}(Z)) \otimes_{\mathbb{Z}G} Q(\mathbb{Z}G)),$$

where $Q(\mathbb{Z}G)$ is the quotient field of the ring $\mathbb{Z}G$.

We shall simply write $\tau^\rho(Y)$ for $\tau^\rho(Y, Z)$ if $Z = \emptyset$.

Remark 2.2. As usual, the torsion $\tau^\rho(Y, Z)$ is defined up to a factor $\pm g$ with $g \in G$. Equivalently, if we regard $\mathbb{Z}G = \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ as a Laurent polynomial ring, where n is the rank of G, then any torsion $\tau^\rho(Y, Z)$ is a rational polynomial up to a sign and a factor of monomial. In this sense, the torsion $\tau^\rho(Y, Z)$ is well defined and independent of the choice of liftings of cells in $Y - Z$.

In simply homotopy theory, a CW complex Y is said to *collapse to Z by an elementary collapsing* if (1) $Y = Z \cup e^{n-1} \cup e^n$ is obtained from Z by attaching two cells; (2) there

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is a ball pair (D^n, D^{n-1}) and a characteristic map $\chi: D^n \to Y$ of e^n such that $\chi|_{D^{n-1}}$ is a characteristic map of e^{n-1}, and such that $\chi(Cl(\partial D^n - D^{n-1}))$ is contained in the $n-1$ skeleton Z^{n-1} of Z. Two CW complexes Y and Y' are said to have *the same simple homotopy type* if they are connected by a series of elementary collapsing or its inverse.

Similar to others, our torsion is also simply homotopy invariant and is preserved under subdivision. We shall generalize this invariance to the case of complex pairs, including the exclusion. This concept was introduced by Fried in [**7**].

Definition 2.3. We say that a CW complex pair (Y', Z') is obtained from (Y, Z) by an elementary operation of CW pairs if one of the following conditions is satisfied.

(1) Y collapses to Y' by an elementary collapsing and $Z = Z'$.

(2) Z' is a subcomplex of Z and $Y - Z = Y' - Z'$.

(3) Z collapses to Z' by an elementary collapsing and $Y = Y'$.

Two CW complex pairs (Y, Z) and (Y', Z') are said to be *RSC equivalent* (relative simple homotopy type on the complement) if there is a sequence $(Y, Z) = (Y_0, Z_0), (Y_1, Z_1), \ldots, (Y_n, Z_n) = (Y', Z')$ of CW complex pairs such that (Y_i, Z_i) is obtained from (Y_{i-1}, Z_{i-1}) by an elementary operation or its inverse for $i = 1, 2, \ldots, n$.

Two CW complex pairs (Y, Z) and (Y', Z') which are RSC equivalent have the same relative homotopy type, and therefore have the same homology groups. Moreover, we have the following proposition.

PROPOSITION 2.4. *Let $\rho: \pi_1(X) \to G$ be a homomorphism, determining a covering $p_\rho: \tilde{X} \to X$, where G is a free abelian group. If two CW complex pairs (Y, Z) and (Y', Z') in X are RSC equivalent. Then*

$$\tau^\rho(Y, Z) = \tau^\rho(Y', Z'), \quad H_*(p_\rho^{-1}(Y), p_\rho^{-1}(Z)) = H_*(p_\rho^{-1}(Y'), p_\rho^{-1}(Z')).$$

Proof. By definition, it is sufficient to consider the case that (Y, Z) is obtained from (Y', Z') elementary operation of CW pair.

(1) If Y collapses to Y' by an elementary collapsing and $Z = Z'$, then $Y = Y' \cup e^{n-1} \cup e^n$. Fix a lifting for each cell in Y, especially, the liftings $\tilde{e}^{n-1}$ and $\tilde{e}^n$ of e^{n-1} and e^n respectively. Let $\bar{C}$ denote the free $\mathbb{Z}G$-module generated by $\tilde{e}^{n-1}$ and $\tilde{e}^n$; we then have an exact sequence

$$0 \to \bar{C} \to C_*(p_\rho^{-1}(Y), p_\rho^{-1}(Z)) \to C_*(p_\rho^{-1}(Y'), p_\rho^{-1}(Z')) \to 0.$$

By tensoring the field $Q(\mathbb{Z}G)$, it follows from [**17**, Theorem 0.1.1] that $\tau^\rho(Y, Z) = \tau^\rho(Y', Z')$. Notice that $H_*(\bar{C}) = 0$. The second equality can be derived directly from long exact sequence of homologies.

(2) If Z' is a subcomplex of Z and $Y - Z = Y' - Z'$. The chain complexes $C_*(p_\rho^{-1}(Y), p_\rho^{-1}(Z))$ and $C_*(p_\rho^{-1}(Y'), p_\rho^{-1}(Z'))$ are the same.

(3) If Z collapses to Z' by an elementary collapsing and $Y = Y'$. We may consider the exact sequence

$$0 \to C_*(p_\rho^{-1}(Z), p_\rho^{-1}(Z')) \to C_*(p_\rho^{-1}(Y'), p_\rho^{-1}(Z')) \to C_*(p_\rho^{-1}(Y), p_\rho^{-1}(Z)) \to 0.$$

By [**17**, Theorem 0.1.1] again, we obtain that $\tau^\rho(Y, Z) = \tau^\rho(Y', Z')$. The second conclusion still holds because $H_*(C_*(p_\rho^{-1}(Z), p_\rho^{-1}(Z'))) = 0$. □

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The other important property of the torsion is the following.

PROPOSITION 2.5. (The multiplicativity of torsion) *Let (U, V, W) be a triple of CW sub complexes of a connected CW complex X. Let $\rho: \pi_1(X) \to G$ be a homomorphism from the fundamental group of X to a free abelian group G. Then $\tau^\rho(U, W) = \tau^\rho(U, V)\tau^\rho(V, W)$.*

Proof. See the proof of [**17**, Theorem 0.1.1]. □

3. *Index pairs of invariant sets*

In this section, we shall discuss the Conley index of an isolated invariant set for a smooth flow. Some adjustments will be made for our purpose here. The basic materials on this topic can be found in [**4**].

Let ϕ_t be a smooth flow on a smooth compact manifold M. We always assume that the vector field $d\phi_t/dt$ generated by ϕ_t is transverse on ∂M if $\partial M \neq \emptyset$. In this case, ∂M is partitioned into two collections of connected components, $\partial_+ M$ and $\partial_- M$, where $d\phi_t/dt$ points inward on $\partial_+ M$ and outward on $\partial_- M$. A subset K of M is said to be an invariant set of a flow ϕ_t if $\bigcup_{-\infty<t<\infty} \phi_t(K) = K$. An invariant set is said to be isolated if there is a neighborhood N of K in M such that $\bigcap_{-\infty<t<\infty} \phi_t(N) = K$. Conley index theory shows that any isolated invariant set has an index pair and any two index pairs of an isolated invariant set have the same relative homotopy type. This unique relative homotopy type is said to be the *Conley index* of the given isolated invariant set. In this paper, we consider only the indices in the smooth case and use an early construction of index pairs in [**12**].

Definition 3.1. A sub-manifold B with corners of M is an *isolating block* for the flow ϕ_t if the following hold.

(1) $\partial B = \partial_+ B \cup \partial_0 B \cup \partial_- B$ is a union of submanifolds with boundaries of codimension one.
(2) The vector field $d\phi_t/dt$ points into B on $\partial_+ B$ and points out of B on $\partial_- B$.
(3) For any $x \in \partial_0 B$, there is an interval $[T_1, T_2]$ containing 0 such that $\phi_{[T_1,T_2]}(x) \subseteq \partial_0 B$, $\phi_{T_1}(x) \in \partial_+ B$ and $\phi_{T_2}(x) \in \partial_- B$.

An isolating block B is an isolating neighborhood of its maximal invariant set $K = \bigcap_{-\infty<t<\infty} \phi_t(B)$. We also say that B is an isolating block of this invariant set K.

THEOREM 3.2. *If two isolating blocks B and B' have the same maximal invariant set, then $(B, \partial_- B)$ and $(B', \partial_- B')$ are RSC equivalent.*

Proof. After some modifications to the statements in [**12**, §3] for the smooth case (cf. [**7**]), there are two elementary operations (see Figure 1) among the isolating blocks for a given invariant set.

(1) B_2 is a *shave* of B_1 if B_2 is obtained from B_1 by deleting an open set of orbits joining $\partial_+ B_1$ to $\partial_- B_1$, i.e.

$$B_2 = B_1 - \{\phi_t(x) \mid x \in U, 0 \leq t \leq \bar{T}(x)\},$$

where U is a relative open subset in $\partial_+ B_1$ and $\bar{T}: U \to (0, +\infty)$ is a smooth function with

$$\phi_{(0,\bar{T}(x))}(x) \subset B_1 - (\partial_+ B_1 \cup \partial_- B_1) \quad \text{and} \quad \phi_{T(x)}(x) \in \partial_- B_1.$$

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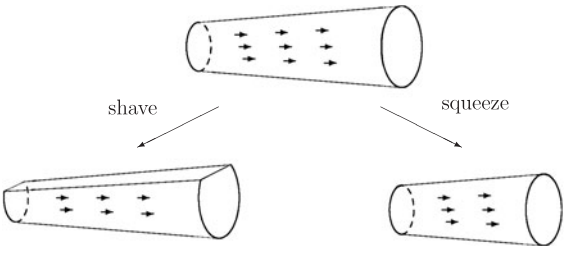


FIGURE 1. Two kinds of elementary operations.

(2) B_2 is a *squeeze* of B_1 if B_2 is obtained from B_1 by compressing B_1 along the flow, i.e.

$$B_2 = B_1 - \{\phi_t(x) \mid x \in \partial_+ B_1,\ 0 \le t < g_+(x)\} - \{\phi_t(x) \mid x \in \partial_- B_1,\ -g_-(x) < t \le 0\},$$

where $g_+ \colon \partial_+ B_1 \to [0, \infty)$ and $g_- \colon \partial_- B_1 \to [0, \infty)$ are two smooth functions satisfying: $\phi_t(x) \ne \phi_s(y)$ for any $0 \le t \le g_+(x)$, $-g_-(y) \le s \le 0$, $x \in \partial_+ B$, $y \in \partial_- B_1$.

Using a CW decomposition which is compatible with the smooth structure of given manifold M and its sub-manifolds B_1 and B_2, we know that B_1 and B_2 are RSC equivalent.

By [**12**, Theorem 3.22], for any two isolating blocks B and B' having common isolating invariant set, there is an isolating block $\bar{B}$ such that $\bar{B}$ can be obtained from either B or B' by a sequence of shaves and squeezes. By Proposition 2.4, we are done. □

By using this result, we can adjust the Conley index to the following.

Definition 3.3. Let B be an isolating block of a flow ϕ_t on a manifold M. Then the RSC equivalent class of $(B, \partial_- B)$ is said to the *RSC index pair* of its maximal invariant set $K = \bigcap_{-\infty < t < \infty} \phi_t(B)$.

In this paper, we will focus on the RSC index pairs of chain-recurrent sets. A point $x \in M$ is called chain recurrent for ϕ_t if, for any ε, $T > 0$, there exist points $x = x_0, x_1, \ldots, x_n = x$ and real numbers $t_0, t_1, \ldots, t_{n-1}$ all greater than T such that $d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon$ for all $0 < i \le n - 1$. The set of all chain-recurrent points is said to be the chain-recurrent set of the flow ϕ_t, denoted $\mathfrak{R}(\phi_t)$. The importance of chain recurrence for the description of asymptotic behavior of orbits is shown in the following theorem of Wilson [**18**]. A great generalization for the continuous case is due to Conley [**4**].

PROPOSITION 3.4. *There exists a smooth real-valued function $g \colon M \to \mathbb{R}$ such that the following hold.*

(1) *If $x \notin \mathfrak{R}(\phi_t)$, then $(d/dt)(g(\phi_t(x))) < 0$.*

(2) *If $x, y \in \mathfrak{R}(\phi_t)$, then $g(x) = g(y)$ if and only if for any ε, $T > 0$, there exist points $x = x_0, x_1, \ldots, x_n = y, x_{n+1}, \ldots, x_{2n} = x$ and real numbers $t_0, t_1, \ldots, t_{2n-1}$ all greater than T such that $d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon$ for all $0 < i \le 2n - 1$.*

A function $g \colon M \to \mathbb{R}$ satisfying the above conclusions is called a *Lyapunov function*. In practice, Conley index pairs are often obtained by using a Lyapunov function; so is our new RSC index pair. For such a Lyapunov function, $(M_a, M_b) =: (\{x \in M \mid g(x) \le a\}, \{x \in M \mid g(x) \le b\})$ is an RSC index pair provided neither of a and b shares the same g value as a chain recurrent point.

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4. *Hyperbolic chain recurrent sets*

In this section, we shall construct RSC index pair for all kinds of basic sets of a Smale flow in a 3-manifold. Some basic definitions are reviewed (see [**6**] for more details).

Definition 4.1. A invariant set K of a smooth flow ϕ_t is said to be hyperbolic if the tangent bundle of M restricted to K is a Whitney sum $E^s \oplus E^u \oplus E^c$ of the three sub-bundles, each sub-bundle is invariant under ϕ_t such that the following hold.

(i) The vector field tangent to ϕ_t spans E^c.

(ii) There are positive real numbers C and λ such that, for all $t \geq 0$, $v \in E^s$ and $w \in E^u$,

$$\|D\phi_t(v)\| \leq Ce^{-\lambda t}\|v\|, \quad \|D\phi_t(w)\| \geq Ce^{\lambda t}\|w\|.$$

The sub-bundles E^s and E^u are respectively said to be stable and unstable bundles of ϕ_t. The fiber dimension of unstable bundle E^u is called the index of hyperbolic set K.

An important consequence of hyperbolicity of chain recurrent set $\mathfrak{R}$ is that $\mathfrak{R}$ is a finite disjoint union of compact invariant sets $\Omega_1, \Omega_2, \ldots, \Omega_n$. Each Ω_i, which said to be a *basic set* of flow ϕ_t, contains an orbit of ϕ_t which is dense in it (see [**15**]).

Definition 4.2. A smooth flow ϕ_t is said to be a Smale flow if the chain recurrent set of ϕ_t is hyperbolic and has zero or one dimension. A Smale flow said to be non-singular if it does not contain any singular point.

It is well-known that each 3-manifold has unique smooth structure. Thus, we shall drop the word 'smooth' for 3-manifolds and flows in the remaining part of this paper.

Note that a one-dimensional basic set Ω of a flow ϕ_t on a 3-manifold having index 0 or 2 must be a closed orbit. Thus, we can choose a RSC index pair $(S^1 \times D^2, \emptyset)$ for an attracting closed orbit and a RSC index pair $(S^1 \times D^2, S^1 \times \partial D^2)$ for a repelling closed orbit. A direct computation shows that the following.

LEMMA 4.3. *Let Ω be a one-dimensional basic set with index 0 or 2 of a flow on a 3-manifold M, and let $p_\rho \colon \tilde{M} \to M$ be the covering determined by a homomorphism $\rho \colon \pi_1(M) \to G$ from the fundamental group to a free abelian group. Then, for any RSC index pair (Y, Z) of Ω,*

$$\tau^\rho(Y, Z) = \begin{cases} \dfrac{1}{1-\rho(\Omega)} & \text{if } \rho(\Omega) \neq 1 \in G, \\ 0 & \text{if } \rho(\Omega) = 1 \in G, \end{cases}$$

$$H_k(p_\rho^{-1}(Y), p_\rho^{-1}(Z)) = \begin{cases} \dfrac{\mathbb{Z}G}{\langle 1-\rho(\Omega)\rangle} & \text{if } \rho(\Omega) \neq 1 \in G,\ k = i(\Omega), \\ \mathbb{Z}G & \text{if } \rho(\Omega) = 1 \in G,\ k = i(\Omega), i(\Omega)+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho(\Omega)$ indicates the ρ-image of the conjugacy class in $\pi_1(M)$ determined by the circle Ω, $\langle 1-\rho(\Omega)\rangle$ is the additive subgroup of $\mathbb{Z}G$ generated by $1-\rho(\Omega)$, and $i(\Omega)$ is the index of basic set Ω.

In contrast to the case of index 0 or 2, a basic set of index 1 may have very complicated structure. Bowen in [**2**] and Bowen and Walters in [**3**] considered this situation and show

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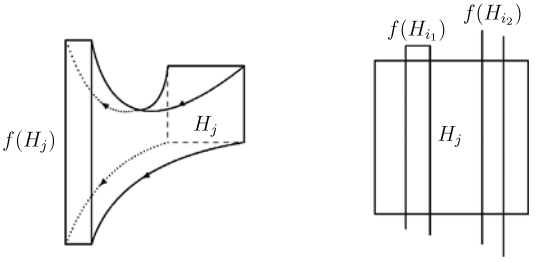


FIGURE 2. $\bigcup_{x\in H_j}(\bigcup_{0\le t<T(x)}\phi_t(x))$ and $H_j\cap f(H)$.

that one can construct a Poincaré section, which can be taken to be a finite union of disjoint discs $\sqcup Q_i$. The importance lies in the behaviors of return map on some disjoint rectangles $\{H_i\}$, which are said to be handles, in $\sqcup Q_i$. We summarize as follows.

PROPOSITION 4.4. ([**8**, Lemma 2.2.6], cf. [**1**]) *Let Ω be a basic with index 1 of a flow on a 3-manifold M. Then there is a Poincaré section Q to the basic set Ω with a return map $f: Q\to Q$. Moreover, there are finitely many disjoint rectangles $\{H_i\}$ in Q such that the following hold.*

(1) $\bigcup_i(\Omega\cap H_i)=Q\cap\Omega$.
(2) *As a rectangle, each H_i is coordinated by local stable and unstable manifolds.*
(3) *If $f(H_i)\cap H_j\neq\emptyset$, then $f(H_i)$ stretches completely across H_j in the unstable direction, and $f^{-1}(H_j)$ stretches completely across H_i in the stable direction.*
(4) *$f(H_i)\cap H_j$ has at most one connected component.*

We wish to construct an isolating block of the given basic set Ω of index 1, and calculate its torsion. Write $H=\bigcup_i H_i$ for the union of all handles. As in [**8**, §2.2], we have a flow box neighborhood $N(\Omega)$ of Ω by flowing H forwards and backwards. Adding some new handles if necessary, this neighborhood is union $Y=\bigcup_{x\in H}(\bigcup_{0\le t<T(x)}\phi_t(x))$ of orbit segments of H in the first recurrent interval (see Figure 2), which is an isolated neighborhood of Ω. Here, $T: Q\to\mathbb{R}$ is the return time map, i.e. $T(x)$ is the smallest $t>0$ with $\phi_t(x)\in Q$.

It should be noticed that for a point $y_0\in f(H)\cap\partial H$, there is a small $\delta>0$ satisfying that $\phi_{(0,\delta)}(y_0)\subset\partial Y$ and $\phi_{(-\delta,0)}(y_0)\subset Y-\partial Y$ (see the left part of Figure 3). Thus, the condition (3) in Definition 3.1 is not satisfied, and hence Y is not an isolating block. We need to do some adjustment on Y. By adding some 'chocks'

$$C_-^\varepsilon=\left\{\phi_t(x)\;\middle|\;x\in f(H),\,d(x,\partial H)\le\varepsilon,\,0\le t\le T(x)\frac{\varepsilon-d(x,\partial H)}{\varepsilon}\right\},$$

covering the points like y_0 (see the right part of Figure 3) and those

$$C_+^\varepsilon=\left\{\phi_t(x)\;\middle|\;x\in H-f(H),\,d(x,f(\partial H))\le\varepsilon,\right.$$
$$\left.-T(f^{-1}(x))\frac{\varepsilon-d(x,f(\partial H))}{\varepsilon}\le t\le 0\right\},$$

corresponding to the inverse flow (please note that $\phi_{-T(f^{-1}(x))}(x)=f^{-1}(x)$ for any $x\in f(Q)$), we obtain an isolating block $\hat{Y}=Y\cup C_+^\varepsilon\cup C_-^\varepsilon$ of Ω, where ε is a small

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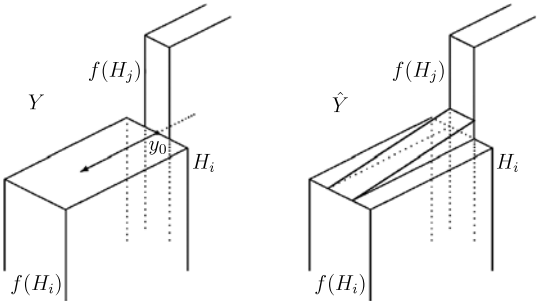


FIGURE 3. Adjusting the isolated neighborhood.

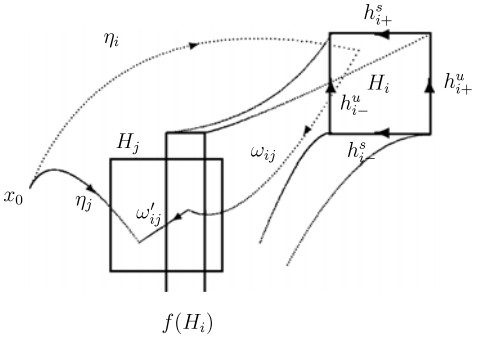


FIGURE 4. Constructing the linking matrix.

positive number. Moreover,

$$\partial_- \hat{Y} = (f(H) - H) \cup \left\{ \phi_t(x) \;\middle|\; x \in f(H) - H,\, d(x, \partial H) \le \varepsilon,\, t = T(x)\frac{\varepsilon - d(x, \partial H)}{\varepsilon} \right\}.$$

By collapsing each 'chock' down to its bottom, $(\hat{Y}, \partial_- \hat{Y})$ is RSC equivalent to (Y, Z), where $Z = (f(H) - H) \cup \{\phi_t(x) \mid x \in f(H) \cap \partial H,\, 0 \le t \le T(x)\}$.

In each H_i, the edges h^u_{i+} and h^u_{i-} in unstable direction and the edges h^s_{i+} and h^s_{i-} in stable direction are oriented so that the direction of h^u_{i+} and h^s_{i+} together with the natural direction of the flow coincides with the given orientation of manifold M, where the superscripts s and u are respectively represent the edges along the stable and unstable direction (see the upper-right corner of Figure 4).

Pick a base point $x_0 \in M$ and a path η_i from x_0 to some point in H_i for each H_i. For any two handles H_i and H_j, there is an element $l(i, j) = \langle \eta_i \omega_{ij} \omega'_{ij} \eta_j^{-1} \rangle$ in $\pi_1(M, x_0)$, where ω_{ij} is an orbit segment of ϕ_t from the terminal point of η_i to H_j, ω'_{ij} is the line segment in H_j from the terminal point of ω_{ij} to the terminal point of η_j. Since the H_i are small disks, $l(i, j)$ is a well-defined element in $\pi_1(M, x_0)$.

Since each handle is very small, the map $Df_x \colon E^u_x \to E^u_{f(x)}$ will preserve (or reverses) the unstable and stable orientation simultaneously for all $x \in H_i \cap f^{-1}(H_j)$ if $f(H_i) \cap H_j \neq \emptyset$. We shall write $b^{\pm}_{ij} = 1$ or -1 accordingly. The $b^{\pm}_{ij}$ is defined to be 0 if

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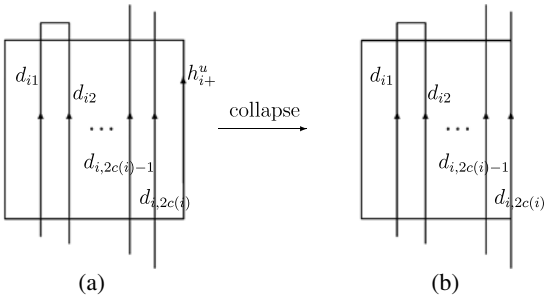


FIGURE 5. Collapsing the out-most part.

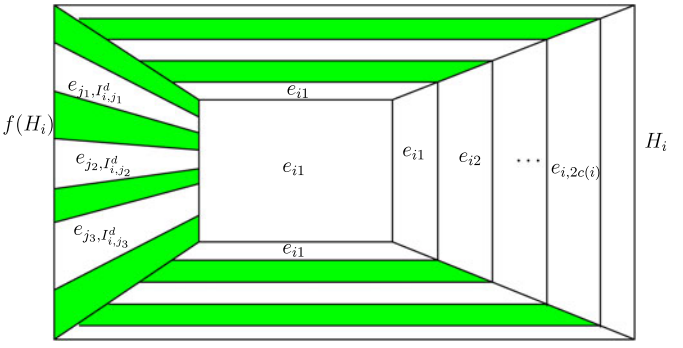


FIGURE 6. Flattened two-dimensional part.

$f(H_i) \cap H_j = \emptyset$. Given a homomorphism $\rho\colon \pi_1(M, x_0) \to G$, the matrix

$$B^\rho = (b^\rho_{ij})_{n\times n} = (b^\pm_{ij}\rho(l(i,\, j)))_{n\times n},$$

is said to be a linking matrix of Ω in M with respect to the homomorphism ρ (cf. [**5**, Definitions 2.2, 2.5]).

Let us go back to the RSC index pair (Y, Z). For the sake of simplification, we collapse, for each H_i, the unique three-dimensional interior together with a part of two-dimensional boundary: the union of orbit segments starting at h^u_{i+}. The intersection $H_i \cap (\bigcup_j f(h^u_{j+} \cup h^u_{j-}))$ form naturally 1-cells: $d_{i1}, d_{i2}, \ldots, d_{i,2c(i)}$, where $c(i)$ is just the number of non-zero entries in ith column of the linking matrix B^ρ. The second subscripts are indexed according to the order in H_i near to h^u_{i-}. We choose the same orientation for all d_{i*} as that of h^u_{i+} (see Figure 5(a)).

These 1-cells divide the remaining two-dimensional part (see Figure 6) of $\bigcup_{x\in H_i}(\bigcup_{0\le t<T(x)} \phi_t(x))$, which are not in Z, into $2c(i)+1$ 2-cells (unshaded regions in Figure 6).

Collapse the outermost 2-cell. The other 2-cells are written as $e_{i1}, e_{i2}, \ldots, e_{i,2c(j)}$, each being chosen such that the orientation coincides with H_i. The cell e_{ik} lies in $H_i \cap f(H)$ if k is even. However, when k is odd, the cell e_{ik} is much larger because it contains some flow lines initial at $\partial H_i - f(H)$; moreover, the cell e_{i1} contains all flow line segments starting at h^u_{i-}. The 2-cells in $f(H_i)$ will be labeled as cells in other handles, say $e_{j_1,I^e_{i,j_1}}$, $e_{j_2,I^e_{i,j_2}}$, etc.

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We write Y' for the resulting complex; hence (Y', Z) and (Y, Z) are RSC equivalent. We can use (Y', Z) to compute the torsion of RSC index pair of Ω. Let us consider the chain complex $C_*(p_\rho^{-1}(Y'), p_\rho^{-1}(Z))$. Pick a point $\tilde{x}_0 \in p^{-1}(x_0)$. For each given path η_i, there is a unique lifting $\tilde{\eta}_i$ of η_i with initial point at $\tilde{x}_0$. The lifting $\tilde{\eta}_i$ determines the unique component $\tilde{H}_i$ of $p_\rho^{-1}(H_i)$. For each i, we choose the liftings $\tilde{e}_{ik}$ and $\tilde{d}_{il}$ with $\tilde{e}_{ik} \cap \tilde{H}_i \neq \emptyset$ and $\tilde{d}_{il} \cap \tilde{H}_i \neq \emptyset$ for all possible k and l. For each i, we have that

$$\partial \tilde{e}_{i1} = \tilde{d}_{i1} - \sum_j b_{ij}^\rho \tilde{d}_{j,I_{ij}^d}, \quad \partial \tilde{e}_{ik} = -\tilde{d}_{i,k-1} + \tilde{d}_{ik}, \quad k = 2, 3, \ldots, 2c(i).$$

Clearly, if $f(H_k) \cap H_l \neq \emptyset$, then $I_{k,l}^d = I_{k,l}^e - (b_{kl}^\pm + 1)/2$, where $I_{k,l}^e$ is always even and $(1/2)I_{k,l}^e$ is the position of $f(H_k) \cap H_l$ in H_l according to the order near to h_{l-}^u. Thus, the unique non-trivial boundary homomorphism in $C_*(p_\rho^{-1}(Y'), p_\rho^{-1}(Z))$ is given by

$$\partial \begin{pmatrix} \tilde{e}_{11} \\ \vdots \\ \tilde{e}_{1,2c(1)} \\ \vdots \\ \tilde{e}_{n1} \\ \vdots \\ \tilde{e}_{n,2c(n)} \end{pmatrix} = \begin{pmatrix} J(2c(1)) - E^{2c(1)}_{1,I^d_{1,1}}(b^\rho_{11}) & \cdots & -E^{2c(n)}_{1,I^d_{1n}}(b^\rho_{1n}) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ -E^{2c(1)}_{1,I^d_{n1}}(b^\rho_{n1}) & \cdots & J(2c(n)) - E^{2c(n)}_{1,I^d_{nn}}(b^\rho_{nn}) \end{pmatrix} \times \begin{pmatrix} \tilde{d}_{11} \\ \vdots \\ \tilde{d}_{1,2c(1)} \\ \vdots \\ \tilde{d}_{n1} \\ \vdots \\ \tilde{d}_{n,2c(n)} \end{pmatrix}$$

where $E_{ij}^k(a)$ is the $k \times k$ matrix with (i, j)th entry a and the other entries 0, and

$$J(m) = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}$$

is a square matrix of size m. Thus, we obtain the following.

LEMMA 4.5. *Let Ω be a one-dimensional basic set with index 1 of a flow on a 3-manifold M, and let $p_\rho \colon \tilde{M} \to M$ be the covering determined by a homomorphism $\rho \colon \pi_1(M) \to G$ from the fundamental group to a free abelian group. Then, for any RSC index pair (Y, Z) of Ω, we have the following.*

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(1) $\tau^\rho(Y, Z) = \det(I - B^\rho)$.

(2) $$H_k(p_\rho^{-1}(Y), p_\rho^{-1}(Z)) = \begin{cases} \operatorname{coker}(I - B^\rho : (\mathbb{Z}G)^n \to (\mathbb{Z}G)^n) & \text{if } k = 1, \\ \ker(I - B^\rho : (\mathbb{Z}G)^n \to (\mathbb{Z}G)^n) & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where B^ρ is the linking matrix of Ω in M with respect to the homomorphism $\rho : \pi_1(M) \to G$ and n is the size of the matrix B^ρ.

From this lemma, we know that up to a multiple $\pm g$ with $g \in G$, the determinate $\det(I - B^\rho)$ is independent of the choices of handles $\{H_i\}$, base point x_0, and base paths η_i.

5. *Main results*

Now we are ready to present our main result in this paper, lying in the following.

THEOREM 5.1. *Let ϕ_t be a non-singular Smale flow on an oriented connected compact 3-manifold M. Suppose that $\rho : \pi_1(M) \to G$ is non-trivial homomorphism from the fundamental group of M to a free abelian group. If $\tau^\rho(M, \partial_- M) \neq 0$, then*

$$\prod_i \det(I - B_i^\rho) = \tau^\rho(M, \partial_- M) \prod_j (1 - \rho(\gamma_j)),$$

where B_i^ρ ranges over the linking matrixes of all basic sets of index 1 with respect to ρ, and γ_j ranges over all the closed orbits which are attractors or repellers.

Proof. We may choose a Smale filtration $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_m = M$ such that the indices of basic sets $\Omega_i = \bigcap_t \phi_t(M_i - M_{i-1})$ are non-decreasing. Let i' and i'' be the integers such that the index of Ω_i is 0 for $1 \le i \le i'$, is 1 for $i' < i \le i''$, and is 2 for $i'' < i \le m$.

For any CW complex pair (Y, Z) in M, we shall write $H_*^Q(Y, Z)$ for homology of the chain complex $C_*(p_\rho^{-1}(Y), p_\rho^{-1}(Y)) \otimes_{\mathbb{Z}G} Q(\mathbb{Z}G)$, where $Q(\mathbb{Z}G)$ is the field of quotient of the ring $\mathbb{Z}G$.

Since it is an isolating block of the union of all basic sets of indices 0, $(M_{i'}, \partial_- M)$ is RSC equivalent to $(\bigsqcup_{i=1}^{i'} W_i, \emptyset)$, where each W_i is a tubular neighborhood of the attracting closed orbit $\Omega_i = \bigcap_t \phi_t(M_i - M_{i-1})$ and hence is a solid torus. If $\rho(\Omega_i) \neq 1$ for any i with $1 \le i \le i'$, by Lemma 4.3, we have $\tau^\rho(M_{i'}, \partial_- M) = \prod_{i=1}^{i'} 1/(1 - \rho(\Omega_i))$. Applying the multiplicativity of torsion to $(M, M_{i'}, \partial_- M)$, we obtain that

$$\tau^\rho(M, \partial_- M) \prod_{i=1}^{i'} (1 - \rho(\Omega_i)) = \tau^\rho(M, M_{i'}). \tag{1}$$

If $\rho(\Omega_{i_0}) = 1$ for some i_0 with $1 \le i_0 \le i'$, then, by Lemma 4.3 again, $H_0^Q(M_{i'}, \partial_- M) = \bigoplus_{i=1}^{i'} H_0^Q(W_i, \emptyset) = (Q(\mathbb{Z}G))^s \neq 0$, where s is the number of Ω_i with $1 \le i \le i'$ and $\rho(\Omega_i) = 1$. Since ρ is non-trivial and M is connected, we have that $H_0^Q(M, \partial_- M) = 0$. It follows from the exact sequence

$$\cdots \to H_1^Q(M, M_{i'}) \to H_0^Q(M_{i'}, \partial_- M) \to H_0^Q(M, \partial_- M) \to \cdots$$

that $H_1^Q(M, M_{i'}) \neq 0$. Thus, $\tau^\rho(M, M_{i'}) = 0$, and hence the equation (1) still holds.

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Now, we claim that

$$\tau^{\rho}(M_k, M_{i'}) = \prod_{i=1}^{k} \det(I - B_i^{\rho}), \quad k = i'+1, i'+2, \ldots, i''. \tag{2}$$

In fact, this equation turns out to be Lemma 4.5 if $k = i'+1$. We can prove the general case by induction. Assume that $\tau^{\rho}(M_{k-1}, M_{i'}) = \prod_{i=1}^{k-1} \det(I - B_i^{\rho})$. The multiplicativity of torsion and Lemma 4.5 will imply the equation (2) if either $\prod_{i=1}^{k-1} \det(I - B_i^{\rho})$ or $\det(I - B_k^{\rho})$ is not zero. In the case of $\prod_{i=1}^{k-1} \det(I - B_i^{\rho}) = \det(I - B_k^{\rho}) = 0$, by Lemma 4.5, we know that $H_1^{Q}(M_k, M_{k-1}) \neq 0$, but $H_0^{Q}(M_{k-1}, M_{i'}) = 0$. From the exact sequence

$$\cdots \to H_1^{Q}(M_k, M_{i'}) \to H_1^{Q}(M_k, M_{k-1}) \to H_0^{Q}(M_{k-1}, M_{i'}) \to \cdots,$$

we obtain that $H_1^{Q}(M_k, M_{i'}) \neq 0$. It follows that $\tau^{\rho}(M_k, M_{i'}) = 0$. Hence, the equation (2) still holds.

Consider the triple $(M, M_{i''}, M_{i'})$. Notice that the invariant set of $M - M_{i''}$ is a disjoint union $\bigsqcup_{j=i''+1}^{m} \Omega_j$ of $m - i''$ repelling closed orbits. If $\rho(\Omega_j) \neq 1$ for all j with $i'' < j \leq m$, by Lemma 4.3, we have that $\tau^{\rho}(M, M_{i''}) = \prod_{j=i''+1}^{m} 1/(1 - \rho(\Omega_j))$. The multiplicativity of torsion shows that

$$\tau^{\rho}(M, M_{i'}) = \tau^{\rho}(M_{i''}, M_{i'}) \prod_{j=i''+1}^{m} \frac{1}{1 - \rho(\Omega_j)}.$$

By equations (1) and (2), we have

$$\prod_{i=i'+1}^{i''} \det(I - B_i^{\rho}) = \tau^{\rho}(M, \partial_- M) \prod_{i=1}^{i'} (1 - \rho(\Omega_i)) \prod_{j=i''+1}^{m} (1 - \rho(\Omega_j)). \tag{3}$$

If there is a basic set Ω_{j_0} with $\rho(\Omega_{j_0}) = 1$ for some $j_0 > i''$, then, by Lemma 4.3, we have

$$H_k^{Q}(M, M_{i''}) = \begin{cases} (Q(\mathbb{Z}G))^{s''} & \text{if } k = 2, 3, \\ 0 & \text{otherwise,} \end{cases}$$

where s is the number of repelling closed orbits Ω_j with $\rho(\Omega_j) = 1$. We claim that $\tau^{\rho}(M_{i''}, M_{i'}) = \prod_{i=i'+1}^{i''} \det(I - B_i^{\rho}) = 0$. Thus, the equation (3) still holds in this case that some $\rho(\Omega_j)$ with $j > i''$ are trivial, because both sides vanish. The equation is our desired conclusion.

In fact, if $\prod_{i=i'+1}^{i''} \det(I - B_i^{\rho}) \neq 0$, then $H_*^{Q}(M_{i''}, M_{i'}) = 0$. The long exact sequence of triple $(M, M_{i''}, M_{i'})$ would imply that

$$H_3^{Q}(M, M_{i'}) = H_3^{Q}(M, M_{i''}) = (Q(\mathbb{Z}G))^{s''}.$$

Since $\tau^{\rho}(M, \partial_- M) \neq 0$, we have $H_*^{Q}(M, \partial_- M) = 0$. By using the homology exact sequence for triple $(M, M_{i'}, \partial_- M)$, we would have that

$$H_2^{Q}(M_{i'}, \partial_- M) = H_3^{Q}(M, M_{i'}) = (Q(\mathbb{Z}G))^{s''}.$$

This is a contradiction, because $H_2^{Q}(M_{i'}, \partial_- M) = \bigoplus_{i=1}^{i'} H_2^{Q}(W_i, \emptyset)$ is always zero from Lemma 4.3. □

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COROLLARY 5.2. *Let ϕ_t be a non-singular Smale flow on an oriented connected compact 3-manifold. If there is a non-trivial homomorphism $\rho\colon \pi_1(M) \to G$ from the fundamental group of M to a free abelian group G such that $\tau^\rho(M, \partial_- M)$ has a non-trivial denominator, then ϕ_t has either an attracting or a repelling closed orbit.*

Following examples illustrate some applications of our results.

Example 5.3. Let ϕ_t be a non-singular Smale flow on an $S^1 \times D^2$. Take $\rho\colon \pi_1(S^1 \times D^2) \to H_1(S^1 \times D^2) = \mathbb{Z}$ to be the Hurewicz homomorphism. By a computation, we have $\tau^\rho(S^1 \times D^2, \partial_-(S^1 \times D^2)) = 1/(1-x)$. It follows from above corollary that ϕ_t must have either an attracting or a repelling closed orbit.

In dimension two, this conclusion becomes the so-called Poincaré–Bendixson theorem where the flow can be arbitrary provided that it keeps the transverse condition on the boundary. From the famous counter-example of Schweitzer [**14**], this phenomenon will not happen for general flow. Our example shows that Poincaré–Bendixson theorem has its three-dimensional analogy if we consider Smale flows.

Example 5.4. Let ϕ_t be a non-singular Smale flow on a solid torus $S^1 \times S^2$. Take $\rho\colon \pi_1(S^1 \times S^2) \to H_1(S^1 \times S^2) = \mathbb{Z}$ to be the Hurewicz homomorphism. By a computation, we have $\tau^\rho(S^1 \times S^2) = 1/(1-x)^2$. It follows from above corollary that ϕ_t must have either an attracting or a repelling closed orbit. Moreover, if all attractors and repellers are non-trivial in $H_1(S^1 \times S^2)$, then there must at least two attracting or repelling closed orbits.

What we like to remark is: this kind of torsion is usually computable. For example, as in [**9**], one can obtain $\tau^\rho(M)$ from a presentation of the fundamental group of M by using Fox calculus if M has non-empty boundary.

6. *Relation to Alexander polynomial*

In this final section, we explain the relation between the torsion we previously define and the classical Alexander polynomial, and hence show that our main result in this paper is really a generalization of Franks' in [**5**].

It should be noticed that if G is chosen as $H_1(M)/\text{Tors } H_1(M)$, i.e. the free part of $H_1(M)$, and if the homomorphism $\rho\colon \pi_1(M) \to G$ is the composition of the Hurewicz homomorphism $\theta\colon \pi_1(M) \to H_1(M)$ and the natural quotient homomorphism from $H_1(M)$ to $H_1(M)/\text{Tors } H_1(M)$, our torsion is the same as that in [**17**]. Using the relation between Alexander polynomial and torsion for 3-manifolds in [**17**, Theorem 1.1.2], we have the following.

PROPOSITION 6.1. *Let L be a link in an oriented closed 3-manifold M, and $X = cl(M - N(L))$ be the exterior of L in M. Suppose that $\rho\colon \pi_1(X) \to H_1(X)/\text{Tors } H_1(X)$. Then the Alexander polynomial of link L in M*

$$\Delta(M, L) = \begin{cases} \tau^\rho(X) & \text{if } b_1(X) > 1, \\ \tau^\rho(X)(x-1) & \text{if } b_1(X) = 1, \end{cases}$$

where $b_1(X)$ is the first Betti number of X, and x is the generator of $H_1(X)/\text{Tors } H_1(X)$ when $b_1(X) = 1$.

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Using this identification, Theorem 5.1 implies the following.

THEOREM 6.2. *Let ϕ_t be a non-singular Smale flow on an oriented closed 3-manifold M, and L be a link consisting of some attracting or repelling closed orbits of ϕ_t with exterior $X = M - \mathrm{int}(N(L))$. Suppose that the homomorphism $i_*\colon H_1(T) \to H_1(X)$ is non-trivial for any component T of $\partial_- X$. Then,*

$$\prod_{i=1}^{m} \det(I - B_i^\rho) = \begin{cases} \Delta(M, L) \prod_{i=1}^{k} (1 - \rho(\gamma_i)) & \text{if } b_1(X) > 1, \\ \Delta(M, L) \prod_{i=1}^{k} (1 - \rho(\gamma_i))(1 - x)^{-1} & \text{if } b_1(X) = 1, \end{cases}$$

where $B_1^\rho, B_2^\rho, \ldots, B_i^\rho$ are the linking matrixes of all basic sets (in X) of indices 1 with respect to the natural homomorphism $\rho\colon \pi_1(X) \to H_1(X)/\mathrm{Tors}\, H_1(X)$, $\gamma_1, \gamma_2, \ldots, \gamma_k$, are all attracting and repelling orbits of ϕ_t that do not lie in L, $b_1(X)$ is the first Betti number of X, and x is the generator of $H_1(X)/\mathrm{Tors}\, H_1(X)$ when $b_1(X) = 1$.

Proof. The restriction of ϕ_t on X is also a non-singular Smale flow. By Theorem 5.1 and Proposition 6.1, we need to prove $\tau^\rho(X, \partial_- X) = \tau^\rho(X)$. By the multiplicativity of torsion, it is sufficient to show that $\tau^\rho(\partial_- X) = 1$.

Note that each component T of $\partial_- X$ is a torus. By the multiplicativity of torsion again, it is also sufficient to show that $\tau^\rho(T) = 1$ for some component T of $\partial_- X$. Choose a standard CW decomposition for T: one 0-cell e^0, two 1-cells e_1^1, e_2^1 and one 2-cell e^2. Then $\pi_1(T) = H_1(T)$ is a free abelian group generated by e_1^1 and e_2^1. Let $p_\rho\colon \tilde{X} \to X$ be the covering determined by the homomorphism $\rho\colon \pi_1(X) \to H_1(X)/\mathrm{Tors}\, H_1(X)$. The chain complex $C_*(p_\rho^{-1}(T))$ vanishes in all dimensions except zero, one and two. Choose liftings $\tilde{e}^0$, $\tilde{e}_1^1$, $\tilde{e}_2^1$ and $\tilde{e}^2$ such that they consist of a square in covering space $\tilde{X}$. The chain groups $C_0(p_\rho^{-1}(T))$, $C_1(p_\rho^{-1}(T))$ and $C_2(p_\rho^{-1}(T))$ can be considered as free $\mathbb{Z}H_1(X)/\mathrm{Tors}\, H_1(X)$-modules generated respectively by $c_0 = \{\tilde{e}^0\}$, $c_1 = \{\tilde{e}_1^1, \tilde{e}_2^1\}$ and $c_2 = \{\tilde{e}^2\}$. The boundary homomorphisms are given by

$$\partial\tilde{e}^2 = (1 - \rho(e_2^1))\tilde{e}_1^1 + (\rho(e_1^1) - 1)\tilde{e}_2^1, \quad \partial\tilde{e}_1^1 = (\rho(e_1^1) - 1)\tilde{e}^0, \quad \partial\tilde{e}_2^1 = (\rho(e_2^1) - 1)\tilde{e}^0.$$

Since $i_*\colon H_1(T) \to H_1(X)$ is non-trivial and each element in $\pi_1(T) = H_1(T)$ is torsion free, we may assume that $\rho(e_1^1) \neq 1$, i.e. $\rho(e_1^1)$ is not the trivial element in $H_1(X)/\mathrm{Tors}\, H_1(X)$. Consider the complex $C_*(p_\rho^{-1}(T)) \otimes Q(\mathbb{Z}H_1(X)/\mathrm{Tors}\, H_1(X))$, $\{\partial\tilde{e}^2\}$ and $\{\partial\tilde{e}_1^1\}$ are respectively the bases of one- and zero-dimensional boundary groups. By definition, we have

$$\begin{aligned} \tau^\rho(T) &= \det[\{\partial\tilde{e}_1^1\}/\{\tilde{e}^0\}]^{(-1)^1} \det[\{\partial\tilde{e}^2\}\{\tilde{e}_1^1\}/\{\tilde{e}_1^1, \tilde{e}_2^1\}]^{(-1)^2} \det[\{\tilde{e}^2\}/\{\tilde{e}^2\}]^{(-1)^3} \\ &= \frac{\det \begin{pmatrix} 1 - \rho(e_2^1) & \rho(e_1^1) - 1 \\ 1 & 0 \end{pmatrix}}{\det(\rho(e_1^1) - 1) \det(1)} \\ &= -1. \end{aligned}$$

Notice that our torsion is defined up to a factor ± 1. We finish our proof. □

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This theorem generalizes Franks' result (see [**5**, Theorem 4.1], because $b_1(X)$ is equal to the component number of link L and the non-triviality of homomorphism $i_*\colon H_1(T) \to H_1(X)$ is satisfied automatically when $M = S^3$.

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