J. Aust. Math. Soc. **121** (2026), 76–93
doi:10.1017/S1446788726101542

NAKAJIMA QUIVER BUNDLES

LISA JEFFREY, MATTHEW KOBAN and STEVEN RAYAN

(Received 24 December 2024; accepted 20 February 2026; first published online 27 April 2026)

Communicated by Graeme Wilkin

Abstract

We introduce the notion of a *Nakajima bundle representation*. Given a labelled quiver and a variety or manifold X, such a representation involves an assignment of a complex vector bundle on X to each node of the double quiver; to the edges, we assign sections of, and connections on, associated twisted bundles. We for the most part restrict attention in our development to algebraic curves or Riemann surfaces. Our construction simultaneously generalizes ordinary Nakajima quiver representations on the one hand and quiver bundles on the other hand. These representations admit gauge-theoretic characterizations, analogous to the Atiyah–Drinfel'd–Hitchin–Manin equations in the original work of Nakajima, allowing for the construction of these generalized quiver varieties using a reduction procedure with moment maps. We study the deformation theory of Nakajima bundle representations, prove a Hitchin–Kobayashi correspondence between such representations and stable quiver bundles, examine the natural torus action on the resulting moduli varieties, and comment on scenarios where the variety is hyperkähler. Finally, we produce concrete examples that recover known moduli spaces.

2020 *Mathematics subject classification*: primary 14D20, 14D22, 32G13, 16G20, 53C07; secondary 14H60, 14J42, 53D20.

Keywords and phrases: quiver variety, quiver bundle, Hitchin–Kobayashi correspondence, moduli space.

1. Introduction

Motivated by the study of Yang–Mills equations on four-dimensional asymptotically locally Euclidean (ALE) spaces, Nakajima [Nak94] used linear representations of quivers to construct new examples of noncompact hyperkähler varieties in arbitrary dimensions—the so-called *Nakajima quiver varieties*, which can be thought of as moduli spaces of these representations. This construction builds on a particular narrative threaded through the work of Gibbons and Hawking in type A [GH79]; of Atiyah, Drinfel'd, Hitchin and Manin [AHDM78] when the underlying quiver is of type A, D or E; and of Kronheimer and Nakajima [KN90] concerning Yang–Mills instantons over ALE gravitational instantons. The construction is also closely related

© The Author(s), 2026. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

to the McKay correspondence for finite subgroups of SU (2) in the case where the underlying quiver is ADE.

Nakajima quiver varieties have been a mainstay in representation theory, in symplectic and complex algebraic geometry, and in mathematical physics. In foundational work [Nak98], Nakajima connected the aforementioned quiver varieties to integrable highest-weight representations of simply laced Kac–Moody algebras. Specifically, these representations admit an interpretation within the homology ring of these varieties, and the Kac–Moody algebra acts in a certain way via Hecke-type correspondences defined with respect to Lagrangian subvarieties of products of quiver varieties. These subvarieties in turn provide generators for the algebra. At the same time, variation of stability parameters for the moduli problem, viewed either via geometric invariant theory or through symplectic reduction, is a means for resolving symplectic singularities [BS21]—this includes somewhat surprising cases such as where *every* resolution of a particular quotient singularity in four complex dimensions is realizable as a Nakajima quiver variety for a single quiver [BCR$^+$24]. Furthermore, Nakajima quiver varieties are similar in crucial ways to moduli spaces of stable Higgs bundles: both are noncompact hyperkähler varieties equipped with an action by $\mathbb{C}^\times$, although one is a finite-dimensional quotient (Nakajima) and the other is an infinite-dimensional one (Hitchin). The relation between them is explored from the vantage point of character varieties in [HLRV11], as well as in [FR16, RS21] where Nakajima quiver varieties of star- and comet-shaped quivers are embedded into moduli spaces of meromorphic Higgs bundles, thereby eliciting a sub-integrable system of the Hitchin system for certain flag types.

A fundamental question concerns the behaviour of such quiver varieties when the linear representation data are generalized or 'globalized' to other types of representations, such as those valued in vector bundles over a fixed variety. This is already motivated in the literature. In [Hit87] there is the first study of fixed points of the $\mathbb{C}^\times$-action on the moduli space of Higgs bundles on a Riemann surface X. Each fixed point is a Higgs bundle with a splitting of the underlying bundle for which the Higgs field has a strictly sub-diagonal matrix representation. As such, viewed through the lens of representation theory, the fixed points are representations of A-type quivers in the category of holomorphic bundles over X. Here, each vertex is assigned a holomorphic vector bundle E_i, and each edge is assigned a section $\phi_i \in H^0(X, \mathrm{Hom}(E_i, E_{i+1}))$. Such representations are often referred to as 'holomorphic chains'. More generally (beyond A-type) they are referred to as *quiver bundles*. Such objects also appear as solutions to quiver vortex equations [AaCGP03], which themselves capture many well-known correspondences in that the existence of solutions is equivalent to the existence of connections satisfying certain curvature constraints. Some recent works on the physics side involving quiver bundles and emphasizing connections with higher-dimensional branes include [CS15a, CS15b, HHTZ23].

This brings us to the goal of the present paper: to produce a variety parametrizing representations of double quivers by complex vector bundles. Such representations will be referred to as 'Nakajima bundle representations'. In doing so, we construct

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

objects similar to quiver bundles on a double quiver, called $\overline{Q}$-bundles, and we provide a Hitchin–Kobayashi-type correspondence that identifies points in the moduli space of Nakajima representations with certain stable $\overline{Q}$-bundles. These representations differ from solutions of the quiver vortex equations described in [AaCGP03] in that the reverse edges appearing in the double quiver are represented by cohomology classes in the cotangent space of the original data. In this way we make use of the natural holomorphic symplectic structure on the representation space. The resulting objects can be thought of in two ways: providing a relative or global version of Nakajima quiver varieties on the surface X, or as the study of quiver bundles in a holomorphic symplectic context. Representations of double quivers in the category of vector bundles have appeared prior to the current work, having appeared for instance in [Sze08, Dia12] in the contexts of holomorphic D-branes, and local Donaldson–Thomas theory, respectively. These objects are also related to quasimaps [CFKM14]. In these works there is no way to distinguish between a representation thought of as coming from a double quiver, or an ordinary quiver. In the present work, the two are clearly separated. To the knowledge of the authors, the approach and generality provided in the current work have not been emphasized in the literature. Moreover, the current work provides a basis-independent construction involving hermitian matrices (Lemma 13), starting from functions ψ, Ψ that are only C^1-bounded rather than smooth.

We do note the parallel work of Azam and the third-named author [AR24] that develops similar ideas related to quiver bundles, but in a categorical context around moduli stacks with homotopy-theoretic applications in mind, as well as work of Minets [Min20] in which a cohomological Hall algebra was constructed on the homology of certain moduli stacks of Higgs bundles by viewing it as a cotangent stack.

1.1. Set-up. Consider the quiver Q with a single vertex and no arrows. Upon fixing a rank r and degree d, a quiver bundle is a holomorphic vector bundle of rank r and degree d on X. By Narasimhan and Seshadri [NS64], a stable holomorphic vector bundle on X is equivalent to a complex vector bundle E equipped with a hermitian connection A whose curvature satisfies the equation $F_A = 0$. One could then hope to recover the moduli space of stable bundles on X having rank r and degree d, $\mathcal{N}_X(r, d)$, from representations of this quiver. With this in mind, the representation space for the quiver with a single vertex and no arrows is given by the space of connections $\mathcal{A}(E)$ that are compatible with the hermitian metric on E. A variety parametrizing representations is then precisely the moduli space of hermitian Einstein connections.

Consider now the double quiver $\overline{Q}$ constructed from Q by adding an additional copy of each edge, whose orientation is reversed. Since there are no arrows to double, $\overline{Q}$ coincides with the ordinary quiver Q. However, it still makes sense to ask what representations of $\overline{Q}$ should be. The guiding principle for our approach is that in the construction of the Nakajima quiver variety, the representation space of the double quiver is the cotangent space of that of the original quiver. For the ordinary quiver, a representation is (equivalent to) a stable vector bundle on X. Deformations of stable bundles E are given by $H^1(X, \mathrm{End}(E))$, and so by Serre duality the cotangent directions

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

are $H^0(X, \mathrm{End}(E) \otimes K)$, where K is the canonical bundle of X. Thus, the cotangent space of the space of stable bundles consists of all pairs (E, ϕ) where E is a stable holomorphic bundle and ϕ is a holomorphic $\mathrm{End}(E)$-valued 1-form, in other words, a Higgs bundle.

For X of genus at least 2, Simpson [Sim92] proved that a stable Higgs bundle on X admits a hermitian metric h such that the pair (A_h, ϕ), satisfies the Hitchin equations

$$\begin{aligned} \sqrt{-1}\Lambda F_{A_h} + [\phi, \phi^*] &= 0, \\ \overline{\partial}_{A_h}\phi &= 0, \end{aligned} \tag{1-1}$$

where A_h is the Chern connection associated to h and $\Lambda : \Omega^{i,j}(X, E) \to \Omega^{i-1,j-1}(X, E)$ is contraction with the symplectic form. The moduli space $\mathcal{M}_X^{\mathrm{Higgs}}(r, d)$ of Higgs bundles of rank r and degree d is a hyperkähler variety arising as a reduction of Equation (1-1).

In order to recover Simpson's construction of $\mathcal{M}_X^{\mathrm{Higgs}}(r, d)$, a representation of the double quiver $\overline{Q}$ should be a hermitian connection A and a section $\phi \in \Omega^{1,0}(X, \mathrm{End}(E))$. With these data, the space of quiver representations satisfying Equation (1-1) up to equivalence recovers precisely the moduli space of Higgs bundles on $\mathcal{M}_X^{\mathrm{Higgs}}(r, d)$.

Now for a general quiver Q associate a bundle E_v with hermitian connection A_v to each vertex v. If $a : v_i \to v_j$ is an edge, we associate with it a bundle morphism $x_a : E_{v_i} \to E_{v_j}$, which can be identified with a global section of $\mathrm{Hom}(E_{v_i}, E_{v_j})$ on X. By taking the $(0, 1)$ part of the connection, we get a holomorphic structure on E_i allowing us to make sense of holomorphic sections. A section $x \in \Omega^0(X, \mathrm{Hom}(E_{v_i}, E_{v_j}))$ is holomorphic if it is holomorphic with respect to the operators $\overline{\partial}_{A_{v_i}}$ and $\overline{\partial}_{A_{v_j}}$. In [AaCGP03] it was shown that there is a correspondence between representations satisfying the vortex equations and stable Q-sheaves. For the double quiver $\overline{Q}$, to capture the doubled information at the vertices, following the discussion above, in addition to the bundle E, we assign a section $\phi \in \Omega^{1,0}(X, \mathrm{End}(E))$. For the reverse edges, assign a cohomology class y_a that is Serre dual to x_a. Taking all equivalence classes of representations of a given quiver that satisfy the moment map conditions equations (2-1) and (2-2), produces a variety $\mathcal{M}_{\overline{Q}}^{r,d}(\tau)$ we are calling a *Nakajima bundle variety*. These varieties are, in a sense, a global analogue of Nakajima quiver varieties.

1.2. Organization. The organization of this work is as follows. Sections 2–4 provide the theoretical framework and main results of the paper. Section 2 introduces the appropriate representations of double quivers and gives the construction of their moduli spaces, Nakajima bundle varieties. After constructing Nakajima bundle representations, Section 3 introduces the algebro-geometric counterpart $\overline{Q}$-bundles. Stable $\overline{Q}$-bundles are shown to be in correspondence with representations that solve the moment map conditions in Section 4. With the general framework out of the way, Section 5 is devoted to a study of the infinitesimal deformations of a Nakajima bundle representation, combining the work of [GK05] for quiver bundles with that of [Bot00]. Section 6 briefly looks at representations of the framed A_1-quiver. Finally, in Section 7 a $\mathbb{C}^\times$-action is described on Nakajima bundle varieties.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

2. Nakajima bundle varieties

The present section defines the main object of the current paper, Nakajima bundle representations. These are representations of double quivers in which vertices are represented by hermitian vector bundles, and edges by cohomology classes valued in Hom-bundles. There is a hermitian action of the group $\mathcal{G} = \prod_{v\in V} \mathrm{Aut}(E_v)$ on the space of all such representations. Symplectic reduction produces a moduli space $\mathcal{M}_{\overline{Q}}^{r,d}(\tau)$ of Nakajima bundle representations.

A *quiver* Q consists of a tuple (Q_0, Q_1, h, t), where $Q_0 = \{v_0, \dots, v_n\}$ is the set of vertices, $Q_1 = \{a : v_i \to v_j\}$ the set of edges, and $h, t : Q_1 \to Q_0$ are functions specifying the head and tail of each edge.

Given any quiver Q, the double quiver $\overline{Q}$ is a quiver whose sets of vertices and edges satisfy $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \sqcup -Q_1$. Here $-Q_1$ consists of a copy of each element of Q_1 but with opposite orientation.

In order to define representations of a quiver, it is necessary to equip it with a label. Given a quiver Q, a label of Q is a pair (r, d) where $r = (r_v)_{v\in Q_0} \in \mathbb{N}^{|Q_0|}$ and $d = (d_v)_{v\in Q_0} \in \mathbb{Z}^{|Q_0|}$. The data (r, d) are referred to as the type of the representation.

DEFINITION 1. Let $\overline{Q}$ be a double quiver with label (r, d). For each $v \in Q_0$, let E_v be a complex vector bundle over X of rank r_v and degree d_v. A *Nakajima bundle representation* of $\overline{Q}$ is a collection $(A, \phi, x, y) = (A_v, \phi_v, x_a, y_a)$ indexed by the vertices and edges of Q, consisting of the following objects: for every vertex, A_v is a connection on E_v, and $\phi_v \in \Omega^{1,0}(X, \mathrm{End}(E_v))$, and for every edge, $x_a \in \Omega^0(X, \mathrm{Hom}(E_{t(a)}, E_{h(a)}))$, and $y_a \in \Omega^{0,1}(X, \mathrm{Hom}(E_{t(a)}, E_{h(a)}) \otimes K)$ such that $\overline{\partial} x = 0$ with respect to the connections $A_{t(a)}$ and $A_{h(a)}$.

Let $\mathcal{A}(E_v)$ denote the space of unitary connections on E_v. The space $\mathrm{Rep}(\overline{Q})$ of Nakajima bundle representations of $\overline{Q}$ is $\mathrm{Rep}(\overline{Q}) = \mathrm{Rep}(Q_0) \oplus \mathrm{Rep}(Q_1)$, where

$$\mathrm{Rep}(Q_0) := \prod_{v\in Q_0} (\mathcal{A}(E_v) \times \Omega^{1,0}(X, \mathrm{End}(E_v)))$$

and $\mathrm{Rep}(Q_1)$ is the subset $\ker(\overline{\partial})$ of

$$\bigoplus_{a\in Q_1} (\Omega^0(X, \mathrm{Hom}(E_{t(a)}, E_{h(a)})) \ \oplus \ \Omega^{0,1}(X, \mathrm{Hom}(E_{h(a)}, E_{t(a)}) \otimes K)).$$

At each vertex, the group $\mathcal{G}_v := \mathrm{Aut}(E_v)$ of unitary bundle automorphisms acts on E_v, producing an action of $\mathcal{G} = \prod_{v\in Q_0} \mathcal{G}_v$ on $\mathrm{Rep}(\overline{Q})$:

$$g \cdot (A, \phi, x, y) = (g \cdot A, g_v \phi_v g_v^{-1}, g_{h(a)} x_a g_{t(a)}^{-1}, g_{t(a)} y_a g_{h(a)}^{-1}).$$

In [Hit87] it was shown that the action of $\mathcal{G}_v$ on $\mathcal{A}(E_v) \times \Omega^{1,0}(X, \mathrm{End}(E_v))$ is Hamiltonian with real and complex moment maps given by Equation (1-1). This extends to the action of the product group $\mathcal{G}$ on $\mathrm{Rep}(Q_0)$ whose real and complex moment maps are $\mu_{\mathbb{R}}^{\mathfrak{H}} = \bigoplus_{v\in Q_0} (\sqrt{-1}\Lambda F_{A_v} + [\phi_v, \phi_v^*])$ and $\mu_{\mathbb{C}}^{\mathfrak{H}} = \bigoplus_{v\in Q_0} \overline{\partial}_{E_v} \phi_v$.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

Consider the $\mathcal{G}$-action on the space of edge representations $\mathrm{Rep}(\overline{Q}_1)$. For each $a \in Q_1$ let y_a^{harm} be the unique harmonic representative of the class y_a. The representative y_a^{harm} may be viewed as a twisted morphism $E_{h(a)} \to E_{t(a)} \otimes \Omega^{1,1}(X)$. Using the metrics $h_{h(a)}$ and $h_{t(a)}$ on $E_{h(a)}$ and $E_{t(a)}$ respectively, there are adjoints $x^{*h_a} : E_{h(a)} \to E_{t(a)}$ and $(y_a^{\mathrm{harm}})^{*h_a} : E_{t(a)} \to E_{h(a)} \otimes \Omega^{1,1}(X)$. The proof of the following proposition is standard.

PROPOSITION 2. *The $\mathcal{G}$-action is Hamiltonian with real and complex moment maps $\mu_{\mathbb{R}}, \mu_{\mathbb{C}}$ whose components at the vertex v are given by*

$$\mu^{\mathfrak{E}}_{\mathbb{R},v}(x,y) = \sum_{a\in Q_1 : h(a)=v} (x_a \circ x_a^{*h_a} - \Lambda((y_a^{\mathrm{harm}})^{*h_a} \circ y_a^{\mathrm{harm}})) + \sum_{a\in Q_1 : t(a)=v} (x_a^{*h_a} \circ x_a - \Lambda(y_a^{\mathrm{harm}} \circ (y_a^{\mathrm{harm}})^{*h_a})) \tag{2-1}$$

and

$$\mu^{\mathfrak{E}}_{\mathbb{C},v}(x,y) = \Lambda\Big(\sum_{a\in Q_1 : h(a)=v} (x_a \otimes \mathrm{id}_K) y_a^{\mathrm{harm}} - \sum_{a\in Q_1 : t(a)=v} y_a^{\mathrm{harm}} (x_a \otimes \mathrm{id}_K) \Big). \tag{2-2}$$

DEFINITION 3. Let $\mathcal{Z}(\mathfrak{g})$ denote the centre of $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$. For a fixed type (r,d), and $\tau = (\tau_{\mathbb{R}}, \tau_{\mathbb{C}}) \in \mathcal{Z}(\mathfrak{g}) \oplus (\mathcal{Z}(\mathfrak{g}) \otimes \mathbb{C})$ the *Nakajima bundle variety at level* τ, $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$ is the moduli space of Nakajima quiver representations that are solutions of the equations

$$\mu_{\mathbb{R}}(A,\phi,x,y)_v : \mu^{\mathfrak{H}}_{\mathbb{R},v}(A,\phi) + \mu^{\mathfrak{E}}_{\mathbb{R},v}(x,y) = \tau_{\mathbb{R}}, \tag{2-3}$$

$$\mu_{\mathbb{C}}(A,\phi,x,y)_v : \mu^{\mathfrak{H}}_{\mathbb{C},v}(A,\phi) + \mu^{\mathfrak{E}}_{\mathbb{C},v}(x,y) = \tau_{\mathbb{C}}. \tag{2-4}$$

That is, it is the quotient

$$\mathcal{M}^{r,d}_{\overline{Q}}(\tau) := \mathrm{Rep}(\overline{Q}) /\!\!/_{\tau} \mathcal{G} = (\mu_{\mathbb{R}}^{-1}(\tau_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\tau_{\mathbb{C}}))/\mathcal{G}.$$

We close this section with a few remarks regarding the definition above.

REMARK 4.

(1) For nonzero values of the parameter $\tau_{\mathbb{C}}$, the 'Higgs fields' ϕ_v are no longer holomorphic in general. Although the current work deals primarily with the $\tau_{\mathbb{C}} = 0$ case, it would be interesting to see what can be said in full generality.
(2) Upon restricting to a point $p \in X$, a Nakajima bundle representation restricts to a linear representation of a quiver, having vertex set given by that of Q and edge set given by the edge set of $\overline{Q}$ with a loop added at each vertex. These quivers have been referred to as tripled quivers and have appeared in [MN18], as well as in the context of Donaldson–Thomas theory [Moz11].
(3) As is the case for Nakajima quiver varieties, the inclusion $\mu_{\mathbb{R}}^{-1}(\tau) \cap \mu_{\mathbb{C}}^{-1}(0) \to \mu_{\mathbb{C}}^{-1}(0)$, followed by the quotient map $\mu_{\mathbb{C}}^{-1}(0) \to \mu_{\mathbb{C}}^{-1}(0)//G$, gives a map $\mathcal{M}_{\overline{Q}}(\tau) \to \mathcal{M}_{\overline{Q}}(0)$.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

3. $\overline{Q}$-bundles

The main purpose of this section is to define an alternative notion of a quiver bundle for a double quiver $\overline{Q}$, which we refer to as $\overline{Q}$-bundles, such that stable $\overline{Q}$-bundles provide solutions to Equations (2-3) and (2-4). One of the main differences in the current work from that of previous correspondences, such as [AaCGP03], lies in the fact that the data associated to a reverse edge are given by a cohomology class in degree 1 rather than a true morphism of vector bundles. This is similar to the study of p-cohomology triples in [BGP95] and Higgs bundles on higher-dimensional varieties as in [Bot00].

DEFINITION 5. Given a double quiver $\overline{Q}$ with label (r, d), a $\overline{Q}$-bundle (E, ϕ, x, y) over X is a collection (E_v, ϕ_v, x_a, y_a) indexed by the vertices and edges of $\overline{Q}$, where E_v is a holomorphic vector bundle of rank r_v and degree d_v on X, $\phi_v \in H^0(X, \mathrm{End}(E_v) \otimes K)$, $x_a \in H^0(X, \mathrm{Hom}(E_{t(a)}, E_{h(a)}))$, and $y_a \in H^0(X, \mathrm{Hom}(E_{t(a)}, E_{h(a)}))^*$. A $\overline{Q}$-bundle (E, ϕ, x, y) will be denoted by $\mathcal{E}$ when the context is clear.

A morphism of $\overline{Q}$-bundles $f : (E, \phi, x, y) \to (E', \phi', x', y')$, is a collection $f_v : E_v \to E'_v$ of bundle morphisms such that the induced morphisms on cohomology satisfy $f_{t(a)*}(y') = (f_{h(a)} \otimes 1_K)^*(y)$, and the required squares commute for all v and a:

$$\begin{array}{ccc} E_v & \xrightarrow{\phi_v} & E_v \otimes K \\ {\scriptstyle f_v}\downarrow & & \downarrow{\scriptstyle f_v \otimes 1_K} \\ E'_v & \xrightarrow[\phi'_v]{} & E'_v \otimes K \end{array} \qquad \begin{array}{ccc} E_{t(a)} & \xrightarrow{x_a} & E_{h(a)} \\ {\scriptstyle f_{t(a)}}\downarrow & & \downarrow{\scriptstyle f_{h(a)}} \\ E'_{t(a)} & \xrightarrow[x'_a]{} & E'_{h(a)} \end{array}$$

DEFINITION 6. If (E, ϕ, x, y) is a $\overline{Q}$-bundle, a $\overline{Q}$-sub-bundle consists of the data (E', ϕ', x', y') where $E'_v \subseteq E_v$ is a sub-bundle for all v, and such that the sections ϕ_v, x_v, y_v agree with ϕ'_v, x'_v, y'_v when restricted to the E'_v. A $\overline{Q}$-subbundle will be referred to as a subbundle when the context is clear.

In order to identify $\overline{Q}$-bundles with solutions to Equations (2-1) and (2-2), it is necessary to restrict to certain subsets of *stable* $\overline{Q}$-bundles.

DEFINITION 7. Let $\mathcal{E}$ be a $\overline{Q}$-bundle on X. For a tuple of real numbers $\tau = (\tau_v)_{v \in Q_0}$, define the degree and the τ-rank of $\mathcal{E}$,

$$\deg(\mathcal{E}) := \sum_{v \in Q_0} \deg(E_v), \quad \mathrm{rk}_\tau(\mathcal{E}) := \sum_{v \in Q_0} \tau_v \, \mathrm{rk}(E_v).$$

The τ-slope of $\mathcal{E}$, is then defined as

$$\mu_\tau(\mathcal{E}) = \frac{\deg(\mathcal{E}) + \mathrm{rk}_\tau(\mathcal{E})}{\mathrm{rk}_1(\mathcal{E})}.$$

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

A $\overline{Q}$-bundle $\mathcal{E}$ is *semi-stable* if for all subbundles $\mathcal{E}'$, the slope satisfies $\mu_\tau(\mathcal{E}') \le \mu_\tau(\mathcal{E})$. It is said to be *stable* if the inequality is strict, and *polystable* if $\mathcal{E}$ is a direct sum of stable $\overline{Q}$-bundles all of which have the same slope.

REMARK 8.

(1) Given a constant $c \in \mathbb{R}$, transforming the stability parameter τ as $\tau'_v = \tau_v + c$ has the effect of transforming the slope as $\mu_{\tau'} = \mu_\tau - c$. This does not change the stability condition, hence it is always possible to set $\mu_\tau(\mathcal{E}) = 0$, and proving stability amounts to showing $\mu_\tau(\mathcal{E}') < 0$ for all subbundles $\mathcal{E}'$.
(2) The stability condition used here is not as general as that found in [AaCGP03], which depends on two stability parameters (σ, τ). However, it is expected that the current framework can be generalized without much difficulty.

Stable $\overline{Q}$-bundles behave similarly to stable vector bundles. In particular, a standard proof as in [Kob87, Corollary 5.7.12] shows the following proposition.

PROPOSITION 9. *If $\mathcal{E}$ is stable, then it is simple.*

4. Stable $\overline{Q}$-bundles and Nakajima Representations

The ideas behind the correspondence of Nakajima quiver bundles with $\overline{Q}$-bundles are not new, having been used to prove many related correspondences (see, for example, [UY86, Hit87, Sim92, AaCGP03]). However, greater care is required in order to account for the higher cohomology classes that appear in the current setting. In order to do this, we work with the unique harmonic representatives y_a^{harm} of the classes y_a. This will reduce the problem to the correspondence for quiver bundles. The end result is the following correspondence relating stable $\overline{Q}$-bundles with solutions of the moment map equations.

PROPOSITION 10. *Given a tuple of real numbers $\tau = (\tau_v)_{v \in Q_0}$, a $\overline{Q}$-bundle $\mathcal{E} = (E, \phi, x, y)$ is τ-polystable if and only if it admits a hermitian metric h such that the Nakajima bundle representation $\mathcal{R} = (A_h, \phi, x, y^{\text{harm}})$ satisfies the moment map equations (2-3) and (2-4) for $(\tau, 0)$.*

One direction of the correspondence is a straightforward generalization of the argument in [Nak94, AaCGP03].

LEMMA 11. *If (A, ϕ, x, y) is a solution of the moment map equations, then the corresponding $\overline{Q}$-bundle (E, ϕ, x, y) is polystable.*

The remainder of this section is devoted to proving the second half of Proposition 10,

LEMMA 12. *If $\mathcal{E}$ is a $\tau_{\mathbb{R}}$-polystable $\overline{Q}$ bundle over a compact Riemann surface X such that $\mu_{\mathbb{C}}^{\mathfrak{C}} = 0$, then $\mathcal{E}$ admits a hermitian connection satisfying Equation (2-3).*

4.1. Preliminaries. Fix a compact Riemann surface X, and a $\overline{Q}$-bundle $\mathcal{E}$ such that $\mu_{\mathbb{C}}^{\mathfrak{C}} = 0$. For each $v \in Q_0$, let Met_v denote the space of hermitian metrics on the

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

bundle E_v. Fix a $k_v \in \mathrm{Met}_v$ such that the metric on $\det(E_v)$ induced by k_v satisfies $\sqrt{-1}F_{\det(k_v)} = d_v$. Given a metric $h_v \in \mathrm{Met}_v$, denote the space of smooth h_v-selfadjoint endomorphisms of E_v by $S(h_v)$. Any $h_v \in \mathrm{Met}_v$ is related to k_v through a section $s_v \in S(k_v)$ by $h_v = k_v e^{s_v}$. This notation extends to the direct sum $E = \oplus E_v$ by dropping the mention of the vertex.

Recall that the adjoints of ϕ_v, x_a and y_a^{harm} depend on the choice of metrics h_v. This dependence will be denoted by ϕ^{*h}, x^{*h} and $(y_a^{\mathrm{harm}})^{*h}$. If the operations are with respect to the fixed k_v, the operations will be denoted by x^*.

For each $h \in \mathrm{Met}$, there is a metric h_a on $\mathrm{Hom}(E_{ta}, E_{ha})$ for each $a \in \overline{Q}_1$, given by $(\zeta_a, \zeta'_a)_{h_a} = \mathrm{tr}(\zeta_a \circ \zeta_a^{*h_a})$. This gives a metric h on $\bigoplus_a \mathrm{Hom}(E_{ta}, E_{ha})$ by $(\zeta, \zeta') = \sum_a (\zeta_a, \zeta'_a)_{h_a}$, and the corresponding L^p-inner products and norms are defined similarly.

Fix an integer $p > 2$. Let $W^{p,2}S(k_v)$ denote the Sobolev space of k_v-selfadjoint sections whose derivatives up to second order have finite L^p-norm, and $\mathrm{Met}(W^{p,2}S(k_v)) = \{k_v e^{s_v} : s_v \in W^{p,2}S(k_v)\}$. Given $h = ke^s \in \mathrm{Met}(W^{p,2}S(k))$, define the h-adjoint of a section ξ by $\xi^{*h} = e^{-s}\xi^* e^s$. Finally, define the connection A_{h_v} by

$$d_{h_v} = d_{k_v} + e^{-s_v}\partial_{k_v}(e^{s_v})$$

with curvature $F_{h_v} = F_{k_v} + \overline{\partial}_{E_v}(e^{-s_v}\partial_{k_v}(e^{s_v}))$.

For each vertex $v \in Q_0$, let $s_v \in S(k_v)$. We consider the spectral resolution $s_v = \sum_i \lambda_{v,i} P_{v,i}$ with selfadjoint projection $P_{v,i}$. Let $\psi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be C^1-bounded functions. We define fibrewise, and basis-independently,

$$\psi(s_v) := \sum_i \psi(\lambda_{v,i})\, P_{v,i}, \quad \Psi(s_v)[T] := \sum_{i,j} \Psi(\lambda_{v,i}, \lambda_{v,j})\, P_{v,i}\, T\, P_{v,j}$$

for any $T \in \mathrm{End}(E_v)$. These definitions remain valid even under eigenvalue multiplicities.

We let $\Psi(s)$ act on the data as bundle maps by

$$\begin{aligned}
(\Psi(s)\,\phi)_v &:= \Psi(s_v)[\phi_v] \in \Omega^{1,0}(X, \mathrm{End}\, E_v),\\
(\Psi(s)\,x)_a &:= \sum_{i,j} \Psi(\lambda_{h(a),i}, \lambda_{t(a),j})\, P_{h(a),i}\, x_a\, P_{t(a),j} \in \Omega^0(X, \mathrm{Hom}(E_{t(a)}, E_{h(a)})),\\
(\Phi(s)\, y^{\mathrm{harm}})_a &:= \sum_{i,j} \Phi(\lambda_{t(a),j}, \lambda_{h(a),i})\, P_{t(a),j}\, y_a^{\mathrm{harm}}\, P_{h(a),i} \in \Omega^{0,1}(X, \mathrm{Hom}(E_{h(a)}, E_{t(a)}) \otimes K).
\end{aligned}$$

If $\Psi(p,q) = \psi_1(p)\psi_2(q)$ for functions $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$, then

$$(\Psi(s)x)_a = \psi_1(s_{h(a)})\, x_a\, \psi_2(s_{t(a)}), \quad (\Psi(s)_y^{\mathrm{harm}})_a = \psi_1(s_{t(a)})\, y_a^{\mathrm{harm}}\, \psi_2(s_{h(a)}).$$

For any real number b, let $W_b^{p,2}S(k) \subset W^{p,2}S(k)$ be the closed subset of sections s with $|s| \le b$ almost everywhere in X.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

LEMMA 13 (Surface case). *Assume* $\psi \in C^1_b(\mathbb{R})$ *and* $\Psi \in C^1_b(\mathbb{R}^2)$ *as above. If* $s \in W^{1,2}_b S(k)$ *is selfadjoint, then for any* $q < \infty$ *the following assertions hold.*

(1) $\psi(s) \in W^{1,2}S(k)$ *with* $\|\psi(s)\|_{W^{1,2}} \leq C\,(1 + \|s\|_{W^{1,2}})$.
(2) *For each vertex* $v \in Q_0$, $\Psi(s_v)$ *defines a bounded operator on* $\Omega^0(\mathrm{End}\, E_v)$ *and on* $\Omega^{1,0}(\mathrm{End}\, E_v)$*; likewise on* $\Omega^0(\mathrm{Hom}(E_{t(a)}, E_{h(a)}))$ *and* $\Omega^{0,1}(\mathrm{Hom}(E_{h(a)}, E_{t(a)})\otimes K)$.
(3) *For every* $q \leq 2$*, the actions in* (2) *extend continuously to* $\Psi(s) : L^2 \to L^q$*, with*

$$\|\Psi(s)[T]\|_{L^q} \leq C_{X,q}\|\psi_1\|_{C^1}\,\|\psi_2(s)\|_{C^1}(1 + \|s\|_{W^{1,2}})^2\|T\|_{L^2}.$$

(4) *The actions in* (2) *extend continuously to* $\Psi(s) : L^2 \to L^2$ *with*

$$\|\Psi(s)[T]\|_{L^2} \leq \|\psi_1(s)\|_{L^\infty}\|\psi_2(s)\|_{L^\infty}\|T\|_{L^2}.$$

PROOF. Since $\psi \in C^1_b$, $\|\psi(\lambda_i)\|_{W^{1,2}} \leq C_i + C_i'\|\nabla\lambda_i\|^2_{L^2}$. Then

$$\begin{aligned}\|\psi(s)\|^2_{w^{1,2}} &\leq \sum_i (C_i + C_i'\|\lambda_i\|^2_{W^{1,2}}) \\ &\leq C\sum_i (1 + \|\lambda_i\|^2_{W^{1,2}}) \\ &\leq C_{X,q}(1 + \|s\|^2_{W^{1,2}}).\end{aligned}$$

From $\Psi \in C^1_b(\mathbb{R}^2)$, (2) is clear.
To see (3), when $1/q = 1/2 + 2/p$,

$$\begin{aligned}\|\Psi(s)[T]\|_{L^q} &= \|\psi_1(s)\,T\,\psi_2(s)\|_{L^q} \\ &\leq \|\psi_1(s)\|_{L^p}\,\|T\|_{L^2}\,\|\psi_2(s)\|_{L^p}.\end{aligned}$$

Since $W^{1,2}$ embeds into L^p for any $p < \infty$, the result follows from (1).

Using the assumption that $s \in W^{1,2}_b$, $\psi_1(s), \psi_2(s) \in W^{1,2}_b$ as well. Then (4) follows from Hölder's inequality. □

At this point, thinking of the tuple $(E, \phi, x, y^{\mathrm{harm}})$ as a quiver bundle for the tripled quiver of Q, the majority of the proof is the same as that of Sections 3.3–3.7 of [AaCGP03]. The interested reader is encouraged to refer there for full details. First, construct a functional M_τ on the space of metrics $\mathrm{Met}(W^{p,2}S(k))$ such that the metrics that minimize the functional provide solutions of the moment map equation. The existence of minimizing metrics is then shown to reduce to a particular inequality on M_τ. Finally, the regularity results of [UY86] are used to show that if this inequality is not satisfied, then there exists a filtration of the quiver bundle, which contradicts stability. To conclude it is necessary to relate the destabilizing quiver subbundles constructed in [AaCGP03] to subbundles of the $\bar{Q}$-bundle. This is accomplished by the following lemma, the proof of which is clear.

LEMMA 14. *Suppose* (E, ϕ, x, y) *is a* $\overline{Q}$*-bundle. If* $(E', \phi', x', y'^{\mathrm{harm}})$ *is a quiver subbundle of* $(E, \phi, x, y^{\mathrm{harm}})$*, then* (E', ϕ', x', y') *is a* $\overline{Q}$*-subbundle of* (E, ϕ, x, y).

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

5. Infinitesimal deformations

Let $\mathcal{R} = (A_v, \phi_v, x_a, y_a)$ be a representation of $\overline{Q}$. In order to study deformations of $\mathcal{R}$, a complex $C^\bullet(\mathcal{R})$ is constructed such that $\mathbb{H}^1(C^\bullet(\mathcal{R}))$ corresponds to the infinitesimal deformations of $\mathcal{R}$. The treatment of this section follows that of [Bot00] in the study of Higgs bundles on n-dimensional varieties. For the deformation theory of ordinary quiver bundles see [GK05]. Due to the growing number of indices in this section, in an effort to improve readability, for the remainder of this section the subscripts indicating vertices and edges are replaced with superscripts.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a Čech cover of X. The cohomology classes (ϕ^v, x^a, y^a) defining the representation $\mathcal{R}$ can be described in terms of Čech cocycles $(\{\phi_i^v\}, \{x_i^a\}, \{y_{ij}^a\})$.

For cocycles $\{\alpha^v_{j_0,\dots,j_p}\}_{v\in Q_0} \in \bigoplus_{v\in Q_0} C^p(\mathrm{End}(E^v))$ define $\{[\alpha, x]^a_{j_0,\dots,j_p}\}_{a\in Q_1} \in \bigoplus_{a\in Q_1} C^p(\mathrm{Hom}(E^{t(a)}, E^{h(a)}))$ by

$$[\alpha, x]^a_{j_0\cdots j_p} = \alpha^{h(a)}_{j_0,\cdots j_p} \circ x^a_{j_p} - x^a_{j_p} \circ \alpha^{t(a)}_{j_0\cdots j_p}$$

and similarly for $[\alpha, y]^a$ and $[\alpha, \phi]^v$. These define morphisms of Čech complexes

$$\begin{aligned}
[\cdot, \phi] &: \bigoplus_{v\in Q_0} C^p(\mathrm{End}(E_v)) \longrightarrow \bigoplus_{v\in Q_0} C^p(\mathrm{End}(E^v)\otimes K),\\
[\cdot, x] &: \bigoplus_{v\in Q_0} C^p(\mathrm{End}(E^v)) \longrightarrow \bigoplus_{a\in Q_1} C^p(\mathrm{Hom}(E^{t(a)}, E^{h(a)})),\\
[\cdot, y] &: \bigoplus_{v\in Q_0} C^p(\mathrm{End}(E^v)) \longrightarrow \bigoplus_{a\in Q_1} C^{p+1}(\mathrm{Hom}(E^{h(a)}, E^{t(a)})\otimes K).
\end{aligned}$$

Putting these together, we define a map

$$\bigoplus_{v\in Q_0} C^p(\mathrm{End}(E^v)) \xrightarrow{\ \Psi\ } \begin{array}{c} \bigoplus_{v\in Q_0} C^p(\mathrm{End}(E^v)\otimes K) \\ \bigoplus \\ \left[\bigoplus_{a\in Q_1} C^p(\mathrm{Hom}(E^{t(a)}, E^{h(a)})) \bigoplus C^{p+1}(\mathrm{Hom}(E^{h(a)}, E^{t(a)})\otimes K)\right] \end{array}$$

by $\Psi(f) = ([f, \phi], [f, x], [f, y])$. Additionally, there exists a map Φ from the codomain of Ψ to $\bigoplus_{v\in Q_0} C^{p+1}(\mathrm{End}(E^v)\otimes K)$ given by

$$\Phi(\beta, \alpha, \mu) = \delta\beta + [\alpha, y] + [\mu, x].$$

The desired complex of sheaves $C^\bullet(\mathcal{R})$ can be defined as follows. Let $C^0(\mathcal{R}) = \bigoplus_{v\in Q_0} C^0(\mathrm{End}(E^v))$, and for $p = 1, 2$,

$$\begin{aligned}
C^p(\mathcal{R}) = &\bigoplus_{v\in Q_0} (C^p(\mathrm{End}(E^v)) \oplus C^{p-1}(\mathrm{End}(E^v)\otimes K))\\
&\oplus \bigoplus_{a\in Q_1} (C^{p-1}(\mathrm{Hom}(E^{t(a)}, E^{h(a)})) \oplus C^p(\mathrm{Hom}(E^{h(a)}, E^{t(a)})\otimes K)),
\end{aligned}$$

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

with differentials $d^0 : C^0(\mathcal{R}) \longrightarrow C^1(\mathcal{R})$ given by $d^0(f) = [\begin{smallmatrix}\delta\\ \Psi\end{smallmatrix}]$ and $d^1 : C^1(\mathcal{R}) \longrightarrow C^2(\mathcal{R})$ by

$$d^1 = \begin{bmatrix} \delta & 0 \\ \Psi & -\delta \\ 0 & \Phi \end{bmatrix}.$$

Above, $-\delta$ is understood as taking in a triple of cocycles and applying minus the appropriate Čech differential to each.

LEMMA 15. *The first cohomology group $\mathbb{H}^1(C^\bullet(\mathcal{R}))$ does not depend on the choice of representatives for the y^a.*

PROOF. Without loss of generality we may assume Q consists of a pair of vertices connected by a single edge. Let $(E^1, E^2, \phi^1, \phi^2, x, y)$ be a $\overline{Q}$-bundle and y', y'' two cocycles representing y. Write $y'' = y' + \delta\nu$ for $\nu \in C^0(\mathrm{Hom}(E^1, E^2) \otimes K)$. Denote by $C^\bullet(\overline{Q}, y')$ and $C^\bullet(\overline{Q}, y'')$ the corresponding complexes. It is straightforward to check that the map

$$\mathbb{H}^1(C^\bullet(\overline{Q}, y'')) \longrightarrow \mathbb{H}^1(C^\bullet(\overline{Q}, y'))$$

given by sending $(\eta^1, \eta^2, \beta^1, \beta^2, \alpha, \mu)$ to $(\eta^1, \eta^2, \beta^1, \beta^2, \alpha, \mu + [\eta, \nu])$ is an isomorphism. □

It is now not difficult to extend the proof of [Bot00, BR94] to arrive at the main result.

THEOREM 16. *The first cohomology $\mathbb{H}^1(C^\bullet(\mathcal{R}))$ corresponds to the set of infinitesimal deformations of the representation $\mathcal{R}$.*

PROOF. Given a vector bundle E^v, a cohomology class $\{\eta^v_{ij}\} \in H^1(\mathrm{End}(E^v))$ uniquely determines a vector bundle $E^v[\epsilon]$ on $X_\epsilon := X \times \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$. The affine sets $U[\epsilon]_j := U_j \times \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$ form a Čech cover of X_ϵ.

Consider a deformation $x^a[\epsilon] : E^{t(a)}[\epsilon] \longrightarrow E^{h(a)}[\epsilon]$. This can be described by morphisms $x^a[\epsilon]_i : U^{t(a)}[\epsilon]_i \longrightarrow U^{h(a)}[\epsilon]_i$ that restrict to x^a modulo ϵ and are compatible with the gluing isomorphisms $1 + \epsilon\eta^v_{ij}$. Thus, we may write $x^a[\epsilon]_i = x^a + \epsilon\alpha^a_i$ for some $\alpha^a_i : U_i^{t(a)} \longrightarrow U_i^{h(a)}$, and for each pair (i, j), the following diagram commutes:

$$\begin{array}{ccc} U^{t(a)}[\epsilon]_{ij} & \xrightarrow{x^a[\epsilon]_i} & U^{h(a)}[\epsilon]_{ij} \\ {\scriptstyle 1+\epsilon\eta_{ij}^{t(a)}}\downarrow & & \downarrow{\scriptstyle 1+\epsilon\eta_{ij}^{h(a)}} \\ U^{t(a)}[\epsilon]_{ij} & \xrightarrow[x^a[\epsilon]_j]{} & U^{h(a)}[\epsilon]_{ij} \end{array}$$

Replacing the maps $x^a[\epsilon]_j$ with $x^a + \epsilon\alpha^a_i$, and using the fact that $(1 + \epsilon\eta^v_{ij})^{-1} = 1 - \epsilon\eta^v_{ij}$, the above diagram is equivalent to

$$(\alpha^a_j - \alpha^a_i)|_{U_{ij}} = \eta_{ij}^{h(a)} x^a - x^a \eta_{ij}^{t(a)} = [\eta_{ij}, x]^a.$$

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

A similar argument shows that

$$(\beta_j^v - \beta_i^v)|_{U_{ij}^v} = (\eta_{ij}^v \otimes \mathrm{id}_K)\phi^v - \phi^v \eta_{ij}^v = [\eta_{ij}, \phi]^v.$$

Now consider a deformation of y^a, which can be described in terms of cocycles as $y^a[\epsilon]_{ij} = y^a + \epsilon\mu_{ij}^a$. Compatibility gives the condition

$$(1 - \epsilon\eta_{ij}^{t(a)})(y^a[\epsilon]_{ij} + y^a[\epsilon]_{jk})(1 + \epsilon\eta_{ij}^{h(a)}) = y^a[\epsilon]_{ik}$$

from which it follows that

$$(\mu_{ij} + \mu_{jk} - \mu_{ik})|_{U_{ijk}} = [\eta_{ij}, y]^a.$$

Being a deformation of (ϕ^v, x^a, y^a), the data $(\phi^v[\epsilon], x^a[\epsilon], y^a[\epsilon])$ must solve the moment map equation. Explicitly, we must have that

$$\sum_{v \in Q_0} \overline{\partial}\phi^v[\epsilon] + \sum_{a:ha=v} (x^a \otimes \mathrm{id}_K) \circ y^a - \sum_{a:ta=v} y^a(x^a \otimes \mathrm{id}_K) = 0.$$

Expanding the cocycles, and using the fact that (ϕ^v, x^a, y^a) satisfy the moment map equation, the result follows.

Now let $(\tilde{E}^v[\epsilon], \tilde{\phi}^v[\epsilon], \tilde{x}^a[\epsilon], \tilde{y}^a[\epsilon])$ be a deformation that is isomorphic to $(E^v[\epsilon], \phi^v[\epsilon], x^a[\epsilon], y^a[\epsilon])$, and corresponds to $(\{\tilde{\eta}_{ij}^v\}, \{\tilde{\beta}_i^v\}, \{\tilde{\alpha}_i^a\}, \{\tilde{\mu}_{ij}^a\})$. Then it follows that $\tilde{\eta}_{ij}^v = \eta_{ij}^v + (\lambda_j^v - \lambda_i^v)|_{U_{ij}[\epsilon]}$, where $\lambda^v[\epsilon] = 1 + \epsilon\lambda^v$ is the isomorphism $E^{v'}[\epsilon] \cong E^v[\epsilon]$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} U^{t(a)}[\epsilon]_{ij} & \xrightarrow{x^a[\epsilon]} & U^{h(a)}[\epsilon]_{ij} \\ {\scriptstyle 1+\epsilon\lambda^{t(a)}}\downarrow & & \downarrow{\scriptstyle 1+\epsilon\lambda^{h(a)}} \\ U^{t(a)}[\epsilon]_{ij} & \xrightarrow[\tilde{x}^a[\epsilon]]{} & U^{h(a)}[\epsilon]_{ij} \end{array}$$

from which we determine that $\alpha_i^a - \tilde{\alpha}_i^a = [\lambda, x]^a$. Similarly, it can be shown that $\beta_i^v - \tilde{\beta}_i^v = [\lambda, \phi]^v$ and $\tilde{\mu}_{ij}^a = \mu_{ij}^a + [\lambda, y]^a$. Thus, deformations correspond to cohomology classes in $\mathbb{H}^1(C^\bullet(\mathcal{R}))$. □

6. $\overline{Q}$-bundles of type A_1

We now consider framed quiver representations. Using the notation of [Gin12], associated to any quiver Q, one can construct a new quiver $Q^\heartsuit$, where $Q_0^\heartsuit = Q_0 \sqcup Q_0$. The vertices in the second copy of Q_0 are called the framing vertices and are denoted by w_i. The edge set $Q_1^\heartsuit$ consists of the edges of Q together with an edge $v_i \to w_i$. A type vector for $Q^\heartsuit$ is vector $((v, d), (w, d_w))$. Then a framed representation of $\overline{Q}$ is a representation of $\overline{Q^\heartsuit}$ such that the data associated to the framing vertices is fixed. Thus, choosing a framing is equivalent to choosing an additional vector bundle W_v of rank w_v for each $v \in Q_0$.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

Let $Q^\heartsuit$ be the framed A_1-quiver, pictured below with framing shown as a square vertex, and label $((1,d),(n,0))$ for $d>0$ and $n>1$:



In [Nak94] it was shown that linear representations of $\overline{Q^\heartsuit}$ are isomorphic to cotangent spaces of Grassmannians.

Take $X=\mathbb{P}^1$ and $\mathcal{O}^n$ as framing bundle. A framed Nakajima bundle representation of $\overline{Q}$ consists of a line bundle L on $\mathbb{P}^1$, together with a pair $x\in H^0(\mathbb{P}^1,\mathrm{Hom}(\mathcal{O}^n,L))$ and $y\in H^1(\mathbb{P}^1,\mathrm{Hom}(L,\mathcal{O}^n)\otimes K)\cong H^0(\mathbb{P}^1,\mathrm{Hom}(\mathcal{O}^n,L))^*$. For $\tau=(\tau_{\mathbb{R}},0)$ satisfying $\tau_{\mathbb{R}}>d$, it follows from the real moment map that x must be surjective, hence it defines a rank 1 quotient bundle F of $\mathcal{O}^n$ having degree d. Moreover, from the complex moment map $xy=0$ we see that $yx\in H^1(\mathbb{P}^1,\mathrm{Hom}(F,\ker(x))\otimes K)$. As shown in [Ser06], the tangent space at a point F in the Grassmannian can be identified with $\mathrm{Hom}_{\mathbb{P}^1}(F,\mathcal{O}^n/F)$, so the map yx lies in the cotangent space to the point defined by x. This describes a map from $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$ to $T^*\mathrm{Gr}^d(1,\mathcal{O}^n)$.

Conversely, assume that (F,Ω) is a point in $T^*\mathrm{Gr}^d(1,\mathcal{O}^n)$. So $q:\mathcal{O}^n\twoheadrightarrow F$ is a quotient bundle of $\mathcal{O}^n$ having degree d and $\Omega\in H^1(\mathbb{P}^1,\mathrm{Hom}(F,\ker(q))\otimes K)$ lies in the cotangent space at $[q:\mathcal{O}^n\twoheadrightarrow F]$. Consider the exact sequence

$$0\longrightarrow \ker(q)\overset{\iota}{\hookrightarrow}\mathcal{O}^n\overset{q}{\twoheadrightarrow}F\longrightarrow 0.$$

We may apply the functor $\mathrm{Hom}(F,-)\otimes K$ to obtain a new short exact sequence

$$0\longrightarrow \mathrm{Hom}(F,\ker(q))\otimes K\longrightarrow \mathrm{Hom}(F,\mathcal{O}^n)\otimes K\longrightarrow \mathrm{Hom}(F,F)\otimes K\longrightarrow 0,$$

with the associated long exact sequence in cohomology

$$\begin{aligned}\cdots\longrightarrow H^0(\mathbb{P}^1,\mathrm{Hom}(F,F)\otimes K)&\longrightarrow H^1(\mathbb{P}^1,\mathrm{Hom}(F,\ker(q))\otimes K)\\ \longrightarrow H^1(\mathbb{P}^1,\mathrm{Hom}(F,\mathcal{O}^n)\otimes K)&\longrightarrow H^1(\mathbb{P}^1,\mathrm{Hom}(F,F)\otimes K)\longrightarrow 0.\end{aligned}$$

Since we are working over $\mathbb{P}^1$, the $H^0(\mathbb{P}^1,\mathrm{Hom}(F,F)\otimes K)$ vanishes and so the map $H^1(\mathbb{P}^1,\mathrm{Hom}(F,\ker(q))\otimes K)\longrightarrow H^1(\mathbb{P}^1,\mathrm{Hom}(F,\mathcal{O}^n)\otimes K)$ is injective. Therefore, we can push forward Ω to get an element $y\in H^1(\mathbb{P}^1,\mathrm{Hom}(F,\mathcal{O}^n)\otimes K)$. The complex moment map condition together with the above sequence implies that y is in the image of the map $H^1(\mathbb{P}^1,\mathrm{Hom}(F,\ker(q))\otimes K)\longrightarrow H^1(\mathbb{P}^1,\mathrm{Hom}(F,\mathcal{O}^n)\otimes K)$. This defines a map $T^*\mathrm{Gr}^d(1,\mathcal{O}^n)$ to $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$. This describes an isomorphism between $T^*\mathrm{Gr}^d(1,\mathcal{O}^n)$ and $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$.

When the genus of X is $g\geq 1$, there exist nonzero sections of $H^0(X,\mathrm{End}(L)\otimes K)$ and so a representation includes a twisted endomorphism ϕ. Thinking now in terms of stability, the $\overline{Q}$-bundle (L,ϕ,x,y) is stable if and only if (L,ϕ) is a stable Higgs bundle of rank 1, and x is surjective.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

7. $\mathbb{C}^\times$-action

In the presence of a suitable action of a torus $\mathbb{T}$ on a (possibly singular) variety X, the theory of Białynicki-Birula [ByB73] establishes that the rational (co)homology of X is determined by the rational (co)homologies of the connected components of the fixed point set $X^{\mathbb{T}}$. In this section we define a $\mathbb{C}^\times$-action on Nakajima bundle varieties in an attempt to better understand the fixed points, and as a result the topology of $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$. In general, even in the case of vector space representations, a full description of the fixed points is difficult. Therefore, we focus on the specific case of the framed A_1 quiver.

Recall the $\mathbb{C}^\times$-actions on the moduli space of Higgs bundles and on Nakajima quiver varieties that scale the cotangent directions. In the same way there is an action of $\mathbb{C}^\times$ on $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$ that preserves the moment map conditions $\mu_{\mathbb{R}} = \tau_{\mathbb{R}},\ \mu_{\mathbb{C}} = 0$. If (E, ϕ, x, y) is a $\overline{Q}$-bundle, the action of $\lambda \in \mathbb{C}^\times$ is given by $\lambda \cdot (E, \phi, x, y) = (E, \lambda\phi, x, \lambda y)$.

For the remainder of this section, assume the genus of X is not 1. If (E, ϕ, x, y) is a fixed point, there exists a one-parameter family of gauge transformations g_λ such that

$$g_\lambda(E, \phi, x, y)g_\lambda^{-1} = (E, \lambda\phi, x, \lambda y).$$

For each $E_v \in Q_0$ let $E_v = \oplus_i E_v^{w_i}$ be the weight decomposition of E_v, in which $\mathbb{C}^\times$ acts on $E_v^{w_i}$ with weight w_i. Being a fixed point of the action implies

$$\begin{aligned}
&\phi_v : E_v^{w_i} \longrightarrow E_v^{w_i-1} \otimes K,\\
&x_a : E_{t(a)}^{w_i} \longrightarrow E_{h(a)}^{w_i},\\
&y_a \in H^1(X, \mathrm{Hom}(E_{h(a)}^{w_i}, E_{t(a)}^{w_i-1} \otimes K)).
\end{aligned}$$

We work with the specific example of the $\mathbb{C}^\times$-action on $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$, where Q is the framed A_1 quiver. From the fixed-point equations, a point (ϕ, x, y) is fixed if there exists a one-parameter subgroup g_λ such that for all λ we have

$$\lambda\phi = g_\lambda \phi g_\lambda^{-1}, \quad x = g_\lambda x, \quad \lambda y = y g_\lambda^{-1}.$$

By stability, x is surjective, and therefore the only fixed points are of the form $(0, x, 0)$ and the fixed-point set coincides with the $\mathrm{Gr}^d(1, O^n)$.

Replace the framing vertex with an unframed vertex. We are interested in the action

$$(E_1, E_2, \phi_1, \phi_2, x, y) \mapsto (E_1, E_2, \lambda\phi_1, \lambda\phi_2, x, \lambda y).$$

Using the weight space decomposition of E_1, E_2, it is straightforward to verify that the fixed points can be described as pairs of holomorphic chains given by the $\phi_i : E_i^{w_i} \to E_i^{w_i-1} \otimes K$, intertwined by the x^k and y^k data. Such a fixed point is shown in Figure 1. A more refined description, such as the length of the chains involved, will depend on stability and the topological type of the bundles E_1, E_2.

If $|Q_0| = n$, the action above can be generalized to an action of the product $(\mathbb{C}^\times)^n$ on $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^\times)^n$, the action scales ϕ by λ_v and y by $\lambda_{t(a)}$.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

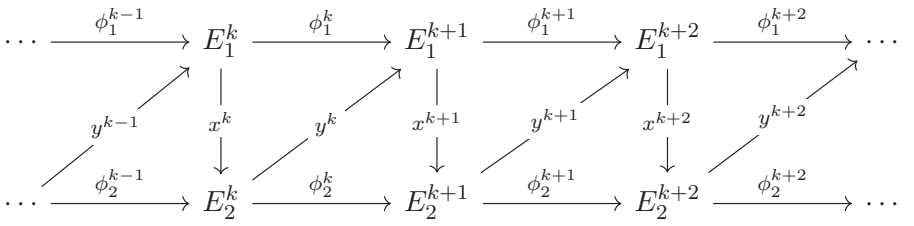


FIGURE 1. A fixed point for the $\mathbb{C}^\times$-action on the unframed A_1 quiver.

A representation (A, ϕ, x, y) is now a fixed point if there exists a homomorphism $g : (\mathbb{C}^\times)^n \to \mathcal{G}$ such that

$$\lambda_v \phi_v = g(\lambda)\phi_v g(\lambda)^{-1}, \quad x_a = g(\lambda)_{h(a)} x_a g(\lambda)^{-1}_{t(a)}, \quad \lambda_{t(a)} y_a = g(\lambda)_{t(a)} y_a g(\lambda)^{-1}_{h(a)}.$$

From here, the same argument as for Higgs bundles shows that in order for $\mathcal{E} = (E, \phi, x, y)$ to be a fixed point, ϕ_v must be nilpotent for all v.

However, unlike in the case of ordinary Higgs bundles, the ϕ_v being nilpotent does not provide a sufficient condition for $\mathcal{E}$ to be fixed. As example, let $\mathcal{E} = (E_1, E_2, \phi_1, \phi_2, x, y)$ be a $\overline{Q}$-bundle on the unframed A_1 quiver. Suppose $E_1 \cong E_2 \cong \mathcal{O}_X \oplus \mathcal{O}_X$ and

$$\phi_1 = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & x_{12} \\ x_{21} & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix}.$$

Defining the nilpotent cone inside of $\mathcal{M}_{\overline{Q}}(\tau)$ to be those Nakajima representations with all Higgs fields ϕ_v nilpotent, all fixed points for the torus action are constrained to live inside the nilpotent cone.

As the underlying quiver becomes more complicated and as the ranks and degrees involved grow larger, the analysis of the fixed points becomes markedly more difficult. In the absence of a systematic approach, the understanding of the fixed points of the action relies on a case-by-case approach. However, once this is accomplished, the Białynicki-Birula decomposition of $\mathcal{M}^{r,d}_{\overline{Q}}(\tau)$ in terms of the fixed-point sets should allow for a more complete understanding of the topology of Nakajima bundle varieties.

On a more speculative note, the fixed points of the $\mathbb{C}^\times$-action on the moduli space of Higgs bundles play an important role in understanding both the Hitchin system and Nakajima quiver varieties. The Hitchin system leads to a proper morphism $\mathcal{H} : \mathcal{M}^{\text{Higgs}}(X) \to \mathcal{B}$ to an affine space $\mathcal{B}$, sending a Higgs field to the coefficients of its characteristic polynomial. The fixed points of the $\mathbb{C}^\times$-action all reside in the nilpotent cone $\mathcal{H}^{-1}(0)$ of Higgs bundles (E, ϕ) with nilpotent ϕ. Given the presence of integrable systems on Nakajima quiver varieties [FR16, RS21], as well as the similarities between Nakajima bundle varieties and Higgs moduli, the fixed points might facilitate a deeper understanding of the relationship between these systems.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

Acknowledgments

The second-named author gratefully acknowledges support from the Simons Center for Geometry and Physics where important progress was made during the second Simons Math Summer Workshop ('Moduli') in July 2024. The second-named author was partially supported by an Ontario Graduate Scholarship and by the NSERC Discovery Grants of the other co-authors. The second- and third-named authors thank Mahmud Azam, Kuntal Banerjee, Eric Boulter, Robert Cornea, and Evan Sundbo for useful discussions, and all three authors acknowledge Eckhard Meinrenken and Nick Rozenblyum for helpful remarks and questions during the early stages of the work, as well as the anonymous referee for valuable feedback on earlier versions of this work.

References

[AaCGP03] L. Álvarez-Cónsul and O. García-Prada, 'Hitchin-Kobayashi correspondence, quivers, and vortices', *Comm. Math. Phys.* **238**(1–2) (2003), 1–33.

[AHDM78] M. F. Atiyah, N. J. Hitchin, V. G. Drinfel'd and Y. I. Manin, 'Construction of instantons', *Phys. Lett. A* **65**(3) (1978), 185–187.

[AR24] M. Azam and S. Rayan. 'Moduli stacks of quiver bundles with applications to Higgs bundles', Preprint, 2024, arXiv:2407.11958.

[BCR+24] G. Bellamy, A. Craw, S. Rayan, T. Schedler and H. Weiss, 'All 81 crepant resolutions of a finite quotient singularity are hyperpolygon spaces', *J. Algebraic Geom.* **33**(4) (2024), 757–793.

[BGP95] S. Bradlow and O. García-Prada, 'Higher cohomology triples and holomorphic extensions', *Comm. Anal. Geom.* **3**(3–4) (1995), 421–463.

[Bot00] F. Bottacin, 'A generalization of Higgs bundles to higher dimensional varieties', *Math. Z.* **233**(2) (2000), 219–250.

[BR94] I. Biswas and S. Ramanan, 'An infinitesimal study of the moduli of Hitchin pairs', *J. Lond. Math. Soc. (2)* **49**(2) (1994), 219–231.

[BS21] G. Bellamy and T. Schedler, 'Symplectic resolutions of quiver varieties', *Selecta Math. (N.S.)* **27**(3) (2021), 36.

[ByB73] A. Białynicki-Birula., 'Some theorems on actions of algebraic groups', *Ann. of Math. (2)* **98** (1973), 480–497.

[CFKM14] I. Ciocan-Fontanine, B. Kim and D. Maulik, 'Stable quasimaps to GIT quotients', *J. Geom. Phys.* **75** (2014), 17–47.

[CS15a] A. Collinucci and R. Savelli, 'F-theory on singular spaces', *J. High Energy Phys.* **100**(9) (2015).

[CS15b] A. Collinucci and R. Savelli, 'T-branes as branes within branes', *J. High Energy Phys.* **161**(9) (2015).

[Dia12] D.-E. Diaconescu, 'Moduli of ADHM sheaves and the local Donaldson-Thomas theory', *J. Geom. Phys.* **62**(4) (2012), 763–799.

[FR16] J. Fisher and S. Rayan, 'Hyperpolygons and Hitchin systems', *Int. Math. Res. Not. IMRN* **2016**(6) (2016), 1839–1870.

[GH79] G. W. Gibbons and S. W. Hawking, 'Classification of gravitational instanton symmetries', *Comm. Math. Phys.* **66**(3) (1979), 291–310.

[Gin12] V. Ginzburg, 'Lectures on Nakajima's quiver varieties', in: *Geometric Methods in Representation Theory. I*, Séminars and Congress, 24-I (ed. M. Brion) (Société Mathématique de France, Paris, 2012), 145–219.

[GK05] P. B. Gothen and A. King, 'Homological algebra of twisted quiver bundles', *J. Lond. Math. Soc. (2)* **71**(1) (2005), 85–99.

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press

[HHTZ23] J. Heckman, M. Hübner, E. Torres, and H. Zhang. 'The branes behind generalized symmetry operators'. *Fortschr. Phys.* **71**(1) (2023), 2200180.
[Hit87] N. J. Hitchin, 'The self-duality equations on a Riemann surface', *Proc. Lond. Math. Soc. (3)* **55**(1) (1987), 59–126.
[HLRV11] T. Hausel, E. Letellier and F. Rodriguez-Villegas, 'Arithmetic harmonic analysis on character and quiver varieties', *Duke Math. J.* **160**(2) (2011), 323–400.
[KN90] P. B. Kronheimer and H. Nakajima, 'Yang-Mills instantons on ALE gravitational instantons', *Math. Ann.* **288**(2) (1990), 263–307.
[Kob87] S. Kobayashi, 'Differential geometry of complex vector bundles', in: *Publications of the Mathematical Society of Japan*, Kanô Memorial Lectures, 5, Vol. 15 (Princeton University Press, Princeton, NJ, 1987).
[Min20] A. Minets, 'Cohomological Hall algebras for Higgs torsion sheaves, moduli of triples and sheaves on surfaces', *Selecta Math. (N.S.)* **26**(2) (2020), 30.
[MN18] K. McGerty and T. Nevins, 'Kirwan surjectivity for quiver varieties', *Invent. Math.* **212**(1) (2018), 161–187.
[Moz11] S. Mozgovoy. 'Motivic Donaldson-Thomas invariants and McKay correspondence', Preprint, 2011, arXiv:1107.6044.
[Nak94] H. Nakajima, 'Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras', *Duke Math. J.* **76**(2) (1994), 365–416.
[Nak98] H. Nakajima, 'Quiver varieties and Kac-Moody algebras', *Duke Math. J.* **91**(3) (1998), 515–560.
[NS64] M. S. Narasimhan and C. S. Seshadri, 'Stable bundles and unitary bundles on a compact Riemann surface', *Proc. Nat. Acad. Sci. USA* **52** (1964), 207–211.
[RS21] S. Rayan and L. Schaposnik, 'Moduli spaces of generalized hyperpolygons', *Quart. J. Math. Oxford Ser. (2)* **72**(1–2) (2021), 137–161.
[Ser06] E. Sernesi, *Deformations of Algebraic Schemes*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 334 (Springer-Verlag, Berlin, 2006).
[Sim92] C. Simpson, 'Higgs bundles and local systems', *Publ. Math. Inst. Hautes_Etudes Sci.* **75** (1992), 5–95.
[Sze08] B. Szendrői, 'Sheaves on fibered threefolds and quiver sheaves', *Comm. Math. Phys.* **278**(3) (2008), 627–641.
[UY86] K. Uhlenbeck and S.-T. Yau, 'On the existence of Hermitian-Yang-Mills connections in stable vector bundles', *Comm. Pure Appl. Math.* **39**(S) (1986), 257–293.

LISA JEFFREY, Department of Mathematics,
University of Toronto, Toronto, Ontario, Canada
e-mail: lisa.jeffrey@gmail.com

MATTHEW KOBAN, Department of Mathematics,
University of Toronto, Toronto, Ontario, Canada
e-mail: matthew.koban@mail.utoronto.ca

STEVEN RAYAN, Centre for Quantum Topology and
its Applications (quanTA) and Department of Mathematics and Statistics,
University of Saskatchewan, Saskatoon, Saskatchewan, Canada
e-mail: rayan@math.usask.ca

https://doi.org/10.1017/S1446788726101542 Published online by Cambridge University Press