European Journal of Applied Mathematics (2026), **37**, pp. 706–720
doi:10.1017/S0956792524000354

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PAPER

Continuum limit for interacting systems on adaptive networks

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Received: 29 January 2024; **Revised:** 28 June 2024; **Accepted:** 06 August 2024; **First published online:** 6 December 2024

Keywords: adaptive networks; continuum limit; graph limit; interacting systems

2020 Mathematics Subject Classification: 05C90, 45J05, 70F45 (Primary); 34C15, 35R02 (Secondary)

Abstract
The article considers systems of interacting particles on networks with adaptively coupled dynamics. Such processes appear frequently in natural processes and applications. Relying on the notion of graph convergence, we prove that for large systems the dynamics can be approximated by the corresponding continuum limit. Well-posedness of the latter is also established.

1. Introduction

Models of collectively interacting particles play a crucial role in many branches of the natural sciences including biological systems, industrial processes and social activities [8, 9, 20, 33, 35]. Many of these real-world examples exhibit an underlying network structure, and consequently, there has been an increasing interest during the last years in corresponding mathematical models both for deterministic and stochastic situations [2–4, 7, 11, 13–15, 23, 28, 30–32, 35]. In the case of finite systems, i.e. where finitely many particles interact, this typically leads to a large system of coupled ordinary differential equations (ODEs):

$$\dot{\phi}_k = \frac{1}{N}\sum_{\ell=1}^{N} \kappa_{k\ell} g(\phi_k, \phi_\ell) \qquad \text{with } k = 1, \dots, N. \tag{1}$$

Here $\phi_k(t)$ describes the state of the k's particle at time t, the function g models the interaction between two particles and $\kappa_{k\ell}$ corresponds to the adjacency matrix of the underlying network. More precisely, each particle is assumed to be located at the node of a graph consisting of N nodes which are labeled by $1, \dots, N$. The quantity $\kappa_{k\ell}$ denotes the weight of the edge between the nodes k and ℓ. One of the most prominent examples is the classical *Kuramoto model* where $g(\phi_k, \phi_\ell) = \sin(\phi_\ell - \phi_k)$ and $\kappa_{k\ell} \equiv \kappa$ [25].

In many applications, the number N of involved particles is so large that the evolution of the whole system is not tractable. Instead, one is interested in continuous limiting descriptions when $N \to \infty$ and, for systems without a network structure, i.e. $\kappa_{k\ell} \equiv const$, there is a well established theory available [18]. Moreover, in recent years, based on the notion of graph convergence, it has been possible to extend these methods to situations with an underlying network [21, 22, 24, 29]. More precisely, assuming that the (stationary) graph structure has for $N \to \infty$ a suitable limiting *graphon*, corresponding continuum and mean-field models have been derived. One advantage of this approach is that these

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continuous limit models are often easier to analyse analytically than their discrete counterparts and thus simplifies the study of, e.g. synchronisation phenomena, phase transitions and pattern formation [1, 10, 16, 24, 29, 34].

1.1 Coupled oscillators on adaptive networks

However, for many systems, the network structure is not fixed, but instead, it evolves in time while this evolution is often also coupled to the particle dynamics. A special case is given by the following adaptively coupled Kuramoto model considered in [5]:

$$\begin{aligned} \dot{\phi}_k &= \omega - \frac{1}{N}\sum_{\ell=1}^{N} \kappa_{k\ell} \sin\left(\phi_\ell - \phi_k + \alpha\right), \quad t > 0 \\ \dot{\kappa}_{k\ell} &= -\varepsilon\left(\sin\left(\phi_k - \phi_\ell + \beta\right) + \kappa_{k\ell}\right) \end{aligned} \tag{2}$$

with $k, \ell = 1, \ldots, N$ and phase parameters $\alpha \in [0, \pi/2)$ and $\beta \in [-\pi, \pi)$. Moreover, in [19], the synchronisation of oscillators following the slightly generalised model

$$\begin{aligned} \dot{\phi}_k &= \frac{1}{N}\sum_{\ell=1}^{N} \kappa_{k\ell} \sin\left(\phi_\ell - \phi_k\right) + \omega_k, \quad t > 0 \\ \dot{\kappa}_{k\ell} &= \Gamma(\phi_\ell - \phi_k) - \gamma\kappa_{k\ell}. \end{aligned} \tag{3}$$

with suitable initial data $\phi_k(0)$ and $\kappa_{k\ell}(0)$ has been considered where $1 \le k, \ell \le N$, ω_k is the natural frequency of the k's oscillator, γ is a non-negative constant and Γ is a 2π periodic function satisfying $\Gamma(-\phi) = \Gamma(\phi)$ for $\phi \in \mathbb{R}$. By means of Duhamel's formula, one can solve the second equation explicitly which yields $\kappa_{k\ell}(t) = \kappa_{k\ell}(0)\mathrm{e}^{-\gamma t} + \int_0^t \Gamma(\phi_\ell(s) - \phi_k(s))\mathrm{e}^{-\gamma(t-s)}\mathrm{d}s$. Plugging this expression back into the first equation reduces the problem again to an equation with a stationary network up to an additional time integration. In [17], the continuum limit has been derived for this kind of graph dynamics.

1.2 A generalised model

In this work, we will consider the following generalised model where the evolution of $\kappa_{k\ell}$ does not only depend on its current state and the dynamics of ϕ_k and ϕ_ℓ, but instead, it might be influenced by the whole system. Moreover, we allow each edge/weight of the network to follow its own dynamics. In fact, we will study the model

$$\begin{aligned} \dot{\phi}_k &= \frac{1}{N}\sum_{\ell=1}^{N} \kappa_{k\ell} g\left(t, \phi_k, \phi_\ell\right) + f_k(t, \phi), \qquad t > 0 \\ \dot{\kappa}_{k\ell} &= \Lambda_{k\ell}(t, \kappa, \phi). \end{aligned} \tag{4}$$

with $1 \le k, \ell \le N$ and continuous functions $f_k \colon [0, \infty) \times (\mathbb{R}^d)^N \to \mathbb{R}^d$, $g \colon [0, \infty) \times (\mathbb{R}^d)^2 \to \mathbb{R}^d$ and $\Lambda_{k\ell} \colon [0, \infty) \times \mathbb{R}^{N\times N} \times (\mathbb{R}^d)^N \to \mathbb{R}$ whose properties will be specified more closely later and $\phi = (\phi_1, \ldots, \phi_N)$ as well as $\kappa = (\kappa_{k\ell})_{k,\ell=1}^N$.

1.3 Assumptions and main result

In order to derive the continuum limit, we rely on the notion of graphons and corresponding graph convergence ([6, 26, 27]) following the same approach developed, e.g. in [29] which has also been exploited in [2]. For this aim, we parametrise the discrete system and the underlying graph over the sets $I = [0, 1)$ and $I \times I = [0, 1) \times [0, 1)$ respectively. Precisely, denoting $I_k = [(k-1)/N, k/N)$ we set

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$$u^N(t,x) := \sum_{k=1}^{N} \phi_k(t)\chi_{I_k}(x) \tag{5}$$

$$K^N(t,x,y) := \sum_{k,\ell=1}^{N} \kappa_{k\ell}(t)\chi_{I_k}(x)\chi_{I_\ell}(y)$$

where χ_{I_k} is the characteristic function of I_k. We note that throughout this work, a graphon is a measurable, bounded and symmetric function $W\colon I^2 \to \mathbb{R}$ (see also [12]) and (5) thus provides a representation of the family of graphs $\kappa_{k\ell}$ (indexed by the time t) by means of a corresponding family of graphons. Moreover, given $\Lambda\colon [0,\infty)\times I\times I\times L^\infty(I\times I,\mathbb{R})\times L^\infty(I,\mathbb{R}^d)\to\mathbb{R}$ and $f\colon [0,\infty)\times I\times L^\infty(I,\mathbb{R}^d)\to\mathbb{R}^d$ satisfying the properties (8) and (9) below, we can reconstruct a corresponding discrete system via

$$\begin{aligned}\Lambda_{k\ell}(t,\kappa,\phi) &:= N^2\int_{I_k\times I_\ell}\Lambda(t,x,y,K^N(t,\cdot,\cdot),u^N(t,\cdot))\mathrm{d}x\mathrm{d}y\\ f_k(t,\phi) &:= N\int_{I_k} f(t,x,u^N(t,\cdot))\mathrm{d}x.\end{aligned} \tag{6}$$

With this notation, (4) can be rewritten as the following integral equation

$$\begin{aligned}\partial_t u^N(t,x) &= \int_I K^N(t,x,y)g(t,u^N(t,x),u^N(t,y))\mathrm{d}y + N\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N} f(t,\xi,u^N(t,\cdot))\mathrm{d}\xi\\ \partial_t K^N(t,x,y) &= N^2\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N}\int_{\lfloor Ny\rfloor/N}^{(\lfloor Ny\rfloor+1)/N}\Lambda(t,\xi,\eta,K^N(t,\cdot,\cdot),u^N(t,\cdot))\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{7}$$

We assume that $\Lambda\colon [0,\infty)\times I\times I\times L^\infty(I\times I,\mathbb{R})\times L^\infty(I,\mathbb{R}^d)\to\mathbb{R}$ is continuous and satisfies

$$\begin{aligned}\|\Lambda(t,\cdot,\cdot,K_1,u_1)-\Lambda(t,\cdot,\cdot,K_2,u_2)\|_{L^2(I^2)} &\le L_\Lambda\big(\|K_1-K_2\|_{L^2(I^2)}+\|u_1-u_2\|_{L^2(I)}\big)\\ |\Lambda(t,x,y,K,u)| &\le B_\Lambda(1+\|K\|_{L^\infty(I^2)}).\end{aligned} \tag{8}$$

Moreover, we assume that $g\colon [0,T]\times(\mathbb{R}^d)^2\to\mathbb{R}^d$ and $f\colon [0,T]\times I\times L^\infty(I,\mathbb{R}^d)\to\mathbb{R}^d$ are continuous and satisfy the following estimates for all $\xi,\xi_1,\xi_2,\eta,\eta_1,\eta_2\in\mathbb{R}^d$ and $u,u_1,u_2\in L^\infty(I,\mathbb{R}^d)$ uniformly in t:

$$\begin{aligned}&|f(t,\cdot,u)|\le B_f(1+\|u\|_{L^\infty}) \quad\text{and}\quad \|f(t,\cdot,u_1)-f(t,\cdot,u_2)\|_{L^2(I)}\le L_f\|u_1-u_2\|_{L^2(I)}\\ &|g(t,\xi,\eta)|\le B_g \quad\text{and}\quad |g(t,\xi_1,\eta_1)-g(t,\xi_2,\eta_2)|\le L_g(|\xi_1-\xi_2|+|\eta_1-\eta_2|).\end{aligned} \tag{9}$$

Our main result in this work is the following theorem which states that in the limit of infinitely many particles, the discrete system (4) can be approximated by the integro-differential equation (10).

Theorem 1.1. *Let $f,g\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ satisfy (9) and let $\Lambda\colon [0,\infty)\times I\times I\times L^\infty(I\times I,\mathbb{R})\times L^\infty(I,\mathbb{R}^d)\to\mathbb{R}$ satisfy (8). Assume that $K^N(0,\cdot,\cdot)$ has a limiting graphon W with respect to $\|\cdot\|_{L^2}$ which is uniformly bounded, i.e. $\lim_{N\to\infty}\|K^N(0,\cdot,\cdot)-W\|_{L^2(I\times I)}=0$ and $\|W\|_{L^\infty(I\times I)}<\infty$. Then, as $N\to\infty$, the parametrisation (u^N,K^N) given in (7) which corresponds to the discrete system (4) with (6) converges to its continuum limit (u,K), i.e. the unique solution of*

$$\begin{aligned}\partial_t u(t,x) &= \int_I K(t,x,y)g(t,u(t,x),u(t,y))\mathrm{d}y + f(t,x,u(t,\cdot))\\ \partial_t K(t,x,y) &= \Lambda(t,x,y,K(t,\cdot,\cdot),u(t,\cdot))\end{aligned} \tag{10}$$

with $K(0,\cdot,\cdot)=W$ provided that the initial value $u^N(0,\cdot)$ converges to $u_0=u(0,\cdot)$ with respect to $\|\cdot\|_{L^2}$, i.e. $\lim_{N\to\infty}\|u^N(0,\cdot)-u(0,\cdot)\|_{L^2(I)}=0$. More precisely, we have

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\Big(\|u^N-u\|_{L^2(I)}^2+\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|_{L^2}^2\Big)=0.$$

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We note that Theorem 1.1 is restricted to the case of a dense initial graph and assumes L^2 convergence of the latter towards a limiting graphon W which is a stronger notion than the one given via the cut-norm $\|W\|_\square := \max_{A,B\subset I}|\int_{A\times B} W(x,y)\mathrm{d}x\mathrm{d}y|$ and the corresponding cut-distance (we refer to, e.g. [26] for details). Similar restrictions are present in related works [17, 29] while we also note that for a continuous graphon W, if we construct a corresponding sequence of discrete graphs $\kappa^N_{k\ell}$ in analogy to (6), i.e. $\kappa_{k\ell} = W(k/N, \ell/N)$, then actually $K^N(0,\cdot,\cdot) \to W$ in $L^2(I^2)$ (e.g. [24, 29]).

1.4 Relation to previous results

Theorem 1.1 provides the continuum limit for a rather general class of adaptively coupled network dynamics. In particular, it contains as a special case the following system modelling opinion dynamics with time varying weights which has been considered in [2]:

$$\begin{aligned}\dot\phi_k &= \frac{1}{N}\sum_{\ell=1}^N m_\ell(t)\psi(\phi_\ell-\phi_k) \qquad\qquad k=1,\dots,N.\\ \dot m_k &= \Psi_k(\phi,m)\end{aligned} \tag{11}$$

Here the opinions are described by $\phi = (\phi_k)_{k=1}^N\colon [0,T]\to(\mathbb{R}^d)^N$ while the weights are given by $m=(m_k)_{k=1}^N\colon [0,T]\to\mathbb{R}^N$. In fact, for $\kappa_{k\ell}=m_\ell$ for all $k=1,\dots,N$, $g(t,\phi_k,\phi_\ell)=\psi(\phi_\ell-\phi_k)$ and $\Lambda_{k\ell}(t,\kappa,\phi)=\Psi_k(\phi,\kappa_{1\cdot})$ this model is a special case of (4). Moreover, Theorem 1.1 generalises the class of graph dynamics considered in [17].

1.5 Outline

The remainder of the article is structured as follows. In the next section, we will provide the well-posedness of the continuous system (10). The proof relies essentially on an application of the contraction mapping theorem but due to the relatively weak Lipschitz continuity of f and Λ some special care is needed. The proof of well-posedness for (4) follows in the same way and will thus be omitted. In Section 3, we will then provide the proof of Theorem 1.1.

2. Well-posedness

We have the following result on the well-posedness of the discrete system (4).

Proposition 2.1. *Let $N\in\mathbb{N}$, $T>0$ and let $g\colon [0,T]\times(\mathbb{R}^d)^2\to\mathbb{R}^d$ satisfy (9). Assume that $f_k\colon [0,T]\times(\mathbb{R}^d)^N\to\mathbb{R}^d$ satisfies*

$$|f_k(t,\phi)| \le \overline{B}_f(1+|\phi|) \qquad \text{and} \qquad |f_k(t,\phi)-f_k(t,\psi)| \le \overline{L}_f|\phi-\psi|$$

for all $\phi,\psi\in(\mathbb{R}^d)^N$. Moreover, assume that $\Lambda_{k\ell}\colon [0,\infty)\times\mathbb{R}^{N\times N}\times(\mathbb{R}^d)^N\to\mathbb{R}$ is uniformly Lipschitz continuous with respect to the second and third component, i.e. $|\Lambda_{k\ell}(t,\kappa,\phi)-\Lambda_{k\ell}(t,\lambda,\psi)|\le \overline{L}\Lambda(|\kappa-\lambda|+|\phi-\psi|)$ and satisfies the bound $|\Lambda_{k\ell}(t,\kappa,\phi)|\le\overline{B}_\Lambda(1+|\kappa|)$ for all $k,\ell\in\{1,\dots,N\}$ uniformly with respect to t. Then for each initial condition $(\phi_0,\kappa_0)\in(\mathbb{R}^d)^N\times\mathbb{R}^{N\times N}$ the system (4) has a unique solution (ϕ,κ) on $[0,T]$.

We note that $\mathbb{R}^d$, $(\mathbb{R}^d)^N$ and $\mathbb{R}^{N\times N}$ are equipped with the usual Euclidean norm which, by abuse of notation, will be denoted in all cases by $|\cdot|$.

Proposition 2.1 can be proved similarly as Proposition 2.3 dealing with the continuous system (10). We thus omit the proof of Proposition 2.1. However, we state the following lemma which guarantees that the functions defined in (6) satisfy the assumptions in Proposition 2.1.

Lemma 2.2. *Let $N\in\mathbb{N}$ fixed and let f be as in (9) and Λ as in (8). Then $\Lambda_{k\ell}$ and f_k as defined in (6) satisfy the assumptions of Proposition 2.1.*

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Proof. According to (6) and (9), we have together with $I_k \cap I_\ell = \emptyset$ for $k \neq \ell$ that

$$\sum_{k=1}^{N} |f_k(t,\phi)|^2 = N^2 \sum_{k=1}^{N} \left| \int_{I_k} f\big(t, x, \textstyle\sum_{\ell=1}^{N} \phi_\ell \chi_{I_\ell}(\cdot)\big) \mathrm{d}x \right|^2 \leq B_f^2 N \Big(1 + \Big| \sum_{\ell=1}^{N} \phi_\ell \chi_{I_\ell}(\cdot) \Big|_{L^\infty} \Big)^2$$

$$= B_f^2 N \big(1 + \max_{\ell \in \{1 \ldots N\}} |\phi_\ell|\big)^2 \leq B_f^2 N \Big(1 + \Big(\sum_{\ell=1}^{N} |\phi_\ell|^2 \Big)^{1/2} \Big)^2 \leq B_f^2 N \Big(1 + |\phi|\Big)^2.$$

Similarly, using additionally Cauchy's inequality we find

$$\sum_{k=1}^{N} |f_k(t,\phi) - f_k(t,\psi)|^2 = N^2 \sum_{k=1}^{N} \Big| \int_{I_k} f\big(t, x, \textstyle\sum_{\ell=1}^{N} \phi_\ell \chi_{I_\ell}(\cdot)\big) - f\big(t, x, \textstyle\sum_{\ell=1}^{N} \psi_\ell \chi_{I_\ell}(\cdot)\big) \mathrm{d}x \Big|^2$$

$$\leq N \sum_{k=1}^{N} \int_{I_k} \big| f\big(t, x, \textstyle\sum_{\ell=1}^{N} \phi_\ell \chi_{I_\ell}(\cdot)\big) - f\big(t, x, \textstyle\sum_{\ell=1}^{N} \psi_\ell \chi_{I_\ell}(\cdot)\big) \big|^2 \mathrm{d}x$$

$$= N \Big| f\big(t, \cdot, \textstyle\sum_{\ell=1}^{N} \phi_\ell \chi_{I_\ell}(\cdot)\big) - f\big(t, \cdot, \textstyle\sum_{\ell=1}^{N} \psi_\ell \chi_{I_\ell}(\cdot)\big) \Big|_{L^2(I)}^2 \leq N L_f^2 \| \textstyle\sum_{\ell=1}^{N} (\phi_\ell - \psi_\ell) \chi_{I_\ell}(\cdot) \|_{L^2(I)}^2$$

$$= L_f^2 |\phi - \psi|^2.$$

In the same way, we get

$$\sum_{k,\ell=1}^{N} |\Lambda_{k\ell}(t,\kappa,\phi)|^2 = N^4 \sum_{k,\ell=1}^{N} \Big| \int_{I_k \times I_\ell} \Lambda\big(t, x, y, \textstyle\sum_{m,n=1}^{N} \kappa_{mn} \chi_{I_m \times I_n}(\cdot), \sum_{m=1}^{N} \phi_m \chi_{I_m}(\cdot)\big) \Big|^2$$

$$\leq B_\Lambda^2 N^2 \Big(1 + \Big| \sum_{m,n=1}^{N} \kappa_{mn} \chi_{I_m \times I_n}(\cdot) \Big|_{L^\infty(I^2)} \Big)^2 \leq B_\Lambda^2 N^2 \Big(1 + |\kappa|\Big)^2.$$

Finally,

$$\sum_{k,\ell=1}^{N} |\Lambda_{k\ell}(t,\kappa,\phi) - \Lambda_{k\ell}(t,\lambda,\psi)|^2$$

$$= N^4 \sum_{k,\ell=1}^{N} \Big| \int_{I_k \times I_\ell} \Lambda\big(t, x, y, \sum_{m,n=1}^{N} \kappa_{mn} \chi_{I_m \times I_n}(\cdot), \sum_{m=1}^{N} \phi_m \chi_{I_m}(\cdot)\big)$$

$$- \Lambda\big(t, x, y, \sum_{m,n=1}^{N} \lambda_{mn} \chi_{I_m \times I_n}(\cdot), \sum_{m=1}^{N} \psi_m \chi_{I_m}(\cdot)\big) \mathrm{d}x\mathrm{d}y \Big|^2$$

$$\leq N^2 \sum_{k,\ell=1}^{N} \int_{I_k \times I_\ell} \big| \Lambda\big(t, x, y, \sum_{m,n=1}^{N} \kappa_{mn} \chi_{I_m \times I_n}(\cdot), \sum_{m=1}^{N} \phi_m \chi_{I_m}(\cdot)\big)$$

$$- \Lambda\big(t, x, y, \sum_{m,n=1}^{N} \lambda_{mn} \chi_{I_m \times I_n}(\cdot), \sum_{m=1}^{N} \psi_m \chi_{I_m}(\cdot)\big) \big|^2 \mathrm{d}x\mathrm{d}y$$

$$= N^2 \big\| \Lambda\big(t, x, y, \sum_{m,n=1}^{N} \kappa_{mn} \chi_{I_m \times I_n}(\cdot), \sum_{m=1}^{N} \phi_m \chi_{I_m}(\cdot)\big)$$

$$- \Lambda\big(t, x, y, \sum_{m,n=1}^{N} \lambda_{mn} \chi_{I_m \times I_n}(\cdot), \sum_{m=1}^{N} \psi_m \chi_{I_m}(\cdot)\big) \big\|_{L^2(I^2)}^2$$

$$\leq N^2 L_\Lambda^2 \Big(\big| \textstyle\sum_{m,n=1}^{N} (\kappa_{mn} - \lambda_{mn}) \chi_{I_m \times I_n}(\cdot)) \big|_{L^2(I^2)} + \big| \sum_{m=1}^{N} (\phi_m - \psi_m) \chi_{I_m}(\cdot)) \big|_{L^2(I)} \Big)^2$$

$$= L_\Lambda^2 \Big(|\kappa - \lambda| + N^{1/2} |\phi - \psi| \Big)^2.$$

□

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The following proposition guarantees the existence of a unique solution to the continuum limit equation (10).

Proposition 2.3. *Let $T > 0$ and assume that $f\colon [0,T] \times I \times L^\infty(I,\mathbb{R}^d) \to \mathbb{R}^d$ and $g\colon [0,T] \times (\mathbb{R}^d)^2 \to \mathbb{R}^d$ satisfy (9). Moreover, assume that $\Lambda\colon [0,\infty) \times I \times I \times L^2(I \times I, \mathbb{R}) \times L^2(I,\mathbb{R}^d) \to \mathbb{R}$ satisfies (8). Then for each initial condition $(u_0, K_0) \in L^\infty(I,\mathbb{R}^d) \times L^\infty(I^2,\mathbb{R})$ the system (10) has a unique solution $(u,K) \in C^1([0,T], L^\infty(I,\mathbb{R}^d)) \times C^1([0,T], L^\infty(I^2,\mathbb{R}))$.*

The claim will follow from the contraction mapping theorem. Due to the properties in (8), we can only obtain a contractive operator with respect to $\|\cdot\|_{L^2}$. However, by following the proof of the contraction mapping theorem and tracking the iterating sequence, we obtain in fact the existence of a unique solution in L^∞. A similar argument has been used in [2] relying on a two-step procedure, while here, we proceed in one step. For $(u_{t_0}, K_{t_0}) \in L^\infty(I,\mathbb{R}^d) \times L^\infty(I^2,\mathbb{R})$ we define the operator $\mathcal{A} := (\mathcal{A}_1, \mathcal{A}_2)\colon C([t_0,T], L^\infty(I)) \times C([t_0,T], L^\infty(I^2)) \to C([t_0,T], L^\infty(I)) \times C([t_0,T], L^\infty(I^2))$ related to the system (10) via:

$$\begin{aligned}\mathcal{A}_1[u,K](t,x) &:= u_{t_0}(x) + \int_{t_0}^t \int_I K(s,x,y) g(s,u(s,x),u(s,y))\mathrm{d}y\mathrm{d}s + \int_{t_0}^t f(s,x,u(s,\cdot))\mathrm{d}s \\ \mathcal{A}_2[u,K](t,x,y) &:= K_{t_0}(x,y) + \int_{t_0}^t \Lambda(s,x,y,K(s,\cdot,\cdot),u(s,\cdot))\mathrm{d}s.\end{aligned} \tag{12}$$

Lemma 2.4. *The operator*

$$\mathcal{A}\colon C([t_0,T], L^\infty(I)) \times C([t_0,T], L^\infty(I^2)) \longrightarrow C([t_0,T], L^\infty(I)) \times C([t_0,T], L^\infty(I^2))$$

is well defined.

Proof. By definition $\mathcal{A}[u,K]$ is continuous in time. Thus, to show that $\mathcal{A}$ is well defined, it suffices to show the boundedness. For $(u,K) \in C([t_0,T], L^\infty(I)) \times C([t_0,T], L^\infty(I^2))$, we can estimate $\mathcal{A}_1$ as

$$\begin{aligned}\|\mathcal{A}_1[u,K](t,\cdot)\|_{L^\infty(I)} &\le \|u_{t_0}\|_{L^\infty(I)} + B_g \int_{t_0}^t \|K(s,\cdot,\cdot)\|_{L^\infty(I^2)}\mathrm{d}s + B_f \int_{t_0}^t (1 + \|u(s,\cdot)\|_{L^\infty(I)})\mathrm{d}s \\ &\le \|u_{t_0}\|_{L^\infty(I)} + \Big(B_g \|K\|_{C([t_0,T],L^\infty)} + B_f(1 + \|u\|_{C([t_0,T],L^\infty)})\Big)(t - t_0).\end{aligned} \tag{13}$$

Thus,

$$\|\mathcal{A}_1[u,K]\|_{C([t_0,T],L^\infty(I))} \le \|u_{t_0}\|_{L^\infty(I)} + \Big(B_g \|K\|_{C([t_0,T],L^\infty)} + B_f(1 + \|u\|_{C([t_0,T],L^\infty)})\Big)(T - t_0).$$

Moreover, for $\mathcal{A}_2$ we have

$$\begin{aligned}\|\mathcal{A}_2[u,K](t,\cdot)\|_{L^\infty(I^2)} &\le \|K_{t_0}\|_{L^\infty(I^2)} + B_\Lambda \int_{t_0}^t (1 + \|K(s,\cdot,\cdot)\|_{L^\infty})\mathrm{d}s \\ &\le \|K_{t_0}\|_{L^\infty(I^2)} + B_\Lambda \Big(1 + \|K\|_{C([t_0,T],L^\infty)}\Big)(t - t_0).\end{aligned}$$

Thus,

$$\|\mathcal{A}_2[u,K]\|_{C([t_0,T],L^\infty(I^2))} \le \|K_{t_0}\|_{L^\infty(I^2)} + B_\Lambda \Big(1 + \|K\|_{C([t_0,T],L^\infty)}\Big)(T - t_0).$$

□

Lemma 2.4 allows to define the sequence $(v_n, J_n)_{n\in\mathbb{N}} \subset C([t_0,T], L^\infty(I)) \times C([t_0,T], L^\infty(I^2))$ via

$$(v_n, J_n) := \mathcal{A}^n[u_{t_0}, K_{t_0}] \tag{14}$$

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where $\mathcal{A}^n$ denotes the n-th iterate of the operator $\mathcal{A}$. We have the following uniform bounds on $(v_n, J_n)_{n\in\mathbb{N}}$.

Lemma 2.5. *Let Λ satisfy (8) and let $u_{t_0} \in L^\infty(I, \mathbb{R}^d)$ and $K_{t_0} \in L^\infty(I^2)$ such that $1 + \|K_{t_0}\|_{L^\infty} \le (1 + \|K_0\|_{L^\infty})\mathrm{e}^{B_\Lambda t_0}$. Then the sequence $(v_n, J_n)_{n\in\mathbb{N}}$ defined in (14) satisfies*

$$\|J_n(t,\cdot,\cdot)\|_{L^\infty(I^2)} \le (1 + \|K_{t_0}\|_{L^\infty})\mathrm{e}^{B_\Lambda(t-t_0)} - 1 \le (1 + \|K_{0L^\infty})\mathrm{e}^{B_\Lambda t} - 1$$

$$\|v_n(t,\cdot)\|_{L^\infty(I)} \le (1 + \|u_{t_0}\|_{L^\infty})\mathrm{e}^{B_f(t-t_0)} + \frac{B_g}{B_\Lambda - B_f}(1 + \|K_{t_0}\|_{L^\infty})\big(\mathrm{e}^{B_\Lambda(t-t_0)} - \mathrm{e}^{B_f(t-t_0)}\big) - 1$$

for all $n \in \mathbb{N}_0$. In particular, we have

$$\|J_n\|_{C([t_0,T],L^\infty(I^2))} \le (1 + \|K_0\|_{L^\infty})\mathrm{e}^{B_\Lambda T}$$

$$\|v_n(t,\cdot)\|_{C([t_0,T],L^\infty(I))} \le (1 + \|u_{t_0}\|_{L^\infty})\mathrm{e}^{B_f(T-t_0)} + \frac{B_g(1 + \|K_{t_0}\|_{L^\infty})}{B_\Lambda - B_f}\big(\mathrm{e}^{B_\Lambda(T-t_0)} - \mathrm{e}^{B_f(T-t_0)}\big) - 1.$$

Remark 2.6. *Note that the estimate on v_n makes sense and is also valid in the limiting case $B_\Lambda - B_f = 0$ when it reduces to*

$$\|v_n(t,\cdot)\|_{L^\infty(I)} \le (1 + \|u_{t_0}\|_{L^\infty})\mathrm{e}^{B_f(t-t_0)} + B_g(1 + \|K_{t_0}\|_{L^\infty})(t - t_0) - 1.$$

Proof of Lemma 2.5. *The bound on J_n is a direct consequence of the following estimate which we obtain by induction:*

$$1 + \|(\mathcal{A}^n[u_{t_0}, K_{t_0}])_2(t,\cdot,\cdot)\|_{L^\infty} \le \Big(\sum_{\ell=0}^{n} \frac{B_\Lambda^\ell}{\ell!}(t - t_0)^\ell\Big)\big(1 + \|K_{t_0}\|_{L^\infty}\big). \tag{15}$$

Similarly, it follows by induction that

$$\begin{aligned} 1 + \|(\mathcal{A}^n[u_{t_0}, K_{t_0}])_1(t,\cdot)\|_{L^\infty} &\le (1 + \|u_{t_0}\|_{L^\infty}) \sum_{\ell=0}^{n} \frac{B_f^\ell}{\ell!}(t - t_0)^\ell \\ &\quad + \frac{B_g}{B_\Lambda}(1 + \|K_{t_0}\|_{L^\infty}) \sum_{k=0}^{n-1}\Big(\frac{B_f}{B_\Lambda}\Big)^k \sum_{\ell=1+k}^{n} \frac{B_\Lambda^\ell}{\ell!}(t - t_0)^\ell - \frac{B_g}{B_f}\sum_{\ell=1}^{n}\frac{B_f^\ell}{\ell!}(t - t_0)^\ell \end{aligned} \tag{16}$$

with $\sum_{k=0}^{-1}(\cdots) := 0 =: \sum_{\ell=1}^{0}(\cdots)$. Moreover, we note that

$$\sum_{k=0}^{n-1}\Big(\frac{B_f}{B_\Lambda}\Big)^k \sum_{\ell=1+k}^{n} \frac{B_\Lambda^\ell}{\ell!}(t - t_0)^\ell = \sum_{\ell=1}^{n}\sum_{k=0}^{\ell-1}\Big(\frac{B_f}{B_\Lambda}\Big)^k \frac{B_\Lambda^\ell}{\ell!}(t - t_0)^\ell = \frac{B_\Lambda}{B_\Lambda - B_f}\sum_{\ell=1}^{n}\frac{B_\Lambda^\ell - B_f^\ell}{\ell!}(t - t_0)^\ell.$$

Together with (16) we thus get

$$\begin{aligned} &1 + \|v_n(t,\cdot)\|_{L^\infty} \\ &\le (1 + \|u_{t_0}\|_{L^\infty}) \sum_{\ell=0}^{n} \frac{B_f^\ell}{\ell!}(t - t_0)^\ell + \frac{B_g}{B_\Lambda - B_f}(1 + \|K_{t_0}\|_{L^\infty}) \sum_{\ell=1}^{n} \frac{B_\Lambda^\ell - B_f^\ell}{\ell!}(t - t_0)^\ell - \frac{B_g}{B_f}\sum_{\ell=1}^{n}\frac{B_f^\ell}{\ell!}(t - t_0)^\ell \end{aligned}$$

which finishes the proof □

The next lemma shows that the operator is contractive with respect to the L^2 norm.

Lemma 2.7. *Let $K_0 \in L^\infty(I^2)$ and $t_0 \in [0, T)$ and assume (8) and (9). For*

$$0 < T_* \le \frac{1}{2(2^{5/2}L_g(1 + \|K_0\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T} + L_f + \sqrt{2}B_g + L_\Lambda)}$$

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the operator $\mathcal{A}$ is contractive on the set $\mathcal{S}_{K_0} := \{(u,K) \in C([t_0,t_0+T_*],L^\infty(I)\times L^\infty(I^2)) \mid 1+\|K(t,\cdot,\cdot)\|_{C([t_0,t_0+T_*],L^\infty(I^2))} \le (1+\|K_0\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T}\}$ with respect to $\|\cdot\|_{C([t_0,t_0+T_*],L^2(I)\times L^2(I^2))}$ for each $t_0 < T$ as long as $t_0+T_* \le T$. More precisely, under these conditions we have

$$\|\mathcal{A}[u_1,K_1]-\mathcal{A}[u_2,K_2]\|_{C([t_0,t_0+T_*],L^2(I)\times L^2(I^2))} \le \frac{1}{2}\Big(\|u_1-u_2\|_{C([t_0,t_0+T_*],L^2(I))}+\|K_1-K_2\|_{C([t_0,t_0+T_*],L^2(I^2))}\Big).$$

Proof. Let $(u_1,K_1),(u_2,K_2)\in\mathcal{S}_{K_0}$. For $\mathcal{A}_2$, we get together with Cauchy's inequality and Fubini's Theorem that

$$\begin{aligned}
&\|\mathcal{A}_2[u_1,K_1](t,\cdot,\cdot)-\mathcal{A}_2[u_2,K_2](t,\cdot,\cdot)\|_{L^2(I^2)}\\
&\quad=\left(\int_{I^2}\left(\int_{t_0}^t \Lambda(s,x,y,K_1(s,\cdot,\cdot),u_1(s,\cdot))-\Lambda(s,x,y,K_2(s,\cdot,\cdot),u_2(s,\cdot))\mathrm{d}s\right)^2\mathrm{d}x\mathrm{d}y\right)^{1/2}\\
&\le\left(\int_{I^2}(t-t_0)\int_{t_0}^t\left|\Lambda(s,x,y,K_1(s,\cdot,\cdot),u_1(s,\cdot))-\Lambda(s,x,y,K_2(s,\cdot,\cdot),u_2(s,\cdot))\right|^2\mathrm{d}s\mathrm{d}x\mathrm{d}y\right)^{1/2}\\
&\quad=(t-t_0)^{1/2}\left(\int_{t_0}^t\|[\,]\Lambda(s,x,y,K_1(s,\cdot,\cdot),u_1(s,\cdot))-\Lambda(s,x,y,K_2(s,\cdot,\cdot),u_2(s,\cdot))\|_{L^2(I^2)}^2\mathrm{d}s\right)^{1/2}.
\end{aligned}$$

By means of (8), we deduce

$$\begin{aligned}
&\|\mathcal{A}_2[u_1,K_1](t,\cdot,\cdot)-\mathcal{A}_2[u_2,K_2](t,\cdot,\cdot)\|_{L^2(I^2)}\\
&\quad\le L_\Lambda(t-t_0)^{1/2}\left(\int_{t_0}^t\left(\left|K_1(s,\cdot,\cdot)-K_2(s,\cdot,\cdot)\right|_{L^2(I^2)}+\left|u_1(s,\cdot)-u_2(s,\cdot)\right|_{L^2(I)}\right)^2\mathrm{d}s\right)^{1/2}.
\end{aligned}$$

This finally yields

$$\begin{aligned}
&\|\mathcal{A}_2[u_1,K_1]-\mathcal{A}_2[u_2,K_2]\|_{C([t_0,t_0+T_*],L^2(I^2))}\\
&\quad\le L_\Lambda T_*\left(\left|K_1(s,\cdot,\cdot)-K_2(s,\cdot,\cdot)\right|_{C([t_0,t_0+T_*],L^2(I^2))}+\left|u_1(s,\cdot)-u_2(s,\cdot)\right|_{C([t_0,t_0+T_*],L^2(I))}\right).
\end{aligned}\tag{17}$$

For $\mathcal{A}_1$, we find similarly by means of Cauchy's inequality and Fubini's Theorem together with (9) that

$$\begin{aligned}
&\|\mathcal{A}_1[u_1,K_1](t,\cdot)-\mathcal{A}_1[u_2,K_2](t,\cdot)\|_{L^2(I)}\\
&\le\left(\int_I\left|\int_{t_0}^t\int_I K_1(s,x,y)g(s,u_1(s,x),u_1(s,y))-K_2(s,x,y)g(s,u_2(s,x),u_2(s,y))\mathrm{d}y\mathrm{d}s\right|^2\mathrm{d}x\right)^{1/2}\\
&\quad+\left(\int_I\left|\int_{t_0}^t f(s,x,u_1(s,\cdot))-f(s,x,u_2(s,\cdot))\mathrm{d}s\right|^2\mathrm{d}x\right)^{1/2}\\
&\le(t-t_0)^{1/2}\left(\int_I\int_{t_0}^t\int_I\Big(\|K_1\|_{C([t_0,t_0+T_*],L^\infty)}\left|g(s,u_1(s,x),u_1(s,y))-g(s,u_2(s,x),u_2(s,y))\right|\right.\\
&\quad\left.+B_g|K_1(s,x,y)-K_2(s,x,y)|\Big)^2\mathrm{d}y\mathrm{d}s\mathrm{d}x\right)^{1/2}\\
&\quad+L_f(t-t_0)^{1/2}\left(\int_{t_0}^t\|u_1(s,\cdot)-u_2(s,\cdot)\|_{L^2(I)}^2\mathrm{d}s\right)^{1/2}.
\end{aligned}$$

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Using (9) together with Young's inequality and the properties of $\mathcal{S}_{K_0}$, we further deduce

$$\begin{aligned}
&\|\mathcal{A}_1[u_1,K_1](t,\cdot)-\mathcal{A}_1[u_2,K_2](t,\cdot)\|_{L^2(I)}\\
&\le\sqrt{2}(t-t_0)^{1/2}\Bigg[L_g(1+\|K_0\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T}\Bigg(\int_I\int_{t_0}^t\int_I\Big(|u_1(s,y)-u_2(s,y)|+|u_1(s,x)-u_2(s,x)|\Big)^2\Bigg)^{1/2}\\
&\quad+B_g\Bigg(\int_I\int_{t_0}^t\int_I|K_1(s,x,y)-K_2(s,x,y)|^2\mathrm{d}y\mathrm{d}s\mathrm{d}x\Bigg)^{1/2}\Bigg]\\
&\quad+L_f(t-t_0)^{1/2}\Bigg(\int_{t_0}^t\|u_1(s,\cdot)-u_2(s,\cdot)\|_{L^2(I)}^2\mathrm{d}s\Bigg)^{1/2}.
\end{aligned}$$

Cauchy's inequality together with Fubini's Theorem then implies

$$\begin{aligned}
&\|\mathcal{A}_1[u_1,K_1](t,\cdot)-\mathcal{A}_1[u_2,K_2](t,\cdot)\|_{L^2(I)}\\
&\le\sqrt{2}(t-t_0)^{1/2}\Bigg[4L_g(1+\|K_0\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T}\Bigg(\int_{t_0}^t\|u_1(s,\cdot)-u_2(s,\cdot)\|_{L^2(I)}^2\mathrm{d}s\Bigg)^{1/2}\\
&\quad+B_g\Bigg(\int_{t_0}^t\|K_1(s,\cdot,\cdot)-K_2(s,\cdot,\cdot)\|_{L^2(I^2)}^2\mathrm{d}s\Bigg)^{1/2}\Bigg]\\
&\quad+L_f(t-t_0)^{1/2}\Bigg(\int_{t_0}^t\|u_1(s,\cdot)-u_2(s,\cdot)\|_{L^2(I)}^2\mathrm{d}s\Bigg)^{1/2}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\|\mathcal{A}_1[u_1,K_1]-\mathcal{A}_1[u_2,K_2]\|_{C([t_0,t_0+T_*],L^2(I))}\\
&\quad\le T_*\Big(2^{5/2}L_g(1+\|K_0\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T}+L_f\Big)\|u_1-u_2\|_{C([t_0,t_0+T_*],L^2(I))}\\
&\qquad+\sqrt{2}T_*B_g\|K_1-K_2\|_{C([t_0,t_0+T_*],L^2(I^2))}.
\end{aligned}$$

Together with (17) we deduce

$$\begin{aligned}
&\|\mathcal{A}[u_1,K_1]-\mathcal{A}[u_2,K_2]\|_{C([t_0,t_0+T_*],L^2(I)\times L^2(I^2))}\\
&\quad:=\|\mathcal{A}_1[u_1,K_1]-\mathcal{A}_1[u_2,K_2]\|_{C([t_0,t_0+T_*],L^2(I))}+\|\mathcal{A}_2[u_1,K_1]-\mathcal{A}_2[u_2,K_2]\|_{C([t_0,t_0+T_*],L^2(I^2))}\\
&\quad\le T_*\Big(2^{5/2}L_g(1+\|K_0\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T}+L_f+\sqrt{2}B_g+L_\Lambda\Big)\times\\
&\qquad\times\Big(\|u_1-u_2\|_{C([t_0,t_0+T_*],L^2(I))}+\|K_1-K_2\|_{C([t_0,t_0+T_*],L^2(I^2))}\Big).
\end{aligned}$$

Thus, for

$$T_*\le\frac{1}{2(2^{5/2}L_g(1+\|K_0\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T}+L_f+\sqrt{2}B_g+L_\Lambda)}$$

the claim follows. □

Moreover, we have the following a priori estimate on solutions of the system (10).

Lemma 2.8. *Assume that* $f\colon[0,T]\times I\times L^\infty(I,\mathbb{R}^d)\to\mathbb{R}^d$ *and* $g\colon[0,T]\times(\mathbb{R}^d)^2\to\mathbb{R}^d$ *satisfy* (9). *Moreover, assume that* $\Lambda\colon[0,\infty)\times I\times I\times L^\infty(I\times I,\mathbb{R})\times L^\infty(I,\mathbb{R}^d)\to\mathbb{R}$ *satisfies* (8). *Let* (u_{t_0},K_{t_0})

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$\in L^\infty(I,\mathbb{R}^d)\times L^\infty(I^2,\mathbb{R})$. *Let* (u,K) *solve* (10) *on* $[t_0,T_1]$ *with* $0\le t_0<T_1\le T$ *and initial condition* $(u(t_0,\cdot),K(t_0,\cdot,\cdot))=(u_{t_0},K_{t_0})$. *Then, we have the estimates*

$$\|u(t,\cdot)\|_{L^\infty(I)}\le\big(1+\|u_{t_0}\|_{L^\infty}\big)\mathrm{e}^{B_f(t-t_0)}+\frac{B_g}{B_\Lambda-B_f}\big(1+\|K_0\|_{L^\infty}\big)\big(\mathrm{e}^{B_\Lambda(t-t_0)}-\mathrm{e}^{B_f(t-t_0)}\big)-1$$

$$\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}\le(1+\|K_{t_0}\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda(t-t_0)}-1.$$

In particular, we have the bounds

$$\sup_{t\in[t_0,T_1]}\|u(t,\cdot)\|_{L^\infty(I)}\le\big(1+\|u_{t_0}\|_{L^\infty}\big)\mathrm{e}^{B_fT_1}+\frac{B_g}{B_\Lambda-B_f}\big(1+\|K_0\|_{L^\infty}\big)\big(\mathrm{e}^{B_\Lambda T_1}-\mathrm{e}^{B_fT_1}\big)-1$$

$$\sup_{t\in[t_0,T_1]}\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}\le(1+\|K_{t_0}\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda T_1}-1.$$

Proof. We start with the estimate on K. Since (u,K) solves (10), we have

$$K(t,x,y)=K_{t_0}(x,y)+\int_0^t\Lambda(s,x,y,K(s,\cdot,\cdot),u(s,\cdot))\mathrm{d}s.$$

By means of (8), we get

$$\|K(t,\cdot,\cdot)_{L^\infty(I^2)}\le\|K_{t_0}\|_{L^\infty(I^2)}+B_\Lambda\int_{t_0}^t(1+\|K(s,\cdot,\cdot)\|_{L^\infty(I^2)})\mathrm{d}s.$$

Gronwall's inequality then implies

$$1+\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}\le(1+\|K_{t_0}\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda(t-t_0)}.\tag{18}$$

With this, the estimate on u follows similarly noting first that

$$u(t,x)=u_{t_0}+\int_{t_0}^t\int_I K(s,x,y)g(t,u(s,x),u(s,y))\mathrm{d}y\mathrm{d}s+\int_{t_0}^t f(s,x,u(s,\cdot))\mathrm{d}s.$$

Thus, using again (8), we get together with (18) that

$$\|u(t,\cdot)\|_{L^\infty}\le\|u_{t_0}\|_{L^\infty}+B_g\int_{t_0}^t\Big((1+\|K_{t_0}\|_{L^\infty(I^2)})\mathrm{e}^{B_\Lambda(s-t_0)}-1\Big)\mathrm{d}s+B_f\int_{t_0}^t(1+\|u(s,\cdot)\|_{L^\infty})\mathrm{d}s.$$

By means of Gronwall's inequality one deduces that

$$\begin{aligned}&1+\|u(t,\cdot)\|_{L^\infty}\\&\quad\le\big(1+\|u_{t_0}\|_{L^\infty}\big)\mathrm{e}^{B_f(t-t_0)}+\frac{B_g}{B_\Lambda-B_f}\big(1+\|K_{t_0}\|_{L^\infty}\big)\big(\mathrm{e}^{B_\Lambda(t-t_0)}-\mathrm{e}^{B_f(t-t_0)}\big)-\frac{B_g}{B_f}(\mathrm{e}^{B_f(t-t_0)}-1)\end{aligned}\tag{19}$$

from which the claim follows. □

We can now give the proof of Proposition 2.3.

Proof of Proposition 2.3. *As announced earlier, we argue along the lines of the proof of the classical contraction mapping theorem. However, since the operator* $\mathcal{A}$ *is only contractive with respect to the* L^2 *topology, some adjustments are needed. First, we fix* $T_*\le T$ *according to Lemma* 2.7. *Next, we set* $t_0=0$ *and define the corresponding sequence* (v_n,J_n) *as in* (14) *which is well defined according to Lemma* 2.4. *Moreover, due to Lemma* 2.5, *the sequence* $(J_n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $C([0,T_*],L^\infty(I^2))$ *with*

$$1+\|J_n(t,\cdot,\cdot)\|_{C([0,T_*],L^\infty(I^2))}\le(1+\|K_0\|_{L^\infty})\mathrm{e}^{B_\Lambda T_*}\le(1+\|K_0\|_{L^\infty})\mathrm{e}^{B_\Lambda T}\qquad\text{for all } n\in\mathbb{N}_0.$$

Consequently, $(v_n,J_n)\in\mathcal{S}_{K_0}$ *for all* $n\in\mathbb{N}_0$ *with* $\mathcal{S}_{K_0}$ *defined in Lemma* 2.7. *Thus, according to this result, we have*

$$\begin{aligned}&\|v_{n+1}-v_n\|_{C([0,T_*],L^2(I))}+\|J_{n+1}-J_n\|_{C([0,T_*],L^2(I^2))}\\&\quad\le\frac{1}{2}\Big(\|v_n-v_{n-1}\|_{C([0,T_*],L^2(I))}+\|J_n-J_{n-1}\|_{C([0,T_*],L^2(I^2))}\Big)\end{aligned}$$

https://doi.org/10.1017/S0956792524000354 Published online by Cambridge University Press

which yields by iteration that $(v_n, J_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T_*], L^2(I)\times L^2(I^2))$. Consequently, there exists $(u,K)\in C([0,T_*], L^2(I)\times L^2(I^2))$ such that

$$\|v_n-u\|_{C([0,T_*],L^2(I))}+\|J_n-K\|_{C([0,T_*],L^2(I^2))}\longrightarrow 0 \qquad \text{as } n\to\infty. \tag{20}$$

For each $t\in[0,T_*]$, we then have

$$v_n(t,\cdot)\to u(t,\cdot) \quad \text{and} \quad J_n(t,\cdot,\cdot)\to K(t,\cdot,\cdot) \qquad \text{pointwise almost everywhere as } n\to\infty.$$

Thus, by means of Lemma 2.5, we have

$$\begin{aligned}\|u(t,\cdot)\|_{L^\infty(I)}&\le (1+\|u_0\|_{L^\infty})\mathrm{e}^{B_f t}+\frac{B_g}{B_\Lambda-B_f}(1+\|K_0\|_{L^\infty})\big(\mathrm{e}^{B_\Lambda t}-\mathrm{e}^{B_f t}\big)-1\\ \|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}&\le (1+\|K_0\|_{L^\infty})\mathrm{e}^{B_\Lambda t}-1.\end{aligned} \tag{21}$$

Moreover, as a consequence of (20) ,we have $(u,K)=\mathcal{A}[u,K]$ and the structure of $\mathcal{A}$ thus immediately implies $(u,K)\in C^1([0,T_*], L^\infty(I)\times L^\infty(I^2))$ and (u,K) is a solution of (10) on $[0,T_*]$. Due to (21) we have in particular $(u,K)\in\mathcal{S}_{K_0}$ and according to Lemma 2.8 any solution $(\hat u,\hat K)$ to (10) satisfies $(\hat u,\hat K)\in\mathcal{S}_{K_0}$. Thus, uniqueness follows again from the contractivity in Lemma 2.7 analogously to the classical contraction mapping theorem. To finish the proof, it remains to extend the solution to $[0,T]$, which can be done, as usual, by iterating the above procedure while we note that Lemma 2.5 ensures that the condition in the definition of $\mathcal{S}_{K_0}$ is preserved. □

3. The continuum limit

In this section, we will give the proof of Theorem 1.1 using similar arguments as [2, 29].

Proof of Theorem 1.1. *By means of (7) and (10), we have*

$$\begin{aligned}\frac12\partial_t\|u^N-u\|_{L^2(I)}^2&=\int_I\partial_t\big(u^N(t,x)-u(t,x)\big)\big(u^N(t,x)-u(t,x)\big)\mathrm{d}x\\ &=\int_{I^2}\Big[K^N(t,x,y)g(t,u^N(t,x),u^N(t,y))-K(t,x,y)g(t,u(t,x),u(t,y))\Big]\big(u^N(t,x)-u(t,x)\big)\mathrm{d}y\mathrm{d}x\\ &\quad+\int_I\Big[N\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N}f(t,\xi,u^N(t,\cdot))-f(t,x,u(t,\cdot))\mathrm{d}\xi\Big]\big(u^N(t,x)-u(t,x)\big)\mathrm{d}x.\end{aligned}$$

Rewriting, we get

$$\begin{aligned}\frac12\partial_t\|u^N-u\|_{L^2(I)}^2&=\int_{I^2}\Big[\big(K^N(t,x,y)-K(t,x,y)\big)g(t,u^N(t,x),u^N(t,y))\\ &\quad+K(t,x,y)\big(g(t,u^N(t,x),u^N(t,y))-g(t,u(t,x),u(t,y))\big)\Big]\big(u^N(t,x)-u(t,x)\big)\mathrm{d}y\mathrm{d}x\\ &\quad+\int_I\Big[N\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N}f(t,\xi,u^N(t,\cdot))-f(t,\xi,u(t,\cdot))\\ &\quad+f(t,\xi,u(t,\cdot))-f(t,x,u(t,\cdot))\mathrm{d}\xi\Big]\big(u^N(t,x)-u(t,x)\big)\mathrm{d}x.\end{aligned}$$

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Using the bounds on f and g from (9) together with Cauchy's inequality, we can estimate the right-hand side to get

$$\begin{aligned}\frac{1}{2}\partial_t\|u^N-u\|_{L^2(I)}^2 &\le B_g\int_I\Big(\int_I (K^N(t,x,y)-K(t,x,y))^2\mathrm{d}y\Big)^{1/2}|u^N(t,x)-u(t,x)|\mathrm{d}x\\ &+L_g\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}\int_{I^2}\big(|u^N(y)-u(y)|+|u(x)-u^N(x)|\big)|u^N(t,x)-u(t,x)|\mathrm{d}y\mathrm{d}x\\ &+\Bigg(\Big(\int_I\Big|N\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N} f(t,\xi,u^N(t,\cdot))-f(t,\xi,u(t,\cdot))\mathrm{d}\xi\Big|^2\mathrm{d}x\Big)^{1/2}\\ &+\int_I\Big|N\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N} f(t,\xi,u(t,\cdot))-f(t,x,u(t,\cdot))\mathrm{d}\xi\Big|^2\mathrm{d}x\Big)^{1/2}\Bigg)\|u^N(t,\cdot)-u(t,\cdot)\|_{L^2(I)}.\end{aligned}$$

Applying Cauchy's inequality again, we further deduce together with Fubini's Theorem that

$$\begin{aligned}\frac{1}{2}\partial_t\|u^N-u\|_{L^2(I)}^2 &\le B_g\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|_{L^2(I^2)}\|u^N(t,\cdot)-u(t,\cdot)\|_{L^2(I)}\\ &+2L_g\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}\|u^N(t,\cdot)-u(t,\cdot)\|_{L^2(I)}^2\\ &+\Bigg(\Big(\int_I N\int_{\lfloor N\xi\rfloor/N}^{(\lfloor N\xi\rfloor+1)/N}\big|f(t,\xi,u^N(t,\cdot))-f(t,\xi,u(t,\cdot))\big|^2\mathrm{d}x\mathrm{d}\xi\Big)^{1/2}\\ &+\int_I\Big|N\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N} f(t,\xi,u(t,\cdot))-f(t,x,u(t,\cdot))\mathrm{d}\xi\Big|^2\mathrm{d}x\Big)^{1/2}\Bigg)\|u^N(t,\cdot)-u(t,\cdot)\|_{L^2(I)}.\end{aligned}$$

We set

$$r_N := N\int_{\lfloor Nx\rfloor/N}^{(\lfloor Nx\rfloor+1)/N} f(t,\xi,u(t,\cdot))-f(t,x,u(t,\cdot))\mathrm{d}\xi$$

such that Young's inequality together with (9) then implies

$$\begin{aligned}\frac{1}{2}\partial_t\|u^N-u\|_{L^2(I)}^2 &\le \frac{B_g}{2}\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|_{L^2(I^2)}^2\\ &+\Big(2L_g\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}+\frac{B_g}{2}\Big)\|u^N(t,\cdot)-u(t,\cdot)\|_{L^2(I)}^2\\ &+\Big(\|f(t,\cdot,u^N(t,\cdot))-f(t,\cdot,u(t,\cdot))\|_{L^2(I)}+\|r_N\|_{L^2(I)}\Big)\|u^N(t,\cdot)-u(t,\cdot)\|_{L^2(I)}\\ &\le \frac{B_g}{2}\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|_{L^2(I^2)}^2\\ &+\Big(2L_g\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}+L_f+\frac{B_g+1}{2}\Big)\|u^N(t,\cdot)-u(t,\cdot)\|_{L^2(I)}^2+\frac{1}{2}\|r_N\|_{L^2(I)}^2.\end{aligned}\tag{22}$$

Similarly, we deduce from (7) and (10) that

$$\begin{aligned}\frac{1}{2}\partial_t\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|_{L^2}^2 &= \int_{I^2}\partial_t\big(K^N(t,x,y)-K(t,x,y)\big)\big(K^N(t,x,y)-K(t,x,y)\big)\mathrm{d}y\mathrm{d}x\\ &=\int_{I^2}\Bigg[N^2\int_{\frac{\lfloor Nx\rfloor}{N}}^{\frac{\lfloor Nx\rfloor+1}{N}}\int_{\frac{\lfloor Ny\rfloor}{N}}^{\frac{\lfloor Ny\rfloor+1}{N}}\Lambda(t,\xi,\eta,K^N(t,\cdot,\cdot),u^N(t,\cdot))-\Lambda(t,x,y,K(t,\cdot,\cdot),u(t,\cdot))\mathrm{d}\xi\,\mathrm{d}\eta\Bigg]\times\\ &\quad\times\big(K^N(t,x,y)-K(t,x,y)\big)\mathrm{d}y\mathrm{d}x.\end{aligned}$$

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Together with Cauchy's inequality we can estimate the right-hand side as

$$\begin{aligned}
&\frac{1}{2}\partial_t \|K^N(t,\cdot,\cdot) - K(t,\cdot,\cdot)\|_{L^2}^2 \\
&\leq \Bigg[\Bigg(\int_{I^2} \Bigg(N^2 \int_{\frac{\lfloor Nx \rfloor}{N}}^{\frac{\lfloor Nx \rfloor + 1}{N}} \int_{\frac{\lfloor Ny \rfloor}{N}}^{\frac{\lfloor Ny \rfloor + 1}{N}} \Lambda(t,\xi,\eta,K^N(t,\cdot,\cdot),u^N(t,\cdot)) \\
&\quad - \Lambda(t,\xi,\eta,K(t,\cdot,\cdot),u(t,\cdot)) \mathrm{d}\xi \mathrm{d}\eta \Bigg)^2 \mathrm{d}x\mathrm{d}y \Bigg)^{1/2} \\
&\quad + \Bigg(\int_{I^2} \Bigg(N^2 \int_{\frac{\lfloor Nx \rfloor}{N}}^{\frac{\lfloor Nx \rfloor + 1}{N}} \int_{\frac{\lfloor Ny \rfloor}{N}}^{\frac{\lfloor Ny \rfloor + 1}{N}} \Lambda(t,\xi,\eta,K(t,\cdot,\cdot),u(t,\cdot)) \mathrm{d}\xi \mathrm{d}\eta \\
&\quad - \Lambda(t,x,y,K(t,\cdot,\cdot),u(t,\cdot)) \Bigg)^2 \mathrm{d}x\mathrm{d}y \Bigg)^{1/2} \Bigg] \times \\
&\quad \times \|(K^N)(t,\cdot,\cdot) - K(t,\cdot,\cdot)_{L^2}\| \\
&= : \big[\|Q_N\|_{L^2} + \|R_N\|_{L^2} \big] \|(K^N)(t,\cdot,\cdot) - K(t,\cdot,\cdot)\|_{L^2}.
\end{aligned} \tag{23}$$

To estimate the integral given by Q_N further, we apply once more Cauchy's inequality and use Fubini's Theorem to deduce together with (8) that

$$\begin{aligned}
&\|Q_N\|_{L^2} \\
&\leq \Bigg(\int_{I^2} N^2 \int_{\lfloor Nx \rfloor / N}^{(\lfloor Nx \rfloor + 1)/N} \int_{\lfloor Ny \rfloor / N}^{(\lfloor Ny \rfloor + 1)/N} \Big(\Lambda(t,\xi,\eta,K^N(t,\cdot,\cdot),u^N(t,\cdot)) \\
&\quad - \Lambda(t,\xi,\eta,K(t,\cdot,\cdot),u(t,\cdot)) \Big)^2 \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}x\mathrm{d}y \Bigg)^{1/2} \\
&= \Bigg(\int_{I^2} N^2 \int_{\lfloor N\xi \rfloor / N}^{(\lfloor N\xi \rfloor + 1)/N} \int_{\lfloor N\eta \rfloor / N}^{(\lfloor N\eta \rfloor + 1)/N} \Big(\Lambda(t,\xi,\eta,K^N(t,\cdot,\cdot),u^N(t,\cdot)) \\
&\quad - \Lambda(t,\xi,\eta,K(t,\cdot,\cdot),u(t,\cdot)) \Big)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}\xi \mathrm{d}\eta \Bigg)^{1/2} \\
&= \Bigg(\int_{I^2} \Big(\Lambda(t,\xi,\eta,K^N(t,\cdot,\cdot),u^N(t,\cdot)) - \Lambda(t,\xi,\eta,K(t,\cdot,\cdot),u(t,\cdot)) \Big)^2 \mathrm{d}\xi \mathrm{d}\eta \Bigg)^{1/2} \\
&= \Big| \Lambda(t,\cdot,\cdot,K^N(t,\cdot,\cdot),u^N(t,\cdot)) - \Lambda(t,\cdot,\cdot,K(t,\cdot,\cdot),u(t,\cdot)) \Big|_{L^2} \\
&\leq L_\Lambda \big(\|K^N(t,\cdot,\cdot) - K(t,\cdot,\cdot)\|_{L^2} + \|u^N(t,\cdot) - u(t,\cdot)\|_{L^2} \big).
\end{aligned} \tag{24}$$

Summarising (23) and (24), we obtain together with Young's inequality that

$$\begin{aligned}
&\frac{1}{2}\partial_t \|K^N(t,\cdot,\cdot) - K(t,\cdot,\cdot)\|_{L^2}^2 \\
&\quad \leq \big[L_\Lambda \big(\|K^N(t,\cdot,\cdot) - K(t,\cdot,\cdot)\|_{L^2} + \|u^N(t,\cdot) - u(t,\cdot)\|_{L^2} \big) + \|R_N\|_{L^2} \big] \|(K^N(t,\cdot,\cdot) - K(t,\cdot,\cdot)\|_{L^2} \\
&\quad \leq \frac{3L_\Lambda + 1}{2} \|(K^N(t,\cdot,\cdot) - K(t,\cdot,\cdot)\|_{L^2}^2 + \frac{L_\Lambda}{2} \|u^N(t,\cdot) - u(t,\cdot)\|_{L^2}^2 + \frac{1}{2} \|R_N\|_{L^2}^2.
\end{aligned}$$

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Together with (22) this yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\Big(\|u^N-u\|^2_{L^2(I)}+\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|^2_{L^2}\Big)\\
&\quad\le\Big(2L_g\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}+L_f+\frac{B_g+L_\Lambda+1}{2}\Big)\|u^N(t,\cdot)-u(t,\cdot)\|^2_{L^2(I)}\\
&\quad+\frac{3L_\Lambda+B_g+1}{2}\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|^2_{L^2(I^2)}+\frac{1}{2}\big(\|r_N\|^2_{L^2}+\|R_N\|^2_{L^2}\big)\\
&\quad\le\Big(2L_g\|K(t,\cdot,\cdot)\|_{L^\infty(I^2)}+L_f+\frac{3L_\Lambda+B_g+1}{2}\Big)\Big(\|u^N(t,\cdot)-u(t,\cdot)\|^2_{L^2(I)}+\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|^2_{L^2}\Big)\\
&\qquad+\frac{1}{2}\big(\|r_N\|^2_{L^2}+\|R_N\|^2_{L^2}\big).
\end{aligned}$$

Integrating this inequality, we find

$$\begin{aligned}
&\|u^N-u\|^2_{L^2(I)}+\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|^2_{L^2}\\
&\le\Big(\|u^N(0,\cdot)-u(0,\cdot)\|^2_{L^2(I)}+\|K^N(0,\cdot,\cdot)-W(\cdot,\cdot)\|^2_{L^2}\Big)\mathrm{e}^{4L_g\int_0^t\|K(s,\cdot,\cdot)\|_{L^\infty(I^2)}\mathrm{d}s+(2L_f+3L_\Lambda+B_g+1)t}\\
&+\int_0^t\big(\|r_N(s)\|^2_{L^2}+\|R_N(s)\|^2_{L^2}\big)\mathrm{e}^{4L_g\int_{\mathrm{d}}s'\|K(\tau,\cdot,\cdot)\|_{L^\infty(I^2)}\mathrm{d}\tau+(2L_f+3L_\Lambda+B_g+1)(t-s)}\mathrm{d}s.
\end{aligned}$$

On a fixed time interval $[0,T]$*, we can estimate the right-hand side uniformly as*

$$\begin{aligned}
&\|u^N-u\|^2_{L^2(I)}+\|K^N(t,\cdot,\cdot)-K(t,\cdot,\cdot)\|^2_{L^2}\\
&\quad\le\Big(\|u^N(0,\cdot)-u(0,\cdot)\|^2_{L^2(I)}+\|K^N(0,\cdot,\cdot)-W(\cdot,\cdot)\|^2_{L^2}+\int_0^T\big(\|r_N(s)\|^2_{L^2}+\|R_N(s)\|^2_{L^2}\big)\mathrm{d}s\Big)\times\\
&\quad\times\,\mathrm{e}^{4L_g\int_0^T\|K(s,\cdot,\cdot)\|_{L^\infty(I^2)}\mathrm{d}s+(2L_f+3L_\Lambda+B_g+1)T}.
\end{aligned}\tag{25}$$

From Lebesgue's differentiation theorem together with dominated convergence, we deduce

$$\|r_N\|_{L^2(I)}\to 0\quad\textit{and}\quad\|R_N\|_{L^2}\to 0\qquad\textit{as } N\to\infty.$$

Thus, by dominated convergence, for $N\to\infty$ *the term in parenthesis in (25) converges to zero which finishes the proof.* □

Competing interests. The author declares none.

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Cite this article: Throm S. (2026). Continuum limit for interacting systems on adaptive networks. *European Journal of Applied Mathematics*, **37**, 706–720. https://doi.org/10.1017/S0956792524000354

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