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ON THE ASYMPTOTIC POWER OF THE VARIANCE RATIO TEST

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The variance-ratio (VR) test statistic, which is based on k-period differences of the data, is commonly used in empirical finance and economics to test the random walk hypothesis. We obtain the asymptotic power function of the VR test statistic when the differencing period k is increasing with the sample size n such that $k/n \to \delta > 0$. We show that the test is inconsistent against a variety of mean-reverting alternatives, confirm the result in simulations, and then characterize the functional form of the asymptotic power in terms of δ and these alternatives.

1. INTRODUCTION

The random walk is one of the most important models in empirical finance and economics, and several tests have been developed to detect deviations from this model. Parsimonious tests of this model have been developed for low frequency autocorrelations in the data, such as slow mean-reverting processes, by employing multiple horizon data. One of the popular tests of this genre is based on the variance ratio (VR) statistic, which is the sample variance of k-period differences, $x_t - x_{t-k}$, of the time series x_t, divided by k times the sample variance of the first difference, $x_t - x_{t-1}$, for some integer k. Under the null hypothesis that the series x_t is a random walk, the corresponding population variance ratio, given by $PVR(k) = \mathrm{Var}(x_t - x_{t-k})/[k\mathrm{Var}(x_t - x_{t-1})]$, is 1 for any value of k. Arguably, the VR statistic has become the primary tool for testing the random walk hypothesis for low frequency series (see, e.g., among many others, Campbell and Mankiw, 1987; Cochrane, 1988; Lo and Mackinlay, 1988; Poterba and Summers, 1988). The use of the VR statistic can be advantageous when testing against several interesting alternatives to the random walk model, most notably those hypotheses associated with mean reversion. In fact, a number of authors (e.g., Lo and Mackinlay, 1989; Faust, 1992; Richardson and Smith, 1991) have found that the VR statistic has optimal power against such alternatives.[1]

The practical implementation of the VR statistic is, however, less straightforward. The statistical significance has been assessed using conventional asymptotic theory, that is, for fixed k and the sample size n increasing to infinity.

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Unfortunately, the underlying asymptotic theory provides a poor approximation to the small sample distribution of the VR statistic. In fact, rather than being normally distributed (when standardized by $\sqrt{n}$), the statistics are severely biased and right skewed for large k, which makes application of the statistic problematic. A seemingly attractive solution to this problem is to use the result of Richardson and Stock (1989). They derive the asymptotic distribution of the VR statistic under the random walk null, assuming that both k and n increase to infinity but in such a way that k/n converges to a positive constant δ that is strictly less than 1. They show that the VR statistic, without any normalization, converges to a functional of Brownian motion. This new distribution provides a far more robust approximation to the small sample distribution of the VR statistic. Most current applications of the VR statistic cite the $k/n \to \delta$ result as justification for using Monte Carlo distributions (i.e., set at $k = \delta n$) as representative of the VR statistic's sampling distribution.

This paper provides another look at the VR statistic and its statistical properties.[2] In particular, we argue that the $k/n \to \delta$ result has some very poor properties under a wide class of interesting alternatives. Similar to the distribution under the null hypothesis of a random walk, for mean-reverting alternatives with some random walk component, the $k/n \to \delta$ asymptotic theory also implies nondegenerate limiting distributions. Thus, even with a large sample size, the VR statistic should not be expected to provide decisive evidence. Loosely speaking, in terms of the asymptotic theory, there is a trade-off between size and power properties when using the fixed k asymptotics versus the $k/n \to \delta$ asymptotics.

The paper is organized as follows. In Section 2, we define the VR statistic, review the existing literature on its asymptotic distribution, and provide the main theoretical result regarding its lack of power. Section 3 then explores some of the practical implications of this result, in particular, through simulation evidence for relevant alternative models to the random walk theory.

2. THE VR STATISTIC'S POWER

Given $n + 1$ observations $x_0, x_1, \ldots, x_n$ of a time series, the VR statistic with a positive integer $k(< n)$ as differencing period is defined as

$$VR(k) = \hat{\sigma}_b^2(k)/\hat{\sigma}_a^2(k),$$

where

$$\hat{\sigma}_b^2(k) = \frac{n}{k(n-k+1)(n-k)} \sum_{t=k}^{n} (x_t - x_{t-k} - k\hat{\mu})^2,$$

$$\hat{\sigma}_a^2(k) = \frac{1}{n-1} \sum_{t=1}^{n} (x_t - x_{t-1} - \hat{\mu})^2,$$

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and

$$\hat{\mu} = n^{-1} \sum_{t=1}^{n} (x_t - x_{t-1}).$$

In the usual fixed k asymptotic treatment, under the null hypothesis that the $\{x_t\}$ follow a random walk with possible drift, given by

$$x_t = \mu + x_{t-1} + \varepsilon_t, \tag{1}$$

where μ is a real number and $\{\varepsilon_t\}$ is a sequence of zero mean independent random variables, it is possible to show (see, e.g., Lo and Mackinlay, 1988) that

$$\sqrt{n}(VR(k) - 1) \xrightarrow{D} N(0, \sigma_k^2),$$

where σ_k^2 is some simple function of k. This result extends to the case where the $\{\varepsilon_t\}$ are a martingale difference series with conditional heteroskedasticity (see, e.g., Campbell, Lo, and MacKinlay, 1997), though the variance σ_k^2 has to be adjusted a little. The fixed k asymptotic distribution is well known to fare badly when k is large relative to n (see Lo and MacKinlay, 1989).

As an alternative to the fixed k asymptotics, Richardson and Stock (1989) develop an alternative approach under $k \to \infty$, $n \to \infty$, and $k/n \to \delta > 0$ asymptotics. Under the assumption that $\{x_t\}$ follows (1), where $\{\varepsilon_t\}$ is a conditionally heteroskedastic martingale difference series, they show that

$$VR(k) \xrightarrow{D} \frac{1}{(1-\delta)^2 \delta} \int_{\delta}^{1} (W(\lambda) - W(\lambda - \delta) - \delta W(1))^2 \, d\lambda, \tag{2}$$

where $W(\lambda)$ is standard Brownian motion. Thus, whereas the VR statistic converges to one under the fixed k treatment, here it has a nondegenerate limiting distribution. Clearly this distribution is nonnormal and instead can be characterized by a functional of Brownian motions. Through Monte Carlo simulations, Richardson and Stock (1989) demonstrate that this alternate theory provides a much better approximation to the small sample distribution of the VR statistic.

Of course, the problems of generating consistent tests when the statistic converges to a nondegenerate distribution are well known. In fact, as we show in Theorem 1, which follows, under a wide class of mean-reverting models, the VR statistic converges to a positive multiple of the limiting distribution in (2) when $k/n \to \delta > 0$. This implies that the resulting test based on the VR statistic will be inconsistent for such alternatives when k is a constant fraction of the sample size n. In practical terms, this means that no matter how large the sample size, the probability of detecting such an alternative is bounded below some number strictly less than one, which is clearly an undesirable property.

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THEOREM 1. *Let $\{e_t\}$ and $\{u_t\}$ be two series of zero mean independent processes with finite variance σ_e^2 and σ_u^2, respectively, which are independent of each other. Define the process $\{y_t\}$ by $y_t = \lambda y_{t-1} + u_t$ where $|\lambda| < 1$. Let $r_t = \mu + r_{t-1} + e_t$ and*

$$x_t = r_t + y_t. \tag{3}$$

If $k \to \infty$, $n \to \infty$, and $k/n \to \delta > 0$, then

$$VR(k) \xrightarrow{D} \frac{(1+\lambda)\sigma_e^2}{(1+\lambda)\sigma_e^2 + 2\sigma_u^2} \frac{1}{(1-\delta)^2\delta} \int_\delta^1 (W(\lambda) - W(\lambda-\delta) - \delta W(1))^2 \, d\lambda, \tag{4}$$

where $W(\lambda)$ is standard Brownian motion.

Proof of Theorem 1. Let $U_t = x_t - x_{t-k} - k\hat{\mu} = \sum_{j=t-k+1}^{t} e_j - k\bar{e} + y_t - y_{t-k} - kn^{-1}(y_n - y_0)$. Thus,

$$\frac{1}{\sqrt{n}} U_{[n\alpha]} = \frac{1}{\sqrt{n}} \sum_{j=1}^{[n\alpha]} e_j - \frac{1}{\sqrt{n}} \sum_{j=1}^{[n(\alpha-\delta)]} e_j - \frac{k}{n} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e_j + \frac{1}{\sqrt{n}} (y_{[n\alpha]} - y_{[n(\alpha-\delta)]}) - \frac{k}{n} \frac{y_n - y_0}{\sqrt{n}}. \tag{5}$$

By (A.6) of Richardson and Stock (1989, p. 346) for $\alpha > \delta$,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{[n\alpha]} e_j - \frac{1}{\sqrt{n}} \sum_{j=1}^{[n(\alpha-\delta)]} e_j - \frac{k}{n} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e_j \Rightarrow \sigma_e[W(\alpha) - W(\alpha-\delta) - \delta W(1)], \tag{6}$$

where $W(\alpha)$ is standard Brownian motion. Because $\{y_t\}$ is a stationary process, it is trivially true that

$$\frac{1}{\sqrt{n}} (y_{[n\alpha]} - y_{[n(\alpha-\delta)]}) - \frac{k}{n} \frac{y_n - y_0}{\sqrt{n}} \Rightarrow 0. \tag{7}$$

Hence, by (5)–(7).

$$\frac{1}{\sqrt{n}} U_{[n\alpha]} \Rightarrow \sigma_e[W(\alpha) - W(\alpha-\delta) - \delta W(1)].$$

Furthermore, by the weak law of large numbers,

$$\frac{1}{n} \sum_{t=1}^{n} (x_t - x_{t-1} - \hat{\mu})^2 \xrightarrow{P} \sigma_e^2 + \mathrm{Var}(y_t - y_{t-1}) = \sigma_e^2 + \frac{2\sigma_u^2}{1+\lambda}.$$

The same argument as that employed in Richardson and Stock (1989, p. 329) then yields the required result. ■

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Theorem 1 shows that the test based on the VR statistic when k is a fraction of the sample size will result in an inconsistent test for mean-reverting alternatives of the form (3). Such mean-reverting alternatives are of the kind often imposed in finance applications (see, e.g., Poterba and Summers, 1988; Fama and French, 1988). The variable x_t contains a permanent (i.e., random walk) component r_t and a temporary (i.e., stationary) component y_t. It is also obvious from (4) that when the variance of the innovations of the temporary component, σ_u^2, is small relative to σ_e^2 the test based on the VR statistic will have power that is barely larger than the nominal size of the test in detecting the alternative even in infinitely large sample sizes.

A limiting result for VR similar to (4) in Theorem 1 can actually be obtained under much more general conditions on $\{r_t\}$ and $\{y_t\}$. One can allow both the first difference series $\{r_t - r_{t-1}\}$ and $\{y_t\}$ to be arbitrary stationary processes instead of the independent process $\{e_t\}$ and the autoregressive process of order one, respectively, that was imposed in Theorem 1. Thus, the VR statistic can be shown to be inconsistent for a very wide range of alternatives that might be of interest in detecting mean reversion if k is a fraction of the sample size.

3. POWER IMPLICATIONS

The theorem in Section 2 shows that the VR statistic is inconsistent against meaningful alternatives as $k \to \infty$, $n \to \infty$, and $k/n \to \delta > 0$. Moreover, equation (4) provides the form for the asymptotic power function. In this section, we explore the implications of these results for the VR statistic's power.

As a first step, Table 1 reports the small sample power of the VR statistic for the relevant case in which its power is consistent. In particular, we consider the alternative model described in Theorem 1 under the assumption that $\sigma_e^2 = 0$, that is, there is no permanent component to x_t. As seen from the table, as n increases, the power approaches 100%. Interestingly, this increase is greater for smaller δ, which is consistent with Theorem 1's main implication that, for consistency, $k/n \to 0$. Intuitively, in small samples, even consistent alternatives are hard to distinguish, and the large k problem is attenuated.

Table 2 reports the relevant power results for a variety of alternatives described by Theorem 1. Several observations are in order. First, the table reports small sample power for n ranging from 60 through 2,880 observations and also the asymptotic power level. As Theorem 1 shows, and Table 2 demonstrates, the small sample power of the VR statistic is clearly limited, supporting its inconsistency as a test statistic. Second, even though there is some increase in power as n increases, the gains are very limited for alternative specifications with substantial permanent components. For example, when the share of the variance captured by the mean-reverting component, γ, is only 0.25 and 0.5, respectively, the power is 14% and 38% as $n \to \infty$. Third, these power gains are worse for larger δ, which is consistent with the requirement that $k/n \to 0$. As an illustration, for γ equal to 0.25, 0.5, and 0.75, respectively, the power is

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TABLE 1. Small sample power of the VR statistic for the consistent power case (in %)

$\delta = \frac{1}{6}$ (n)	$\lambda = 0.9$	$\lambda = 0.95$	$\lambda = 0.98$
60	7	4	3
120	21	8	4
180	47	11	4
360	98	42	8
720	100	94	22
1,440	100	100	69
2,880	100	100	98
$\delta = \frac{1}{3}$ (n)	$\lambda = 0.9$	$\lambda = 0.95$	$\lambda = 0.98$
60	7	4	3
120	21	8	4
180	41	11	4
360	90	34	5
720	100	79	19
1,440	100	98	47
2,880	100	100	78

Note: Power of the VR statistic is reported for a one-tailed test of the random walk null against a variety of alternatives at the 5% level of significance for $\delta = \frac{1}{6}$ and $\delta = \frac{1}{3}$, where δ represents the ratio of the variance horizon to the data length. The alternative model is $x_t = y_t$ where $y_t = \lambda y_{t-1} + u_t$ and $\sigma_u^2 = 1$.

14%, 38%, and 86% for $\delta = \frac{1}{6}$ versus only 11%, 27%, and 62% for $\delta = \frac{1}{3}$. To get a more complete picture of this point, Figure 1 graphs the three-dimensional relation between power, γ, and δ. As borne out by the figure, lower δ's increase power, with varying impact depending on the permanent component's importance (i.e., level of γ). For a fixed value of γ, the drop-off in the power, as δ increases, is quite dramatic. Finally, note that the mean-reversion parameter, λ, has no effect upon the asymptotic distribution as the VR statistic is consistent against pure mean-reversion alternatives (e.g., see Table 1). Table 2, however, shows that λ is important in small samples.

On the one hand, these results are fairly discouraging for developing tests against slow mean-reverting alternatives. Given a fixed δ, the VR statistic will have limited success. Within the Richardson and Stock (1989) framework, however, Perron and Vodounou (2001) do manage to characterize the maximal possible power by taking a continuous-time limit given a fixed data span, n. On the other hand, and in contrast to the Perron and Vodounou (2001) result, Theo-

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TABLE 2. Relevant power results for alternatives described by Theorem 1 (in %)

$\delta = \frac{1}{6}$	$\lambda = 0.9$			$\lambda = 0.95$			$\lambda = 0.98$		
(n)	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$
60	4	5	6	4	5	4	4	3	4
120	6	9	14	6	4	6	3	4	3
180	9	12	23	6	7	9	6	4	3
360	9	21	51	7	13	23	6	5	5
720	12	28	69	10	21	48	5	8	12
1,440	13	32	78	11	25	69	9	14	30
2,880	16	37	83	14	30	81	10	26	55
∞	14	38	86	14	38	86	14	38	86
$\delta = \frac{1}{3}$	$\lambda = 0.9$			$\lambda = 0.95$			$\lambda = 0.98$		
(n)	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$
60	5	5	6	5	5	5	4	4	4
120	6	9	11	6	5	6	4	4	4
180	8	11	18	5	6	9	4	4	3
360	7	15	33	5	10	18	5	5	4
720	10	21	47	8	15	35	6	7	11
1,440	12	24	55	10	18	43	7	11	19
2,880	13	25	60	9	21	56	9	18	34
∞	11	27	62	11	27	62	11	27	62

Note: The power of the VR statistic is reported for a one-tailed test of the random walk null against a variety of alternatives at the 5% level of significance for $\delta = \frac{1}{6}$ and $\delta = \frac{1}{3}$, where δ represents the ratio of the variance horizon to the data length. The alternative model is $x_t = r_t + y_t$ where $r_t = r_{t-1} + e_t$, $y_t = \lambda y_{t-1} + u_t$, and $\gamma = 2\sigma_u^2/(1 + \lambda)\sigma_e^2 + 2\sigma_u^2$.

rem 1 suggests an alternative approach in which k is allowed to increase with n but at a slower rate. This guarantees consistency against a wide range of alternatives. Practically, this means that, for power purposes, k cannot be too large in small samples.

The practical issue of how to then adjust the size of the statistics for a given k in small samples remains an important question. Nevertheless, Chen and Deo (2001) provide a possible solution to this issue. They show that one way to alleviate the small sample skewness of linear combinations of positive random variables is to use power transformations. Because VR statistics can be characterized as such a linear combination, their approach allows the researcher to apply the variance ratio in small samples across a wide array of unknown distributions. The transformed VR statistic will maintain its consistent power, yet it approximates the small sample distribution much more closely. This particular problem deserves future analysis.

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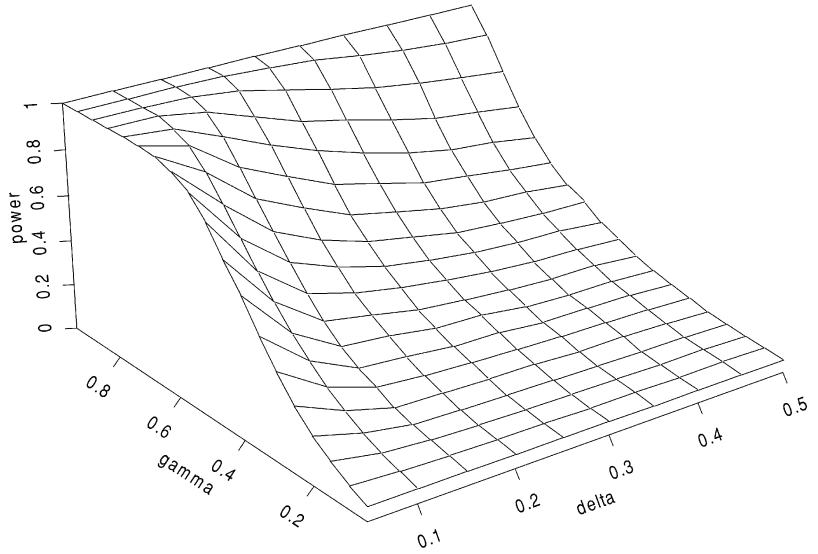


FIGURE 1. Asymptotic power versus γ and δ. Asymptotic power of the VR statistic is shown for a test of the random walk null against a specific alternative model at the 5% level of significance for a range of δ's and γ's, where δ represents the ratio of the variance horizon to the data length and γ represents the proportion of the variance due to the mean-reverting component. In particular, the alternative model is $x_t = r_t + y_t$ where $r_t = r_{t-1} + e_t$, $y_t = \lambda y_{t-1} + u_t$, and $\gamma = 2\sigma_u^2/((1 + \lambda)\sigma_e^2 + 2\sigma_u^2)$.

NOTES

1. Richardson and Smith (1994) and Daniel (2001) explore a wider range of possible test statistics.
2. Perron and Vodounou (2001) also look at the VR statistic's properties under the Richardson and Stock (1989) framework. Their approach, however, is different to the extent that they keep the data span, n, fixed and then take a continuous-time limit within this span. In other words, rather than assuming $n \to \infty$, they assume the sampling frequency of the data goes to zero.

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