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There exists a topologically mixing interval exchange transformation

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Abstract. We prove the existence of a topologically mixing interval exchange transformation and prove that no interval exchange is topologically mixing of all orders.

1. *Introduction*

The main result of this paper is that there exists a topologically mixing interval exchange transformation (IET) answering a question of Boshernitzan. The construction is based on the work of Keane [**11**]. The plan of the paper is as follows. First, we state the results of the paper and introduce the terminology. Second, we describe the Keane construction presented in [**11**], which is used to prove topological mixing. Third, we prove topological mixing. Last, we prove that no IET is topologically mixing of all orders.

Definition 1. Let $\Delta_{n-1} = \{(l_1, \ldots, l_n) : l_i > 0, l_1 + \cdots + l_n = 1\}$ be an $(n-1)$-dimensional simplex. Given $L = (l_1, l_2, \ldots, l_n) \in \Delta_{n-1}$, we can obtain n subintervals of the unit interval $I_1 = [0, l_1)$, $I_2 = [l_1, l_1 + l_2)$, $\ldots$, $I_n = [l_1 + \cdots + l_{n-1}, 1)$. If we are also given a permutation on n letters π, we obtain an *n-interval exchange transformation* $T : [0, 1) \to [0, 1)$, which exchanges the intervals I_i according to π, that is, if $x \in I_j$ then

$$T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.$$

Interval exchange transformation will henceforth be abbreviated to IET.

Remark 1. The various conventions for naming the permutations of IETs can be a source of confusion. The naming convention this paper uses says that if an IET has the permutation (4213) then the fourth interval is placed first by T, the second interval is placed second, the first is placed third, and the third is placed last. It does *not* say that the fourth interval is sent to the second interval.

Definition 2. Let X be a topological space. A dynamical system $T : X \to X$ is said to be *topologically mixing* if for any two non-empty open sets U, V there exists $N_{U,V} := N$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

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THEOREM 1. *There exists a topologically mixing 4-IET.*

Remark 2. It has been shown that no IET satisfies the stronger property of being measure theoretically mixing [**9**]. It has been shown [**8**] that minimal, uniquely ergodic dynamical systems of linear block growth cannot be measure theoretically mixing. No 3-IET can be topologically mixing [**3**]. However, almost every IET that is not of rotation type satisfies the weaker property of being topologically weak mixing [**12**]. In fact, almost every IET that is not of rotation type is measure theoretically weak mixing [**2**]. This was earlier shown for almost every 3-IET [**10**], almost every type W IET [**13**] and every type W IET satisfying an explicit Diophantine condition (Boshernitzan type, which almost every IET satisfies) [**4**]. For type W IETs it has been shown that idoc (the Keane condition) implies topologically weak mixing [**5**]. The formulation of type W used in the two previous results is described in [**6**].

Remark 3. Since writing this paper, we have been able to show that the IETs with permutation $(n, n-1, \ldots, 1)$ of a residual (dense G_δ) set are topologically mixing for any $n \geq 4$. The methods of this proof are different.

Remark 4. The IET exhibited will be topologically mixing with respect to the standard topology on [0,1) and also the finer topology given by considering the IET as a symbolic dynamical system. As a symbolic dynamical system, the IET is continuous.

A complementary result to Theorem 1 is the following theorem.

THEOREM 2. *No IET is topologically mixing of all orders.*

Remark 5. Another result along these lines is proved in [**7**]; this shows that no substitution dynamical system can be measure theoretically mixing, but that there exists a topologically mixing substitution dynamical system. However, no substitution dynamical system can be topologically mixing of all orders. Some substitution dynamical systems have IETs that are obviously conjugate to them. However, the topologically mixing substitution dynamical system in [**7**] is a very particular substitution on two letters which has a closely related substitution that is not topologically mixing. The IETs obviously corresponding to substitutions on two (or even three) letters are never topologically mixing.

Definition 3. Given an IET $T : [0, 1) \to [0, 1)$ and a subinterval J, let $T|_J : J \to J$ denote the first return map induced by T on J, that is, if $x \in J$, let

$$r(x) = \min\{n > 0 : T^n(x) \in J\} \quad \text{and} \quad T|_J(x) = T^{r(x)}(x) \quad \text{if } r(x) \text{ is finite.}$$

A tower over J is the consecutive images of a subinterval before it returns to J. We call the individual images levels.

Remark 6. The discontinuities for the induced map of an IET on an interval $[a, b)$ come from the preimages of discontinuities of the IET and preimages of the endpoint. Therefore, if one chooses the endpoints carefully, one gets an IET on at most the same number of intervals.

Definition 4. $I_j^{(1)}$ denotes the jth subinterval of the IET given by inducing on the fourth interval. $I_j^{(k+1)}$ denotes the jth subinterval of the IET given by inducing on $I_4^{(k)}$.

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Definition 5. Let $O(I_j^{(k)})$ denote the disjoint images under T of $I_j^{(k)}$ before the first return to $I^{(k)}$.

Remark 7. This is the tower over $I^{(k)}$ given by images of $I_j^{(k)}$.

Definition 6. Set

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Remark 8. Some of the above terminology is non-standard, but convenient for our purposes.

2. *The Keane construction*

This construction is given in [**11**] and it is included here for completeness. Consider IETs with the permutation (4213). Observe that the second interval gets shifted by $l_4 - l_1$. If this difference is small relative to l_2 then much of I_2 gets sent to itself. At the same time, pieces of I_3 do not reach I_2 until they have first reached I_4. This is the heart of the Keane construction. The details of the Keane construction are centered around iterating this procedure. Keane showed that by choosing the lengths appropriately, one could ensure that $T|_{I_4^{(1)}}$ is a (2413) IET. Name the intervals in reverse order and we once again get a (4213) IET. Moreover, Keane showed that for any choice $m, n \in \mathbb{N}$ one can find an IET whose landing pattern of $I_j^{(1)}$ is given by the following matrix:

$$A_{m,n} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ m-1 & m & 0 & 0 \\ n & n & n-1 & n \\ 1 & 1 & 1 & 1 \end{pmatrix}; \quad m, n \in \mathbb{N} = \{1, 2, \ldots\}.$$

In order to see this, pick lengths for $I^{(1)}$ and write them as a column vector. Now assign lengths to the original IET by multiplying this column vector by $A_{m,n}$. The induced map will travel according to this matrix by construction. For instance, if one chooses lengths $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$ for $I_4^{(1)}, \ldots, I_1^{(1)}$, one gets lengths of

$$\left[\frac{2}{2+2m-1+4n-1+4}, \frac{2m-1}{2m+4n+4}, \frac{4n-1}{2m+4n+4}\frac{4}{2m+4n+4}\right]$$

for the original IET (after renormalizing). For any finite collection of matrices one can just iterate this construction (assign lengths for $I^{(k)}$ by multiplying the lengths of $I^{(k+1)}$ by A_{m_k,n_k} and multiplying the resulting column vector by $A_{m_{k-1},n_{k-1}}, \ldots$). Compactness (of Δ_3, which can be thought of as the parameterizing space of (4213) IETs) ensures that we can pass to an infinite sequence of these matrices.

Since the intervals are named in reverse order, the discontinuity (under the induced map) between $I_2^{(1)}$ and $I_3^{(1)}$ is given by $T^{-1}(\delta_1)$, where δ_1 denotes the discontinuity between I_1 and I_2. As the first row of the matrix suggests, $I_1 = T(I_4^{(1)} \cup I_3^{(1)})$. The discontinuity (under the induced map) between $I_1^{(1)}$ and $I_2^{(1)}$ is given by $T^{-m}(\delta_2)$, where δ_2 denotes the

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discontinuity between I_2 and I_3. As the second row of the matrix suggests,

$$I_2 = T(I_2^{(1)} \cup I_1^{(1)}) \cup T^2(I_2^{(1)} \cup I_1^{(1)}) \cup \cdots \cup T^{m-1}(I_2^{(1)} \cup I_1^{(1)}) \cup T^m(I_2).$$

The discontinuity (under the induced map) between $I_3^{(1)}$ and $I_4^{(1)}$ is given by $T^{-n-1}(\delta_3)$, where δ_3 denotes the discontinuity between I_3 and I_4. As the third row of the matrix suggests,

$$\begin{aligned} I_3 = {} & T^m(I_1^{(1)}) \cup T^{m+1}(I_2^{(1)}) \cup T^2(I_4^{(1)} \cup I_3^{(1)}) \cup T^{m+1}(I_1^{(1)}) \\ & \cup T^{m+2}(I_2^{(1)}) \cup T^3(I_4^{(1)} \cup I_3^{(1)}) \cup \cdots \cup T^{m+n-1}(I_1^{(1)}) \cup T^{m+n}(I_2^{(1)}) \\ & \cup T^n(I_4^{(1)} \cup I_3^{(1)}) \cup T^{m+n}(I_1^{(1)}) \cup T^{m+n+1}(I_2^{(1)}) \cup T^{n+1}(I_4^{(1)}). \end{aligned}$$

$I_4 = I_4^{(1)} \cup I_3^{(1)} \cup I_2^{(1)} \cup I_1^{(1)}$. As the columns of the matrix suggest, this is also

$$I_4 = T^{n+1}(I_3^{(1)}) \cup T^{m+n+1}(I_2^{(1)}) \cup T^{m+n}(I_1^{(1)}) \cup T^{n+2}(I_4^{(1)}).$$

To summarize, the composition of I_j can be given by the jth row of the matrix. The travel before the first return of $I_j^{(1)}$ can be given by the jth column. Additionally, because the intervals were named in reverse order, the permutation of the induced map is once again (4213).

It is important for this construction that everything be iterated. The composition of $I_j^{(k)}$ in pieces of $I^{(k+r)}$ is given by $e_j^\tau A_{m_k,n_k}, \ldots, A_{m_{k+r-1},n_{k+r-1}}$ (where e_j^τ denotes the transpose of e_j). Likewise, the travel of $I_j^{(k+r)}$ under $T|_{I^{(k)}}$ before the first return to $I^{(k+r)}$ is given by $A_{m_k,n_k}, \ldots, A_{m_{k+r-1},n_{k+r-1}} e_j$.

Now follow some explicit statements about the travel of subintervals of $I^{(k)}$ under the induced map $T|_{I^{(k)}}$. When $I_3^{(k)}$ returns to $I^{(k)}$, it entirely covers $I_4^{(k)}$. It is a subset of $I_3^{(k)} \cup I_4^{(k)}$. When $I_4^{(k)}$ returns to $I^{(k)}$, it entirely covers $I_1^{(k)}$. It intersects $I_2^{(k)}$. Moreover, part of this intersection will stay in $O(I_2^{(k)})$ for the next $m_k b_{k,2}$ images (the other part stays for $(m_k - 1)b_{k,2}$ images). When $I_2^{(k)}$ returns to $I^{(k)}$, it intersects $I_3^{(k)}$. Moreover, this piece of intersection will stay in $O(I_3^{(k)})$ for the next $n_k b_{k,3}$ images.

Definition 7. Let $b_{k,i}$ be the first return time of $I_i^{(k)}$ to $I^{(k)}$.

Remark 9. $b_{k,2} = m_{k-1} b_{k-1,2} + n_{k-1} b_{k-1,3} + b_{k-1,4}$ and $b_{k,3} = b_{k-1,1} + (n_{k-1} - 1) b_{k-1,3} + b_{k-1,4}$.

Some facts to keep in mind.

(1) The choice of n_k has no effect on $b_{i,2}$ for $i \leq k$.
(2) The choice of n_k has no effect on $b_{i,3}$ for $i \leq k$.
(3) The choice of m_k has no effect on $b_{i,2}$ for $i \leq k$.
(4) The choice of m_k has no effect on $b_{i,3}$ for $i \leq k+1$.

3. *There exists a topologically mixing Keane IET*

The conditions for $b_{k,2}$ and $b_{k,3}$ to ensure topological mixing are as follows.

(1) $b_{k,2}$ is prime for all k.
(2) $b_{i,2} \not| b_{k,3}$ for all $i < k$.
(3) The group of multiplicative units modulo $\Pi_{i=1}^k b_{i,2}$ has more than $\frac{1}{2}\Pi_{i=1}^k b_{i,2}$ elements.

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(4) $b_{k,2}b_{k+1,3} + b_{k+1,3} + b_{k,4} + b_{k-1,4} < m_k b_{k,2}$.
(5) $b_{k,3}b_{k,2} + b_{k,2} < n_k b_{k,3}$.

Theorem 1 will be proven by showing that any Keane IET chosen in this way is topologically mixing. We first show that the set of such IETs is non-empty.

LEMMA 1. *We can choose $b_{k,2}$ and $b_{k,3}$ to fulfill these conditions.*

Proof. The proof is by induction. Assume we have chosen $n_1, m_1, n_2, m_2, \ldots, n_{k-1}, m_{k-1}$; we describe how to choose n_k and then, given this n_k, how to choose m_k. Consider congruence modulo $g := \Pi_{i=1}^{k} b_{i,2}$. Choose a congruence class $[f]$ that is in the group of multiplicative units and so that $[f + b_{k,3} - b_{k,1}]$ is in the multiplicative group of units. This can be done by the pigeon hole principle (by condition (3)). Pick n_k so that $b_{k+1,3} \in [f]$ and so that

$$n_k > \frac{b_{k,3}b_{k,2} + b_{k,2}}{b_{k,3}}.$$

This can be done because $b_{k,3}$ is relatively prime to $b_{i,2}$. Next we pick m_k so that $b_{k+1,2}$ is prime,

$$m_k > \frac{b_{k,2}b_{k+1,3} + b_{k,4} + b_{k+1,3}}{b_{k,2}},$$

and condition (3) is satisfied. This can be done because we wish to find a prime in the arithmetic progression $n_k b_{k,3} + b_{k,4} + b_{k,2}\mathbb{N}$, and the starting point and the increment are relatively prime and the other conditions merely require choosing m_k large enough. This is Dirichlet's theorem (see [**1**, Ch. 7], for example). □

Let

$$c_k = b_{k+1,2}b_{k+2,3} + b_{k+2,3} + b_{k+2,4} + b_{k+1,4}$$

and

$$d_k = b_{k+2,3}b_{k+2,2} + b_{k+2,2}.$$

Let J be a subinterval in I containing at least one level of a tower over $I^{(k)}$. This means that it contains at least one level from each of the four towers over $I^{(k+2)}$. For all $j > k$, $i > c_j$, $T^i(J)$ intersects every level of every tower over $I^{(j-1)}$. This is proved in the following lemmas. In these arguments it will be important to pick out a level from $O(I_2^{(k+2)})$ and $O(I_3^{(k+2)})$. These will be denoted by J'.

LEMMA 2. *At times c_k to d_k J', a level in $O(I_3^{(k+2)})$ intersects every level of $O(I_2^{(k+1)})$.*

Proof. There exists $0 < i \leq b_{k+2,3}$ (it is equal to $b_{k+2,3}$ for $I_3^{(k+2)}$ but for pieces of the orbit it is less than $b_{k+2,3}$) such that $I_4^{(k+2)} \subset T^i(I_3^{(k+2)})$. So $T^{i+b_{k+2,4}}(I_3^{(k+2)}) \cap I_2^{(k+2)} \neq \emptyset$. Also,

$$T^{i+b_{k+2,4}+b_{k+1,4}}(I_3^{(k+2)}) \cap I_2^{(k+1)} \neq \emptyset.$$

In fact,

$$T^{i+b_{k+2,4}+b_{k+1,4}+jb_{k+2,3}}(I_3^{(k+2)}) \cap I_2^{(k+1)} \neq \emptyset$$

for $j < n_{k+3}$. Pieces of $I_3^{(k+2)}$ are inserted into $I_2^{(k+1)}$ with a delay of $b_{k+2,3}$, which is coprime to $b_{k+1,2}$. It follows that $T^{c_k}(J')$ intersects every level of $O(I_2^{(k+1)})$. By

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condition (5), it follows that $T^r(J')$ intersects every level of $O(I_2^{(k+1)})$ for $c_k \le r \le d_k$. Moreover, the pieces inserted take $m_{k+1}b_{k+1,2}$ to leave $O(I_2^{(k+2)})$. Because $m_{k+1}b_{k+1,2} > b_{k+2,3}b_{k+1,2}$ (condition (4)) the piece does not leave $O(I_2^{(k+2)})$ before another is inserted into its level. □

LEMMA 3. *At times d_k to c_{k+1} J', a level in $O(I_2^{(k+2)})$ intersects every piece of $O(I_3^{(k+2)})$.*

Proof. There exists $0 < i \le b_{k+2,2}$ (it is equal to $b_{k+2,2}$ for $I_2^{(k+2)}$ but for pieces of the orbit it is less than $b_{k+2,2}$) such that $I_3^{(k+2)} \cap T^i(I_2^{(k+2)}) \ne \emptyset$. Also, $I_3^{(k+2)} \cap T^{i+jb_{k+2,2}} \ne \emptyset$ for $j < m_{k+2}$. Because $b_{k+2,2}$ is relatively prime to $b_{k+2,3}$, we have that $T^{i+jb_{k+2,2}}(I_2^{(k+2)})$ intersects each level of $O(I_3^{(k+2)})$ for $j = b_{k+2,3}$. It follows from condition (4) that $T^r(I_2^{(k+2)})$ intersects each level of $O(I_3^{(k+2)})$ for $d_k \le r \le c_{k+1}$. Moreover, the pieces inserted take $(n_{k+2}-1)b_{k+2,3}$ to leave $O(I_3^{(k+2)})$. Because $(n_{k+2}-1)b_{k+2,3} > b_{k+2,2}b_{k+2,3}$ (condition (5)), the piece does not leave $O(I_3^{(k+1)})$ before another is inserted into its level. □

Proof of Theorem 1. Let J_1, J_2 be any two non-empty intervals in I. There therefore exists k_0 such that both intervals contain some level of a tower over $I^{(k_0)}$. This implies that they contain a level from each tower over $I^{(k)}$ for all $k > k_0 + 1$. This implies that $T^n(J_1) \cap J_2 \ne \emptyset$ for $n \in [c_k, d_k]$ because J_1 contains a level of $I_3^{(k+2)}$ and J_2 contains a level of $I_2^{(k+1)}$. Also, $T^n(J_1) \cap J_2 \ne \emptyset$ for $n \in [d_k, c_{k+1}]$ because J_1 contains a level of $I_2^{(k+2)}$ and J_2 contains a level of $I_3^{(k+2)}$. It follows that $T^n(J_1) \cap J_2 \ne \emptyset$ for any $n > c_{k_0+1}$. □

4. *No IET is topologically mixing of all orders*

The argument is a straightforward application of [**9**]. Let T be a d-IET. Observe that a topologically mixing IET must be minimal (otherwise it splits into disjoint invariant components). Let J, J' be any disjoint intervals bounded by discontinuities of T^l for some l, and $n_1, \ldots, n_{d^2}$ be natural numbers. We will find a violation of topological mixing of order $d^2 + 1$ at bigger times. Pick an interval V such that all of the first returns to V are greater than $\max\{l, n_1, \ldots, n_{d^2}\}$. We may also choose V so that $T|_V$ is an s-IET for some $s \le d$. By our assumption that the return times to V are larger than l, each level of a tower over V is either contained in J or disjoint from J. Let $U_1, U_2, \ldots, U_s$ be its subintervals. $T|_{U_i}$ is an s_i-IET for $s_i \le s$. Call its intervals $U_{i,1}, \ldots, U_{i,s_i}$ and their return times $r_{i,1}, \ldots, r_{i,s_i}$. If $x \in O(U_i) \cap J$ and $x \in O(U_{i,j})$ then $T^{r_{i,j}}(x) \in J$. This is because $x \in T^k(U_i) \subset J$ for some $k < r_i$, in fact $x \in T^k(U_{i,j})$. $T^{r_{i,j}-k}(x) \in U_i$. So $T^k(T^{r_{i,j}-k}(x)) \in T^k(U_i) \subset J$. Therefore $\bigcap_{i,j=1}^{d} T^{r_{i,j}}(J) \cap J' = \emptyset$.

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