Probability in the Engineering and Informational Sciences, **27**, 2013, 367–374.
doi:10.1017/S0269964813000089

A NOTE ON BIVARIATE DUAL GENERALIZED MARSHALL–OLKIN DISTRIBUTIONS WITH APPLICATIONS

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This note introduces bivariate dual generalized Marshall–Olkin distribution and builds a comparison result on the copula of this distribution. Several applications in survival analysis and actuarial science are presented as well.

1. INTRODUCTION

Assume S_1, S_2, and S_3 are independent non-negative random variables. The random vector $(\min\{S_1, S_3\},\ \min\{S_2, S_3\})$ has bivariate generalized Marshall–Olkin (GMO) distribution. Because this structure of dependence has very good applications in reliability theory and finance, bundles of interesting research works have been accomplished ever since it was first introduced by Marshall and Olkin (1967) [12]. For instance, Muliere and Scarsini [9], Li [7], Lin and Li [8], etc. Recently, in the fields of credit risk, for example, actuary, finance, and economics, researchers found that it is also of interest to understand the dependence structure

$$\big(\max\{S_1, S_3\},\ \max\{S_2, S_3\}\big), \tag{1}$$

a dual version of Marshall–Olkin dependence. Here are two practical examples:

- In network security, suppose a pair of nodes of degree two share exactly one common neighbor and each works properly if and only if at least one of its neighbors is working. If S_1, S_2 denote the time to the first failure of their distinct neighbors and S_3 is that of the common neighbor, then the vector of times to the first failures of the two nodes will take the form of (1);
- In insurance, consider two policy holders, which are subject to the common risk with potential loss S_3 and their own risks having potential losses S_1 and S_2, respectively. If only the larger loss will be covered by the insurer (this is often the case in practice), then, the loss vector of the two insureds is just given by (1).

In general, the dual MO structure of dependence may be employed to model many phenomena either in risk analysis or financial engineering. However, few results on this topic

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https://doi.org/10.1017/S0269964813000089 Published online by Cambridge University Press

except for the following ones may be found in literature. Li [5] investigated the multivariate Marshall–Olkin exponential distributions, and Li [6] pointed out that the dual MO copula is just the MO survival copula when the three concerned distributions are of *proportional reversed hazard rates*. As one application in statistics, Sarhan et al. [10] employed this dual MO structure to construct the so-called *generalized bivariate linear failure rate* distribution, by which a better fit to the UEFA Champion's League data was achieved.

This note studies the order of the bivariate dual GMO copula and presents some applications in insurance industry and survival analysis. Section 2 introduces the dual GMO distribution and builds the order of dual GMO copulas. Those applications are presented in Section 3.

2. DUAL GENERALIZED MARSHALL–OLKIN DISTRIBUTIONS

Suppose mutually independent random variables S_1, S_2, and S_3 have their respective distribution functions F_1, F_2, and F_3. As is well-known, the random vector $\boldsymbol{Z} = (\min\{S_1, S_3\}, \min\{S_2, S_3\})$ has the GMO distribution, Li and Pellerey [7] derived its survival copula

$$\hat{C}_{\boldsymbol{Z}}(u,v) = \frac{uv}{\bar{F}_3\Big(\min\big\{\bar{K}_1^{-1}(u), \bar{K}_2^{-1}(v)\big\}\Big)},$$

here $\bar{F} = 1 - F$ and $\bar{K}_i = \bar{F}_i\bar{F}_3$ for $i = 1, 2$, and built one interesting result on association.

Recall S_1 is said to be *stochastically larger* than T_1 (denoted by $S_1 \geq_{st} T_1$) if $P(S_1 > x) \geq P(T_1 > x)$ for all x.

THEOREM 1 (Li and Pellerey [7]): *For GMO random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ with their respective generators S_j and T_j, $j = 1, 2, 3$, if $S_3 \leq_{st} T_3$ and $S_i \geq_{st} T_i$ for $i = 1, 2$, then, the two survival copulas satisfy*

$$\hat{C}_{\boldsymbol{X}}(u,v) \geq \hat{C}_{\boldsymbol{Y}}(u,v), \quad \textit{for all } 0 \leq u, v \leq 1,$$

If further $\min\{S_i, S_3\} \geq_{st} \min\{T_i, T_3\}$ for $i = 1, 2$, then $\bar{F}_{\boldsymbol{X}}(x_1, x_2) \geq \bar{F}_{\boldsymbol{Y}}(x_1, x_2)$ for all x_1, x_2.

For more on GMO distributions, we refer readers to Li and Pellerey [7], Lin and Li [8]. As a dual version, the bivariate random vector

$$\boldsymbol{X} = (X_1, X_2) = (\max\{S_1, S_3\}, \max\{S_2, S_3\}) \tag{2}$$

has the joint distribution function

$$\begin{aligned} F_{\boldsymbol{X}}(x,y) &= P(X_1 \leq x, X_2 \leq y) \\ &= P(S_1 \leq x, S_2 \leq y, S_3 \leq \min\{x,y\}) \\ &= F_1(x)F_2(y)F_3(\min\{x,y\}) \\ &= \frac{K_1(x)K_2(y)F_3(\min\{x,y\})}{F_3(x)F_3(y)}, \end{aligned} \tag{3}$$

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where $K_i(x) = F_i(x)F_3(x)$ for $i = 1, 2$. For convenience, a random vector with joint distribution (3) is said to be of the dual GMO distribution. Evidently, $\boldsymbol{X}$ has the copula

$$\begin{aligned} C_{\boldsymbol{X}}(u,v) &= uv\frac{F_3\left(\min\left\{K_1^{-1}(u), K_2^{-1}(v)\right\}\right)}{F_3\big(K_1^{-1}(u)\big)F_3\big(K_2^{-1}(v)\big)} \\ &= uv\min\left\{\frac{1}{F_3\big(K_1^{-1}(u)\big)}, \frac{1}{F_3\big(K_2^{-1}(v)\big)}\right\} \\ &= \frac{uv}{F_3\left(\max\left\{K_1^{-1}(u), K_2^{-1}(v)\right\}\right)}. \end{aligned} \tag{4}$$

Suppose another dual GMO random vector $\boldsymbol{Y} = (\max\{T_1, T_3\}, \max\{T_2, T_3\})$, which is generated by independent random variables $\{T_1, T_2, T_3\}$, and denote G_i the distribution function of T_i, $i = 1, 2, 3$. Next theorem builds the relationship between stochastic ordering of the generating components and the resulted copulas of the dual GMO distributions.

THEOREM 2: *Suppose*

$$S_3 \le_{st} T_3, \quad S_i \ge_{st} T_i, \quad i = 1, 2. \tag{5}$$

Then,

$$C_{\boldsymbol{X}}(u,v) \le C_{\boldsymbol{Y}}(u,v), \quad \textit{for all } 0 \le u, v \le 1.$$

PROOF: Let F_j and G_j be distribution functions of S_j and T_j, respectively, $j = 1, 2, 3$. Denote, for $i = 1, 2$,

$$K_i(x) = F_i(x)F_3(x), \quad L_i(x) = G_i(x)G_3(x),$$

Then, by (4), we have

$$C_{\boldsymbol{X}}(u,v) = uv\min\left\{\frac{1}{F_3\big(K_1^{-1}(u)\big)}, \frac{1}{F_3\big(K_2^{-1}(v)\big)}\right\}$$

and

$$C_{\boldsymbol{Y}}(u,v) = uv\min\left\{\frac{1}{G_3\big(L_1^{-1}(u)\big)}, \frac{1}{G_3\big(L_2^{-1}(v)\big)}\right\}.$$

Since $S_3 \le_{st} T_3$ implies $F_3^{-1}(u) \le G_3^{-1}(u)$ for any $u \in [0,1]$ and $S_i \ge_{st} T_i$ implies $F_i(x) \le G_i(x)$ for all $x \ge 0$, $i = 1, 2$, we have

$$F_i\big(F_3^{-1}(u)\big) \le G_i\big(G_3^{-1}(u)\big), \quad \text{for any } u \in [0,1] \text{ and } i = 1, 2.$$

Equivalently, for any $u \in [0,1]$ and $i = 1, 2$,

$$K_i\big(F_3^{-1}(u)\big) = uF_i\big(F_3^{-1}(u)\big) \le uG_i\big(G_3^{-1}(u)\big) = L_i\big(G_3^{-1}(u)\big),$$

That is, just

$$F_3\big(K_i^{-1}(u)\big) \ge G_3\big(L_i^{-1}(u)\big), \qquad \text{for any } u \in [0,1] \text{ and } i = 1, 2.$$

This directly invokes

$$\min\left\{\frac{1}{F_3\big(K_1^{-1}(u)\big)}, \frac{1}{F_3\big(K_2^{-1}(v)\big)}\right\} \le \min\left\{\frac{1}{G_3\big(L_1^{-1}(u)\big)}, \frac{1}{G_3\big(L_2^{-1}(v)\big)}\right\}$$

and hence the desired result. ■

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As a direct consequence of Theorem 2, it is easy to obtain Corollary 3.

COROLLARY 3: *Under the assumption in* (5)*, if further the* S_j*'s and* T_j*'s satisfy*

$$\min\{S_i, S_3\} \geq_{st} \min\{T_i, T_3\}, \qquad i = 1, 2, \tag{6}$$

then, $F_{\boldsymbol{X}}(x_1, x_2) \leq F_{\boldsymbol{Y}}(x_1, x_2)$ *for all* x_1 *and* x_2*.*

PROOF: Since X_i and Y_i have the marginal distributions $K_i(x) = F_i(x)F_3(x)$ and $L_i(x) = G_i(x)G_3(x)$, $i = 1, 2$, the stochastic order $\min\{S_i, S_3\} \geq_{st} \min\{T_i, T_3\}$ implies $K_i(x) \leq L_i(x)$, $i = 1, 2$. Thus, by Sklar's theorem and Theorem 2, $\boldsymbol{X}$ has the joint distribution function

$$\begin{aligned} F_{\boldsymbol{X}}(x_1, x_2) &= C_{\boldsymbol{X}}(K_1(x_1), K_2(x_2)) \\ &\leq C_{\boldsymbol{X}}(L_1(x_1), L_2(x_2)) \\ &\leq C_{\boldsymbol{Y}}(L_1(x_1), L_2(x_2)) \\ &= F_{\boldsymbol{Y}}(x_1, x_2), \end{aligned}$$

which is just the joint distribution function of $\boldsymbol{Y}$. ■

The following example is concerned with one subclass of the dual GMO distributions.

Example 4 (Proportional reversed hazard rate): Suppose S_1, S_2, S_3 follow the proportional reversed hazard rate model with the baseline distribution $H(x)$ and proportional parameters $\lambda_1, \lambda_2, \lambda_3$, namely,

$$F_i(x) = H^{\lambda_i}(x), \quad i = 1, 2, 3.$$

Then, the copula in (4) reduces to

$$C_{\boldsymbol{X}}(u, v) = uv \min\left\{u^{\frac{-\lambda_3}{\lambda_1+\lambda_3}},\ v^{\frac{-\lambda_3}{\lambda_2+\lambda_3}}\right\}, \qquad 0 \leq u,\ v \leq 1.$$

This is just the survival copula of Marshall–Olkin-type distribution (see Muliere and Scarsini [9]). Incidentally, this copula is also the bivariate version of Li [5].

Assume further the other dual GMO random vector $\boldsymbol{Y}$, whose generating components T_1, T_2, T_3 also follow the proportional reversed hazard rate model with the same baseline distribution $H(x)$ and proportional parameters μ_1, μ_2, μ_3. In this context, the assumption in (5) becomes

$$\lambda_3 \leq \mu_3, \quad \lambda_i \geq \mu_i, \quad i = 1, 2,$$

and the assumption in (6) is just simplified as

$$\lambda_i + \lambda_3 \geq \mu_i + \mu_3, \quad i = 1, 2.$$

One may directly verify the inequality between the two copula functions and the uniform order between the two joint distribution functions.

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3. SOME APPLICATIONS

Before proceeding to some applications of the dual GMO distributions in survival analysis and actuarial science, let us recall one useful result.

A bivariate random vector $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the lower orthant (upper orthant) order (denoted as $\boldsymbol{X} \leq_{\text{lo}} (\leq_{\text{uo}}) \boldsymbol{Y}$) if $F_{\boldsymbol{X}}(x_1, x_2) \geq F_{\boldsymbol{Y}}(x_1, x_2)$ $\big(\bar{F}_{\boldsymbol{X}}(x_1, x_2) \leq \bar{F}_{\boldsymbol{Y}}(x_1, x_2)\big)$ for all x_1, x_2. Readers may refer to Shaked and Shanthikumar [11] for a comprehensive discussion on upper and lower orthant orders. The following characterization result is useful in presenting applications.

THEOREM 5 (Shaked and Shanthikumar [11]): *For bivariate random vectors* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$,

(i) $\boldsymbol{X} \leq_{uo} \boldsymbol{Y}$ *if and only if* $\mathsf{E}[\phi_1(X_1)\phi_2(X_2)] \leq \mathsf{E}[\phi_1(Y_1)\phi_2(Y_2)]$ *for any non-negative non-decreasing functions* ϕ_1 *and* ϕ_2, *and*

(ii) $\boldsymbol{X} \leq_{lo} \boldsymbol{Y}$ *if and only if* $\mathsf{E}[\psi_1(X_1)\psi_2(X_2)] \geq \mathsf{E}[\psi_1(Y_1)\psi_2(Y_2)]$ *for any non-negative non-increasing functions* ψ_1 *and* ψ_2.

3.1. Bivariate Frailty Model

In reliability and survival analysis, it is often of interest to study a pair of random lifetimes with association. Suppose $\boldsymbol{X} = (X_1, X_2)$ is the vector of life spans of two individuals under study, the bivariate frailty model assumes $\boldsymbol{X}$ has the joint survival function

$$\bar{F}(x_1, x_2) = \mathsf{E}\Big[\bar{H}_1^{V_1}(x_1)\bar{H}_2^{V_2}(x_2)\Big],$$

here $\bar{H}_i$ is the baseline survival function for X_i, and V_i is the corresponding unobservable *frailty* variable, $i = 1, 2$. Since the two individual lifetimes X_1 and X_2 are usually subject to some common factors, the frailty vector $\boldsymbol{V}$ is usually assumed to be of association. As is well known, there is no firm theoretical basis for selecting the distribution of the frailty vector in practical problems, thus, it is of critical importance to know how the variation of the distribution of the frailty vector affects that of the overall population vector. In the literature, the univariate and multivariate versions of the frailty model have been extensively investigated, readers may refer to Duchateau and Janssen [2] for more details.

In some practical situations, such as lifetimes of relatives (twins or parent-child) and components sharing a common stress, the association between V_1 and V_2 is unavoidable. Specifically, suppose that the individual with lifetime X_i is subject to a common frailty S_3 and the frailty S_i only affecting itself, $i = 1, 2$. Instead of being accumulated, frailties for ith individual are assumed to be dominated by the maximum $\max\{S_i, S_3\}$, $i = 1, 2$. Then, it holds that

$$\boldsymbol{V} = (V_1, V_2) = (\max\{S_1, S_3\}, \max\{S_2, S_3\})$$

has a dual GMO distribution.

Let $\boldsymbol{Y} = (Y_1, Y_2)$ be another pair of lifetimes having the same baseline survival functions $\bar{H}_1$, $\bar{H}_2$, and the corresponding frailty vector $\boldsymbol{W} = (W_1, W_2)$ has a dual GMO distribution with generating random variables (T_1, T_2, T_3). If $S_3 \leq_{st} T_3$, $S_i \geq_{st} T_i$ and $\max\{S_i, S_3\} \geq_{st} \max\{T_i, T_3\}$ for $i = 1, 2$, then the frailty vector $\boldsymbol{V}$ is smaller than $\boldsymbol{W}$ in the sense of the lower orthant order. Since the function $\bar{H}_i^{v_i}(x_i)$ is nonnegative and non-increasing in v_i for all $x_i \geq 0$, $i = 1, 2$, by Theorem 5 (ii), we may conclude that

$$\mathsf{E}\Big[\bar{H}_1^{V_1}(x_1)\bar{H}_2^{V_2}(x_2)\Big] \geq \mathsf{E}\Big[\bar{H}_1^{W_1}(x_1)\bar{H}_2^{W_2}(x_2)\Big],$$

and hence the overall population vector $\boldsymbol{X}$ with frailty vector $\boldsymbol{V}$ is larger than the one $\boldsymbol{Y}$ with frailty vector $\boldsymbol{W}$ in the sense of the upper orthant order. Namely, the smaller the frailty

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in the sense of the lower orthant order, the higher the probability of having simultaneously larger lifetimes.

3.2. Deductibles and Policy Limits

Consider an insurance policy covering a pair of random losses (X_1, X_2) subject to their respective risks with potential losses S_1 and S_2 and a common risk having potential loss S_3. In some situations, for example, the passenger bodily injury liability insurance, only the larger loss is covered by the policy, and thus the random losses (X_1, X_2) here takes the form of the dual GMO random vector. The retained loss of the insurance company is

$$g_1(\max\{S_1, S_3\}) + g_2(\max\{S_2, S_3\}),$$

where g_i, $i = 1, 2$, are determined by the form of coverage offered by the insurer. The following two forms of coverage are commonly used in practice.

- Upper limit is allocated to each risk loss: $g_i(x) = \min\{x, l_i\}$, here l_i is some predetermined constant, $i = 1, 2$.
- Deductible is designated to each risk loss: $g_i(x) = \max\{x - d_i, 0\}$, here d_i is some predetermined constant, $i = 1, 2$.

In the literature, there are many research results on the optimal policy limits and deductibles. See, for example, Cheung [1] and Hua and Cheung [4] and references therein.

Suppose the same insurance policy is applied to another pair of random losses (Y_1, Y_2), which has the dual GMO distribution generated by T_j, $j = 1, 2, 3$. One problem of practical interest for the insurance company is to compare the retained losses due to the coverage for these two pairs of random losses. In the framework of expected utility theory, this is equivalent to compare the expected utility of the retained losses

$$\mathsf{E}\Big[u\big(g_1(X_1) + g_2(X_2)\big)\Big] \quad \text{and} \quad \mathsf{E}\Big[u\big(g_1(Y_1) + g_2(Y_2)\big)\Big],$$

here u is the utility of the insurance company. Specifically, assume that the insurer has exponential utility, that is, $u(x) = 1 - \exp\{-\lambda x\}$ with some $\lambda > 0$. Then, the previous two expected utilities of the retained losses become

$$1 - \mathsf{E}\Big[\exp\big\{-\lambda g_1(X_1) - \lambda g_2(X_2)\big\}\Big], \quad 1 - \mathsf{E}\Big[\exp\big\{-\lambda g_1(Y_1) - \lambda g_2(Y_2)\big\}\Big].$$

Under the assumption (5) and (6), it may concludes that $(X_1, X_2) \leq_{lo} (Y_1, Y_2)$. Note that for both deductible and upper limit coverage, the function $\exp\{-\lambda g_i\}$ is non-negative and non-increasing functions for $i = 1, 2$. According to Theorem 5 (ii), we have

$$\mathsf{E}\Big[\exp\big\{-\lambda g_1(X_1) - \lambda g_2(X_2)\big\}\Big] \geq \mathsf{E}\Big[\exp\big\{-\lambda g_1(Y_1) - \lambda g_2(Y_2)\big\}\Big],$$

and hence

$$\mathsf{E}\Big[u\big(g_1(X_1) + g_2(X_2)\big)\Big] \leq \mathsf{E}\Big[u\big(g_1(Y_1) + g_2(Y_2)\big)\Big].$$

That is, the retained loss of the first risk pair (X_1, X_2) has smaller expected utility. Therefore, for the insurance company, it is riskier to take the second pair of random losses (Y_1, Y_2) whatever the coverage is.

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3.3. Exponential Premium Principle

The premium calculation is of prominent interest in its own right in actuarial science, since an inappropriate premiums quoted for the risks under coverage may lead to bankruptcy or even market failure. Under different situations and criteria, the insurance company may adopt different premium principles, and it is well-known that no premium principle uniformly outperforms others. By applying the *exponential premium* principle, the insurance company quotes the amount of premium for a pair of random risks $\boldsymbol{X} = (X_1, X_2)$

$$\frac{1}{r} \ln \mathsf{E}\big[\exp\{rX_1 + rX_2\}\big],$$

where $r \geq 0$ is the risk-averse coefficient. For more details on exponential premium principle and other premium principles, readers are referred to Goovaerts et al. [3] and the references therein.

Suppose that we wish to model two lifetimes that we suspect are subject to some common disaster that may induce an association between the lives. For simplicity, let us assume that S_1 and S_2 are two independent natural lifetimes, and there exists an independent random variable S_3 representing the time until some common disaster. Since both lives are subject to this disaster, the actual ages-at-death are represented as $X_1 = \min\{S_1, S_3\}$ and $X_2 = \min\{S_2, S_3\}$. Thus, the pair $\boldsymbol{X} = (X_1, X_2)$ is of GMO distribution. According to exponential premium principle, the premium for this pair of risk lifetimes is quoted as

$$\frac{1}{r} \ln \mathsf{E}\Big[\exp\big\{rX_1 + rX_2\big\}\Big] = \frac{1}{r} \ln \mathsf{E}\Big[\exp\big\{r \min\{S_1, S_3\} + r \min\{S_2, S_3\}\big\}\Big].$$

Likewise, the premium for another pair of lifetimes $\boldsymbol{Y} = (Y_1, Y_2)$ also having a GMO distribution with generating random variables (T_1, T_2, T_3) is quoted as

$$\frac{1}{r} \ln \mathsf{E}\Big[\exp\big\{rY_1 + rY_2\big\}\Big] = \frac{1}{r} \ln \mathsf{E}\Big[\exp\big\{r \min\{T_1, T_3\} + r \min\{T_2, T_3\}\big\}\Big].$$

Under assumptions $S_3 \leq_{st} T_3$, $S_i \geq_{st} T_i$ and $\min\{S_i, S_3\} \geq_{st} \min\{T_i, T_3\}$ for $i = 1, 2$, one may conclude $\boldsymbol{X} \geq_{uo} \boldsymbol{Y}$. Since the $\exp\{rx\}$ is nonnegative and non-decreasing in $x \geq 0$, from Theorem 5 (i), we have

$$\mathsf{E}\Big[\exp\big\{rX_1 + rX_2\big\}\Big] \geq \mathsf{E}\Big[\exp\big\{rY_1 + rY_2\big\}\Big].$$

And thus, the insurance company sets a higher premium to the pair of risk lifetimes $\boldsymbol{X}$.

Acknowledgments

The authors would like to thank the referee's valuable comments on the former version of this manuscript, which led us to the current presentation of this note. Supported by National Science Foundation of China (11171278).

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