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Fixed-point permanence under actions by finite quantum groups

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Abstract. Given an action by a finite quantum group $\mathbb{G}$ on a von Neumann algebra M, we prove that a number of familiar W^* properties are equivalent for M and the fixed-point algebra $M^{\mathbb{G}}$ (i.e., hold or not simultaneously for the two algebras); these include being hyperfinite, atomic, diffuse, and of type I, II, or III. Moreover, in all cases, the canonical central projections of M and $M^{\mathbb{G}}$ cutting out the summand with the respective property coincide. The result generalizes its classical-$\mathbb{G}$ analog due to Jones–Takesaki.

1 Introduction

Recall [12, Proposition 2.1.4] that for any action of a finite group $\mathbb{G}$ on a von Neumann algebra M, the fixed-point subalgebra $M^{\mathbb{G}}$ is or not of type I, II, III, or finite, along with M (i.e., these properties are equivalent for the two von Neumann algebras). The present note is aimed primarily at extending that result to its analog for actions of finite *quantum* groups $\mathbb{G}$ in the sense of [13, Definition 1.1.1]: unital C^*-algebras $A := C(\mathbb{G})$ equipped with a unital C^* morphism

$$A \longrightarrow A \otimes A \quad (\text{minimal } C^* \text{ tensor product [21, Definition IV.4.8]}),$$

coassociative in the obvious sense and such that

$$\overline{(A \otimes \mathbb{C}) \cdot \Delta A}^{\|\cdot\|} = A \otimes A = \overline{(\mathbb{C} \otimes A) \cdot \Delta A}^{\|\cdot\|}$$

(products of sets meaning spans of products). $\mathbb{G}$ is finite when $C(\mathbb{G})$ is finite-dimensional, and actions on von Neumann algebras are as in [23, Definition 2.2]: writing $L^\infty(\mathbb{G})$ for the von Neumann closure of $C(\mathbb{G})$ in the GNS representation attached to the *Haar state* [13, Theorem 1.2.1] (A, h), an action of $\mathbb{G}$ on M is a W^* morphism

$$M \xrightarrow{\rho} M \otimes L^\infty(\mathbb{G}) \quad (\textit{spatial } W^* \text{ tensor product [21, Definition IV.5.1]})$$

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such that

- $(\mathrm{id}\otimes\Delta)\circ\rho = (\rho\otimes\mathrm{id})\circ\rho$;
- and

$$\overline{(\mathbb{C}\otimes L^\infty(\mathbb{G}))\cdot\rho M}^{W^*} = M\otimes L^\infty(\mathbb{G}).$$

That in place, the quantum analog of the aforementioned [12, Proposition 2.1.4] reads the following.

Theorem 1.1. *Let M be a W^*-algebra equipped with an action by a finite quantum group $\mathbb{G}$. M and the fixed-point subalgebra $M^{\mathbb{G}}$ then have the same canonical finite hyperfinite, atomic, diffuse, and type-τ central projections for $\tau\in\{I,\ II,\ III\}$.*

References for the various properties mentioned in the statement are provided in Example 2.3, and we refer to [21, Section V.1] for type-decomposition theory.

Theorem 1.1 follows fairly easily from work done in [4] on *finite-index expectations* attached to quantum actions. A brief reminder on the various notions of finiteness introduced in [1, Définition 3.6] for *conditional expectations* $M\xrightarrow{E}N$ onto W^*-subalgebras (i.e., [2, Section II.6.10] σ-weakly continuous norm-1 idempotents) will help provide some context.

- E is *of finite index* if $E-\lambda$ is *completely positive* [2, Section II.6.9] for some $\lambda>0$:

  (1-1)
  $$E(x^*x)\geq\lambda x^*x,\quad \forall x\in M\otimes M_n \quad \text{for some fixed } \lambda>0 \text{ and arbitrary } n\in\mathbb{Z}_{>0};$$

  Equivalently [9, Theorem 1], it is enough to require (1-1) for $n=1$ alone, i.e., the positivity of $E-\lambda$ entails its complete positivity. In the language of [1, Définition 3.6], being of finite index is equivalent to being *weakly* of finite index.
- E is *strongly of finite index* if furthermore M, regarded as an M–N-*correspondence* [1, Définition 2.1(i)] via E, has a finite *orthonormal basis* ([1, Définition 1.6], [5, Section 1.1], and [15, Theorem 3.12]; *(complete) quasi-orthonormal system* in [19, Definitions 4.8 and 4.10 and Theorem 4.11]).

We will also substitute *Pimsner–Popa* or *PP* for the phrase *of finite index*, following [18, Section 2.1].

The proof of Theorem 1.1 follows a small detour meant to abstract away from the specifics of type decompositions and the like, proving that PP expectations preserve certain classes of von-Neumann-algebra properties of which being of type $\tau\in\{I,\ II,\ III\}$, or diffuse, or hyperfinite are examples.

Definition 1.2 Consider a W^*-algebra property $\mathcal{P}$, transferred to projections $p\in N$ by saying that p has (or is) $\mathcal{P}$ if pNp does (or is).

(1) $\mathcal{P}$ is *ideal* if for every von Neumann algebra N

(1-2) $\textsc{Proj}_{\mathcal{P}}(N) := \{\mathcal{P}\text{-projections } p\in N\}\subseteq \textsc{Proj}(N) := \{\text{all } N\text{-projections}\}$

is a (lattice-theoretic [3, Definition post Example 2.15]) *ideal*:

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- closed under taking suprema;
- and a *down-set* [3, Definition post Example 1.13]:

$$p \text{ has } \mathcal{P} \text{ and } q \leq p \Longrightarrow q \text{ has } \mathcal{P}.$$

(2) $\mathcal{P}$ is *completely* ideal if it is ideal and (1-2) is furthermore *complete*, i.e., closed under *arbitrary* suprema.

(3) $\mathcal{P}$ is instead (only) *Z-completely* ideal if it is ideal and closed under *centrally orthogonal* [21, pre Theorem V.1.8] sums (i.e., sums of projections whose respective *central supports* [21, pre Corollary IV.5.6] are orthogonal).

Theorem 2.5 below shows that completely ideal properties travel well along PP expectations, Corollary 2.6 specializes this to the properties listed in Theorem 1.1, and the latter follows. A version of Theorem 2.5 applicable to *strongly* PP expectations and Z-completely ideal properties is proven in Theorem 2.7, and might be of some independent interest given the context.

2 Permanence under sufficiently non-degenerate expectations

Recall the Introduction's finiteness expectation conditions.

Remark 2.1 One simple observation that it will be convenient to take for granted now and then is that being strongly PP is equivalent to being PP and M being finitely generated as a right N-module in the purely algebraic sense that

$$M = x_1 N + \cdots + x_s N \quad \text{for some} \quad x_i \in M.$$

One direction is obvious; as for the other ($\Leftarrow$), note that in general, an N–W^*-module X finitely generated as plain N-module has a finite orthonormal basis. One can induct on the number s of generators:

- $p_1 := \langle x_1 \mid x_1 \rangle \in N$ can be assumed a projection after polar-decomposing the original x_1 [15, Proposition 3.11];
- this then gives a projection

$$X \ni x \xmapsto{Q := |x_1\rangle\langle x_1|} x_1 \langle x_1 \mid x \rangle \in X$$

  in $\mathcal{L}_N(X)$;
- and the summand $(1-Q)X \leq X$ complementary to QX is generated by the (images of the) $s-1$ elements x_j, $2 \leq j \leq s$, propelling the induction.

A few simple preliminary remarks, useful enough to set out explicitly, but left as an exercise (in abstracting away from, say, [21, Theorem V.1.19]).

Lemma 2.2. *Let N be a von Neumann algebra and $\mathcal{P}$ a W^*-algebra property.*

(1) *If $\mathcal{P}$ is completely ideal the strong negation $\mathcal{P}'$ of $\mathcal{P}$ defined by*

$$N \text{ is } \mathcal{P}' \Longleftrightarrow \text{it has no nonzero } \mathcal{P}\text{-projections}$$

is also completely ideal, and $z_{N,\mathcal{P}'} = 1 - z_{N,\mathcal{P}}$.

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(2) *If $\mathcal{P}$ is completely ideal there is a largest $\mathcal{P}$-projection $z_{N,\mathcal{P}} \in N$, automatically central.*

(3) *If $\mathcal{P}$ is only Z-completely ideal the Z-strong negation*

$$N \text{ is } \mathcal{P}'_Z \iff \text{it has no nonzero central } \mathcal{P}\text{-projections}$$

of $\mathcal{P}$ is closed under arbitrary suprema.

(4) *If $\mathcal{P}$ is Z-completely ideal then there still is a largest central $\mathcal{P}$-projection $z = z_{N,\mathcal{P}} \in N$, and $1 - z$ is the largest $\mathcal{P}'_Z$-projection.* ∎

Examples 2.3

(1) The property of being type $\tau \in \{I,\ II,\ III\}$ is completely ideal, as follows from downward closure [6, Proposition 11 and Corollary 4 to Proposition 13 of Section I.6.8; Proposition 4 of Section I.8.3] and the central decomposition [21, Theorem V.1.19] $1 = z_I + z_{II} + z_{III}$.

(2) Recall (see [2, Section IV.2.2.1] and [20, Section 10.21]) that a W^*-algebra is *atomic* if every nonzero projection dominates a minimal nonzero projection. Atomicity is completely ideal, with closure under suprema following from the general discussion on the relative position of two projections in [21, discussion preceding Theorem V.1.41].

The strong negation of atomicity is *diffuseness* (see [20, Section 29.2] and [2, Theorem III.4.8.8]); it too is completely ideal, either as a direct simple exercise or by Lemma 2.2(1).

(3) Another completely ideal property is that of being *AFD* (short for *approximately finite-dimensional* [22, Definition XIV.2.3] and an alternative for *hyperfinite*, following [7, p. 175]). It means that arbitrary finite subsets of the von Neumann algebra in question (N, say) are arbitrarily approximable in the σ-strong* *topology* [2, Section I.3.1.6] by finite-dimensional von Neumann subalgebras, and is equivalent (see [8, Theorem 2 and Corollary 5] and [10, Section 6.4]) to the *injectivity* [2, Section IV.2.2] of N.

As to the fact that AFD-ness is completely ideal:

- Closure under projection cutting is noted in [2, Example IV.2.1.2(iii)].
- For AFD projections $p, q \in N$, the supremum $p \vee q$ is again AFD:
  - the already-mentioned relative-position analysis of [21, discussion preceding Theorem V.1.41] will afford a decomposition

$$(p \vee q)N(p \vee q) \cong N_0 \otimes M_2 \times \prod_{i=1}^{4} N_i$$

    with all N_i projection cuts of either pNp or qNq and hence AFD by the preceding point;
  - while tensoring with finite-dimensional W^*-algebras and finite products preserve the AFD property by [2, Corollary IV.2.1.5 and Example IV.2.1.2(ii)], respectively.
- Finally, closure under filtered inclusions of AFD subalgebras is self-evident from the very definition of approximate finite-dimensionality, hence closure under *arbitrary* suprema.

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(4) One example of a commonly-discussed property that of course is *not* completely ideal is finiteness: for infinite-dimensional H, the identity of $\mathcal{L}(H)$ is the supremum of finite-rank (hence finite) projections. Finiteness *is* ideal though [21, Theorem V.1.37], and in fact [21, Lemma V.1.18] Z-completely so.

(5) *Proper infinitude* [21, Definition V.1.15], on the other hand, is not ideal (let alone completely or Z-completely so) because it is not closed under projection-cutting: the same example $N := \mathcal{L}(H)$ above is properly infinite (i.e., its nonzero projections are all infinite), but pNp is finite for finite-rank p.

Note in passing that proper infinitude is precisely the Z-strong negation of finiteness in the sense of Lemma 2.2(4).

Remark 2.4 Lemma 2.2(4) is intended precisely as stated: for Z-completely ideal $\mathcal{P}$ $z_{N,\mathcal{P}}$ is by definition the largest *central* $\mathcal{P}$-projection in N, while its complement $z_{N,\mathcal{P}'_Z} = 1 - z_{N,\mathcal{P}}$ is the largest $\mathcal{P}'_Z$-projection period, central or not.

For illustration, consider Examples 2.3(4) and 2.3(5) above: in the central decomposition

$$N \ni 1 = z + z' := z_{\text{FINITE}} + z_{\text{PROPERLY INFINITE}}$$

of [21, Theorem V.1.19] the second central summand z' dominates every properly infinite projection in N.

Theorem 2.5. *Let $M \xrightarrow{E} N$ be a PP conditional expectation and $\mathcal{P}$ a completely ideal property that descends along such expectations. The maximal central projections $z_{N,\mathcal{P}}$ and $z_{M,\mathcal{P}}$ then coincide.*

Proof Under the PP assumption we also have [1, Théorème 3.5, Lemme 2.15, and Section 3.10] a PP expectation $\mathcal{L}_N(X_E) \xrightarrow{E_1} M$ from the von Neumann algebra [15, Proposition 3.10] of *adjointable* continuous N-module endomorphisms of the M–N-correspondence (hence also a right N-W^*-module) attached [1, Proposition 2.8] to the expectation E.

Note furthermore that X_E is *non-degenerate* [5, Section 1.1] as an N-W^*-module: the ideal of N generated by $\langle x \mid y\rangle$, $x, y \in X_E$ is (dense in) N.

(I) : $\mathcal{P}$**or its absence are equivalent for** N **and** $\mathcal{L}_N(X_E)$. This is true of N and $\mathcal{L}_N(X)$ for *any* non-degenerate N-W^*-module X: by the general structure [15, Theorem 3.12] we have

$$\mathcal{L}_N(X) \cong p\left(N \otimes \mathcal{L}(\ell^2(S))\right)p, \quad S \text{ a set and } p \in N \otimes \mathcal{L}(\ell^2(S)) \text{ a projection.}$$

Indeed, if $(x_s)_{s\in S}$ is an orthonormal basis for X (what [15, Theorem 3.12] provides) with corresponding projections $p_s := \langle x_s \mid x_s\rangle \in N$, then

(2-1)
$$X \cong \bigoplus_{s\in S} p_s N \quad \text{and} \quad \mathcal{L}_N(X) = p\left(N \otimes \mathcal{L}(\ell^2(S))\right)p \quad \text{with} \quad p := \operatorname{diag}(p_s,\ s \in S).$$

The operations of projection-cutting (i.e., $\bullet \mapsto p \bullet p$) and tensoring with a type-I factor both preserve completely ideal properties, transporting $\mathcal{P}$ from N to

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$\mathcal{L}_N(X)$. Conversely, if the latter has property $\mathcal{P}$ then so do its corners p_sNp_s and non-degeneracy means that $\bigvee_s p_s = 1 \in N$, hence the claim.

(II) : $\mathcal{P}$**lifts from** N **to** M. For it lifts from N to $\mathcal{L}_N(X_E)$ by (I), and then descends to M along E_1 by hypothesis.

(III) : **The strong negation** $\mathcal{P}'$ **descends along PP expectations.** Suppose N is *not* $\mathcal{P}'$, i.e., $z_{N,\mathcal{P}} \neq 0$. (II) applied to

$$z_{N,\mathcal{P}}Mz_{N,\mathcal{P}} \xrightarrow{E \text{ restricted}} z_{N,\mathcal{P}}N$$

then implies that the nonzero W^*-algebra $z_{N,\mathcal{P}}Mz_{N,\mathcal{P}}$ is $\mathcal{P}$, negating $\mathcal{P}'$ for M.

(IV) : **The strong negation** $\mathcal{P}'$ **lifts from** N to M. Consider the central decomposition $1 = z_{M,\mathcal{P}} + z_{M,\mathcal{P}'}$ of Lemma 2.2(1). We have [1, Section 3.9] a PP expectation

$$zM \xrightarrow{E_z} zN, \quad z := z_{M,\mathcal{P}}, \tag{2-2}$$

and the $\mathcal{P}'$ property for N (hence also for zN) entails it for zM by (II) applied to $\mathcal{P}'$ (completely ideal again by Lemma 2.2(1) and still descending along PP expectations by (III)). $z = z_{M,\mathcal{P}}$ thus vanishes, i.e., the target rephrased.

(V) : **Conclusion.** Applying (II) and (IV) to the expectations

$$z_{N,\mathcal{P}}Mz_{N,\mathcal{P}} \xrightarrow{E \text{ restricted}} z_{N,\mathcal{P}}N \quad \text{and}$$
$$z_{N,\mathcal{P}'}Mz_{N,\mathcal{P}'} \xrightarrow{E \text{ restricted}} z_{N,\mathcal{P}'}N$$

respectively, we obtain an orthogonal decomposition $1 = z_{N,\mathcal{P}} + z_{N,\mathcal{P}'}$ into a $\mathcal{P}$ and a $\mathcal{P}'$ summand. Now, $z := z_{N,\mathcal{P}}$ and $z' := z_{N,\mathcal{P}'}$ cannot have nonzero subprojections

$$p \leq z \quad \text{and} \quad p' \leq z' \quad \text{equivalent in } M,$$

for then $pMp \cong p'Mp'$ would be both $\mathcal{P}$ and non-$\mathcal{P}$. It follows [21, Lemma V.1.7] that z and z' are centrally orthogonal in M. Being themselves orthogonal and complementary, z and z' must also be central in M, so that $1 = z + z'$ is the canonical central decomposition of M attached to property $\mathcal{P}$. ∎

[18, Section 2.1] (essentially) claims, and [9, Propositions 2.1 and 2.2] imply Corollary 2.6 (give or take, minus the AFD property) for PP expectations.

Corollary 2.6. *If there is a PP conditional expectation $M \xrightarrow{E} N$, then the maximal central projections*

(a) *z_τ of respective types $\tau \in \{I,\ II,\ III\}$,*
(b) *z_a (atomic),*
(c) *z_d (diffuse),*
(d) *and z_{AFD} (AFD),*

for M and N coincide.

Proof Given that diffuseness and type *III* are the respective strong negations of atomicity and *semifiniteness* [2, Sections III.1.4.2 and III.1.4.7], the claims are all direct consequences of Theorem 2.5 once we recall that

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- semifiniteness [2, Corollary III.2.5.25(i)]
- being of type I [2, Theorem III.2.5.26] (or *discrete* [2, Section III.1.4.4]),
- atomicity [2, Theorem IV.2.2.3]
- and AFD-ness (obviously by [2, Proposition IV.2.1.4] and the equivalence [8, Theorem 2 and Corollary 5] AFD $\Leftrightarrow$ injective),

all descend along *arbitrary* (not just PP) conditional expectations. ∎

[11, Théorème 2.2(1)] shows that finiteness lifts along strongly PP expectations. [17, Property 1.1.2(iii)] makes the parallel claim under only the weak PP assumption, while [14, Section 2.1] amplifies that claim with the assertion that the finite/properly infinite central decompositions of M and N coincide in the presence of a weakly PP expectation. The weaker version assuming *strong* PP-ness (strengthening [11, Théorème 2.2(1)]) can be recovered by specializing Theorem 2.7 below to $\mathcal{P}$ = finiteness (per Example 2.3(4)).

Theorem 2.7. *Let $M \xrightarrow{E} N$ be a strongly PP conditional expectation and $\mathcal{P}$ a Z-completely ideal property that descends along such expectations. The central decompositions*

$$z_{N,\mathcal{P}} + z_{N,\mathcal{P}'_Z} = 1 = z_{M,\mathcal{P}} + z_{M,\mathcal{P}'_Z}$$

of Lemma 2.2(4) coincide.

Proof The proof plan roughly follows that of Theorem 2.5, with appropriate modifications.

(I) : **$\mathcal{P}$ is equivalent for N and $\mathcal{L}_N(X_E)$.** Precisely as in part (I) of Theorem 2.5, with the S of (2-1) finite this time around.

Section (II) of the proof of Theorem 2.5 now also applies to yield its respective analog.

(II) : **$\mathcal{P}$ lifts from N to M.**

The counterpart to section (IV) of the earlier proof, on the other hand, is easily verified directly.

(III) **: The Z-strong negation $\mathcal{P}'_Z$ lifts from N to M.** If M were to contain a nonzero central $\mathcal{P}$-projection z, the induced expectation (2-2) would ensure by the descent assumption that the nonzero quotient $N \to zN$ of N is $\mathcal{P}$. This amounts to a nonzero central $\mathcal{P}$-projection for N, contradicting $\mathcal{P}'_Z$.

(IV) **: Conclusion.** We now have a decomposition $1 = z_{N,\mathcal{P}} + z_{N,\mathcal{P}'_Z}$ in M, and we will be done as soon as we argue that $z_{N,\mathcal{P}'_Z} \in M$ is central.

Observe first that liftability (step (III)) applied in the context of the (still strong PP) expectation

$$z_{N,\mathcal{P}'_Z} M z_{N,\mathcal{P}'_Z} \longrightarrow\!\!\!\!\!\rightarrow N z_{N,\mathcal{P}'_Z} \tag{2-3}$$

induced by E via [1, Section 3.9] in particular implies that

$$z_{M,\mathcal{P}'_Z} \geq z_{N,\mathcal{P}'_Z} \quad \left(\Longleftrightarrow z_{M,\mathcal{P}} = 1 - z_{M,\mathcal{P}'_Z} \leq 1 - z_{N,\mathcal{P}'_Z} = z_{N,\mathcal{P}}\right);$$

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what we have to rule out is the possibility that this domination be strict. To that end, consider the $\mathcal{P}$ version

$$z_{N,\mathcal{P}} M z_{N,\mathcal{P}} \longrightarrow\!\!\!\!\!\rightarrow N z_{N,\mathcal{P}}$$

of (2-3) and conclude via (II) that the central $\mathcal{P}$-projection $z_{M,\mathcal{P}} \leq z_{N,\mathcal{P}}$ of $z_{N,\mathcal{P}} M z_{N,\mathcal{P}}$ cannot be *strictly* smaller than the unit $z_{N,\mathcal{P}} \in z_{N,\mathcal{P}} M z_{N,\mathcal{P}}$. ∎

Proof of Theorem 1.1 Consider the *spectral projections*

$$M \xrightarrow{E_\alpha} M_\alpha, \quad \alpha \in \operatorname{Irr}\mathbb{G}$$

onto the respective *isotypic subspaces* ([16, Theorem 1.5] or [4, Section 2.2]). In particular, we have a conditional expectation

$$M \xrightarrow{E:=E_{\mathrm{TRIV}}} M_{\mathrm{TRIV}} = M^{\mathbb{G}}.$$

The conclusion is a direct application of Corollary 2.6 for all but finiteness and [9, Proposition 2.1] for the latter, as soon as we verify that E is PP. Now, [4, Lemma 2.5] proves a "local" version of the desired result, valid for arbitrary compact quantum $\mathbb{G}$: for every $\alpha \in \operatorname{Irr}\mathbb{G}$ there is some $\lambda_\alpha > 0$ with

$$E(x^*x) \geq \lambda_\alpha E_\alpha(x)^* E_\alpha(x) \text{ matricially}, \quad \forall x \in M. \tag{2-4}$$

Assuming (as we are) that $\mathbb{G}$ is finite and hence $|\operatorname{Irr}(\mathbb{G})| < \infty$, for every $x \in M$, we have

$$\begin{aligned} x^*x &= \left(\sum_\alpha E_\alpha(x)^*\right)\left(\sum_\alpha E_\alpha(x)\right) \\ &\leq |\operatorname{Irr}(\mathbb{G})|^2 \sum_\alpha E_\alpha(x)^* E_\alpha(x) \quad \text{by } \textit{Cauchy Schwartz } [2\ \S\text{I.1.1.2}] \\ &\leq |\operatorname{Irr}(\mathbb{G})|^2 \left(\max_\alpha \frac{1}{\lambda_\alpha}\right) E(x^*x) \quad \text{by (2-4)}, \end{aligned}$$

all valid matricially. (1-1) thus holds for

$$\lambda := \frac{1}{|\operatorname{Irr}\mathbb{G}|^2} \min_\alpha \lambda_\alpha,$$

and we are done. ∎

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