Proc. Camb. Phil. Soc. (1971), **69**, 297
PCPS 69–28
Printed in Great Britain

Summability methods which include the Riesz typical means. II

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(*Received* 15 *December* 1969)

By making use of a convergence-factor theorem of Bosanquet (3), Cooke ((4), Theorem I) gave conditions for a regular sequence-to-sequence summability matrix B to be at least as strong as Cesàro summability (C, κ) $(\kappa > 0)$, namely:

THEOREM C. *Let* $\kappa > 0$. *In order that the* T*-matrix* $B = (b_{\rho\mu})$ *shall satisfy* $B \supseteq (C, \kappa)$ *it is necessary and sufficient that*

$$|b_{\rho\mu}| \leqslant H_\rho(\mu+1)^{-\kappa}, \tag{1}$$

$$\sup_\rho \sum_\mu \mu^\kappa |\Delta_\mu^\kappa b_{\rho\mu}| < \infty. \tag{2}$$

If $0 < \kappa \leqslant 1$ *then* (2) *alone is necessary and sufficient.*

The case $0 < \kappa \leqslant 1$ is due to Lorentz (5), and in this case (1) (while still necessary) is implied by (2) and the fact that B is a T-matrix, and can therefore be omitted. Generalizations of Lorentz's result have been given in (8), (10), (11). One possible generalization of Theorem C, for $\kappa > 1$, would be to replace (C, κ) by (C, λ, κ), the generalized Cesàro means of type $\lambda = \{\lambda_n\}$ (where $0 \leqslant \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$) and order κ, since Borwein (1) has shown that (C, λ, κ) is equivalent to (C, κ) when $\lambda_n = n$ (see also (2) for definitions and further properties). However, the conditions in Theorem C appear in a relatively simple form, mainly because the inverse of the (C, κ) matrix can be explicitly written down, and we can do this readily for (C, λ, κ) only when κ is an integer. Moreover, if (continuing the work of (11)) the ultimate objective is to obtain necessary and sufficient conditions for B to be at least as strong as Riesz summability (R, λ, κ), then the equivalence of (C, λ, κ) and (R, λ, κ) for arbitrary λ and for $\kappa > 1$ is known so far only for integral κ (Russell (6), Meir (7)); i.e.

$$(C, \lambda, p) \sim (R, \lambda, p) \quad \text{for } \lambda \text{ arbitrary}, \quad p = 0, 1, 2, \ldots. \tag{3}$$

Consequently we shall give here two theorems for $B \supseteq (C, \lambda, p)$ (one for sequence transformations and one for series transformations). For $\lambda_n = n$ these contain the case of Theorem C in which $\kappa = p$, a non-negative integer; for $p = 1$ they contain results of Maddox ((6), Theorems 1, 2).

Let $\bar{A} = (\bar{a}_{n\nu})$ and $A = (a_{n\nu})$ denote the (C, λ, p) γ-matrix and T-matrix respectively; thus

$$\bar{a}_{n\nu} = (1-\lambda_\nu/\lambda_{n+1})\ldots(1-\lambda_\nu/\lambda_{n+p}) \quad (0 \leqslant \nu \leqslant n), \quad \bar{a}_{n\nu} = 0 \quad (\nu > n),$$

$$a_{n\nu} = \Delta_\nu \bar{a}_{n\nu} \equiv \bar{a}_{n\nu} - \bar{a}_{n,\nu+1},$$

and the (C, λ, p)-mean of a series Σc_ν, with partial sums s_ν, is

$$\gamma_n^p = \sum_{\nu=0}^{n} \bar{a}_{n\nu} c_\nu = \sum_{\nu=0}^{n} a_{n\nu} s_\nu.$$

https://doi.org/10.1017/S0305004100046661 Published online by Cambridge University Press

The inverses $A'' = (a''_{\mu k})$ and $A' = (a'_{\mu k})$ of $\bar{A}$ and A respectively, are given by (see (12), (26) and (27))

$$\left.\begin{aligned} a''_{\mu k} &= (-1)^{p+1}(\lambda_{k+p+1}-\lambda_k)\lambda_{k+1}\ldots\lambda_{k+p}/\beta_{k\mu} \quad (0 \leqslant k \leqslant \mu \leqslant k+p+1), \\ a''_{\mu k} &= 0 \quad \textit{otherwise}, \quad \textit{where} \quad \beta_{k\mu} = \prod_{j=k}^{k+p+1}{}' (\lambda_\mu - \lambda_j); \end{aligned}\right\} \tag{4}$$

$$a'_{\mu k} = \sum_{\nu=0}^{\mu} a''_{\nu k} \quad (0 \leqslant k \leqslant \mu \leqslant k+p), \quad a'_{\mu k} = 0 \quad \text{otherwise}. \tag{5}$$

Π' in (4) indicates that the zero factor corresponding to $j = \mu$ is to be omitted. We denote throughout $\Lambda_n \equiv \lambda_{n+1}/(\lambda_{n+1}-\lambda_n)$, and when we write $B \supseteq A$ we require $B\text{-lim}\, s_n = A\text{-lim}\, s_n$ whenever the right-hand side exists.

THEOREM 1. *Let p be a non-negative integer, and when $p > 1$ assume $\Lambda_{n-1} = O(\Lambda_n)$. In order that a sequence-to-sequence or sequence-to-function method $B = (b_{\rho\mu})$ shall be not weaker than (C, λ, p) it is necessary and sufficient that*

$$\text{(i)} \ \exists \lim_\rho b_{\rho\mu} = \beta_\mu \quad (\mu = 0, 1, \ldots), \quad \text{(ii)} \ \exists \lim_\rho \sum_\mu b_{\rho\mu} = \beta, \tag{6}$$

$$|b_{\rho\mu}| \leqslant H_\rho \Lambda_\mu^{-p}, \tag{7}$$

$$\sup_\rho \sum_k |t_{\rho k}| < \infty, \tag{8}$$

where

$$t_{\rho k} = \sum_{\mu=k}^{k+p} b_{\rho\mu} a'_{\mu k} \tag{9}$$

and $a'_{\mu k}$ is given by (4) *and* (5). *In order that $B \supseteq (C, \lambda, p)$, conditions* (6), (7), (8) *are necessary and sufficient, with $\beta_\mu = 0$, $\beta = 1$. If $p = 0$ or $p = 1$ then* (7) *is implied by the other conditions, and may be omitted.*

Proof. Necessity. Since B is not weaker than the regular method (C, λ, p), B must be conservative, so that (6) is necessary (with $\beta_\mu = 0$, $\beta = 1$ if consistency is also required). For the other conditions we appeal to a convergence-factor theorem given in (12), Theorem 2; inserting the extra parameter ρ, this shows that (under the stated hypotheses on p and λ) in order that the series $\sum_\nu b_{\rho\nu} s_\nu$ should converge (for each ρ) whenever the (C, λ, p)-transform $\{\gamma_k^p\}$ of $\{s_\nu\}$ converges, it is necessary and sufficient that (7) should hold, and that

$$\sum_k |t_{\rho k}| \leqslant M_\rho; \tag{10}$$

and then ((12), Lemma 1, (20))

$$\xi_\rho \equiv \sum_\nu b_{\rho\nu} s_\nu = \sum_k t_{\rho k} \gamma_k^p. \tag{11}$$

Consequently, in order that $\lim \xi_\rho$ should exist whenever $\lim \gamma_k^p$ exists, (8) is necessary.

Sufficiency. In addition to (6), we assume (7) and (8) (which imply (10) and (11)) and we have to show that $\gamma_k^p \to s$ implies $\xi_\rho \to s'$; we must therefore show that T is conservative. But this follows since, in the first place, (8) gives $\|T\| < \infty$; secondly, (6) (i) and (9) give

$$\tau_k \equiv \lim_\rho t_{\rho k} = \sum_{\mu=k}^{k+p} \beta_\mu a'_{\mu k} \quad (k = 0, 1, \ldots); \tag{12}$$

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thirdly, by taking $s_\nu = 1$ for every ν, we get† $\gamma_k^p = 1$ for every k, and (6) (ii) and (11) then give

$$\tau \equiv \lim_\rho \sum_k t_{\rho k} = \beta. \tag{13}$$

Also, if $\beta_\mu = 0$, $\beta = 1$ then, by (12) and (13), $\tau_k = 0$, $\tau = 1$, so that T is regular and hence $B \supseteq (C, \lambda, p)$.

Finally, (6) (ii) obviously implies (7) when $p = 0$; and (see (12), Remark 4) (6) (ii) and (8) imply (7) in the case $p = 1$; thus (7) can be omitted in these cases.

THEOREM 2. *Let p be a non-negative integer; for $p > 1$ assume $\Lambda_{n-1} = O(\Lambda_n)$. In order that a series-to-sequence or series-to-function method $\bar{B} = (\bar{b}_{\rho\mu})$ shall be not weaker than (C, λ, p) it is necessary and sufficient that*

$$\exists \lim_\rho \bar{b}_{\rho\mu} = \bar{\beta}_\mu \quad (\mu = 0, 1, \ldots), \tag{14}$$

$$|\bar{b}_{\rho\mu}| \leqslant H_\rho \Lambda_\mu^{-p}, \tag{15}$$

$$\sup_\rho \sum_k |t_{\rho k}| < \infty, \tag{16}$$

where

$$t_{\rho k} = \sum_{\mu=k}^{k+p+1} \bar{b}_{\rho\mu} a''_{\mu k} \quad \textit{and} \quad a''_{\mu k} \textit{ is given by (4).} \tag{17}$$

In order that $\bar{B} \supseteq (C, \lambda, p)$, conditions (14), (15), (16) are necessary and sufficient, with $\bar{\beta}_\mu = 1$. If $p = 0$, (15) may be omitted.

Proof. We shall deduce the result from Theorem 1. Let $b_{\rho\mu} = \Delta_\mu \bar{b}_{\rho\mu}$ and note that (by (5) and partial summation) the $t_{\rho k}$ defined by (9) and (17) are identical, so that (16) $\Leftrightarrow$ (8) $\Rightarrow$ (10). The case $p = 0$ is well known, so assume $p \geqslant 1$.

Sufficiency. Assume first that (10) and (15) hold. Then (15) implies (7), provided that $\Lambda_{n-1} = O(\Lambda_n)$—however, when $p = 1$ we can dispense with the restriction on λ and still obtain (7) from (10) and (15) (see (12), Remark 4). Thus (7) and (10) hold, and so, by (12), Theorem 2, $\sum_\nu b_{\rho\nu} s_\nu$ exists whenever $\{s_\nu\}$ is (C, λ, p)-limitable; in particular, the sequence $s_\nu = 1$ (all ν) is (C, λ, p)-limitable, and so $\sum_\nu b_{\rho\nu} = \bar{b}_{\rho 0} - \lim_\mu \bar{b}_{\rho\mu}$ exists. Also (as in the proof of Theorem 1) (11) follows.

In the case‡ $\Lambda_n = O(1)$, (C, λ, p) is equivalent to convergence ((9), Corollary 3B) and so we have to show that $\bar{B}$ is conservative. Now in order that $\sum_\nu b_{\rho\nu} s_\nu$ should exist for each ρ, and be bounded whenever $\{s_\nu\}$ is bounded, it is necessary and sufficient that

$$\sup_\rho \sum_\mu |b_{\rho\mu}| < \infty; \tag{18}$$

on the other hand, since in this case every bounded $\{\gamma_k^p\}$ is the transform of some bounded $\{s_\nu\}$ (and conversely), we see that if $\Lambda_n = O(1)$ and (11) holds, then (16) and (18) are equivalent. It now follows that (15) and (16) imply (18), and this, together with (14), shows that $\bar{B}$ is conservative.

In the case $\Lambda_n \neq O(1)$, (15) shows that $\bar{b}_{\rho\mu}$ (for fixed ρ) has 0 as a limit point, whence $\lim_\mu \bar{b}_{\rho\mu} = 0$ and $\sum_\nu b_{\rho\nu} = \bar{b}_{\rho 0}$. Thus if (14) also holds, then (6) now follows, with $\beta_\mu = \Delta \bar{\beta}_\mu$,

† Provided $\lambda_0 = 0$; and we can assume this without loss of generality.

‡ I had originally confined myself in Theorem 2 to the most important case $\Lambda_n \neq O(1)$. However, the referee kindly pointed out that this restriction was not needed, and also supplied a proof, similar to that incorporated here, for the case $\Lambda_n = O(1)$.

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$\beta = \bar{\beta}_0$. We may now assume without loss of generality that $\{s_\nu\}$ is (C, λ, p)-limitable to zero and then, by (15) and the limitation theorem for (C, λ, p)-means ((9), Corollary 3A), $\bar{b}_{\rho N} s_N = o(1)$ $(N \to \infty)$; hence, by partial summation, with $s_\nu = c_0 + \ldots + c_\nu$, $\sum_\nu \bar{b}_{\rho\nu} c_\nu$ exists and is equal to $\sum_\nu b_{\rho\nu} s_\nu$. Finally, assuming (16) and appealing to Theorem 1, we see that B, and hence $\bar{B}$, is not weaker than (C, λ, p); also, if $\bar{\beta}_\mu = 1$ then $\beta_\mu = 0$, $\beta = 1$, and B and $\bar{B}$ are consistent with (C, λ, p).

Necessity. If $\bar{B}$ is not weaker than (C, λ, p), then $\bar{B}$ must be conservative and (14) and (18) are necessary (with $\bar{\beta}_\mu = 1$ for consistency); and ((12), Theorem 3) shows that (10) and (15) (and hence (11)) are necessary. In the case $\Lambda_n = O(1)$, (11) and (18) then imply (16). When $\Lambda_n \neq O(1)$, the whole of the argument for the sufficiency part of that case may now be used except for the final sentence; it then follows that B is not weaker than (C, λ, p) and hence, by Theorem 1, (8) (same as (16)) is necessary.

Remark. In (11), Theorems 1 and 2, and Remark (i), conditions are given for methods B or $\bar{B}$ to include (R, λ, κ); the conditions are necessary for all $\kappa > 0$, and sufficient for $0 < \kappa \leqslant 1$. We may now show that for $\kappa = 2, 3, \ldots$ the conditions of those theorems (with (11), Theorem 1 augmented by (11), (27) and the restriction

$$\Lambda_{n-1} = O(\Lambda_n)$$

imposed for $\kappa > 1$) respectively imply the conditions of Theorems 1 and 2 above (we have only to note that ((11), (8)) $\Rightarrow$ ((11), (5)) and that ((11), (5) (with $\kappa = p$)) and $T = BA' = \bar{B}A''$ together imply $\|T\| < \infty$, by (12), Lemma 2). Consequently, by appealing to (3) above, the sufficiency parts of (11), Theorems 1 and 2, are then extended to cover the cases $\kappa = 2, 3, \ldots$.

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