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ARTICLE

Hypergraph independence polynomials with a zero close to the origin

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Abstract

For each uniformity $k \geq 3$, we construct k uniform linear hypergraphs G with arbitrarily large maximum degree Δ whose independence polynomial Z_G has a zero λ with $|\lambda| = O\left(\frac{\log \Delta}{\Delta}\right)$. This disproves a recent conjecture of Galvin, McKinley, Perkins, Sarantis, and Tetali.

Keywords: Independence polynomial; hypergraph; zero-free region

2020 MSC Codes: Primary: 05C31; Secondary: 05C69, 82B20

1. Introduction

A hypergraph $G = (V, E)$ is a set of vertices V together with a set of edges $E \subset 2^V$. A hypergraph is k-uniform if every edge has size k. The degree of a vertex $v \in V$, denoted by $d(v)$, is the number of edges it appears in; in a hypergraph with maximum degree Δ, each vertex appears in at most Δ edges.

An independent set in G is a set of vertices $I \subset V$ such that I contains no edge. Let $\mathcal{I}(G)$ denote the family of all independent sets in G. The independence polynomial of G is defined by

$$Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.$$

This polynomial plays an important role in mathematics, physics and computer science [1, 3, 4, 7–10]. A key property for understanding this polynomial is the largest radius of a disk-shaped zero-free region(ZFR), a region in $\mathbb{C}$ where Z_G has no zero. We refer the reader to the introduction of [2] for a survey of how knowledge of the zeros of Z_G can lead to interesting results about independent set.

When G is a graph with maximum degree Δ, the ZFRs for Z_G are well-understood [6, 8]. Specifically, Shearer [8] showed that Z_G has no zero inside the disk $|\lambda| < \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}}$, and this bound is the best possible.

In a recent paper [2], Galvin, McKinley, Perkins, Sarantis and Tetali studied the zeros of Z_G when G is a general hypergraph with given maximum degree Δ. They showed that Z_G has no zero inside the disk $|\lambda| < \frac{\Delta^{\Delta}}{(\Delta+1)^{(\Delta+1)}}$. Furthermore, for each uniformity $k \geq 2$, they constructed a family of k-uniform hypergraphs with arbitrarily large maximum degree Δ such that Z_G has a zero λ with $|\lambda| < O_k\left(\frac{\log \Delta}{\Delta}\right)$, thereby showing that their bound is tight up to a logarithmic factor if we place no additional assumption on G.

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A hypergraph is linear if each pair of edges intersect in at most one vertex. The aforementioned constructions in [2, Section 4] are far from linear, since they contain edges that intersect in $(k-1)$ vertices. In [2, Conjecture 3], Galvin, McKinley, Perkins, Sarantis and Tetali conjectured that their lower bound on the maximum radius of the zero-free disk for Z_G can be improved under the additional assumption that G is linear.

Conjecture 1.1. *For each $k \geq 2$, there exists a constant $C_k > 0$ such that the following is true. If G is a k-uniform, linear hypergraph with maximum degree Δ and if*

$$|\lambda| \leq C_k \Delta^{-\frac{1}{k-1}}$$

then $Z_G(\lambda) \neq 0$.

This conjecture is motivated by results on asymptotic enumeration in [5]. Galvin, McKinley, Perkins, Sarantis and Tetali verified this conjecture when G is a hypertree [2, Theorem 4],

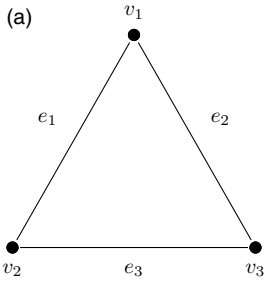
In this note, we disprove Conjecture 1.1 in any uniformity $k \geq 3$. Our counterexample shows that the radius of the disk-shaped ZFR $|\lambda| < \frac{\Delta^\Delta}{(\Delta+1)^{(\Delta+1)}}$ is tight up to a logarithmic factor even if we assume that G is a k-uniform linear hypergraph.

Theorem 1.2. *For each uniformity $k \geq 3$ and $\Delta > 100k^2$, there exists a k-uniform linear hypergraph G with maximum degree Δ, such that Z_G has a negative real zero λ with $\lambda \in [-\frac{6k \log \Delta}{\Delta}, 0]$.*

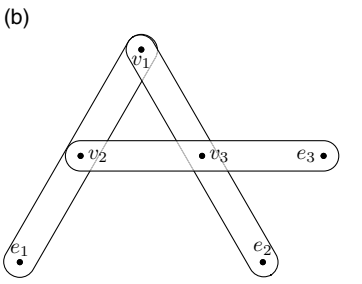
2. The counterexample

We begin by describing a general construction.

Definition 2.1. *Let $G = (V, E)$ be a hypergraph. We define S_G as the hypergraph whose vertex set is $V \sqcup E$, and whose edge set is $\{e \cup \{e\} : e \in E\}$.*



A graph G.



The 3-uniform hypergraph S_G.

We begin by showing a basic property of S_G.

Proposition 2.2. *If $G = (V, E)$ is a $(k-1)$ uniform linear hypergraph with maximum degree Δ, then S_G is a k uniform linear hypergraph with maximum degree Δ.*

Proof. Each edge in S_G has the form $e \cup \{e\}$ for some $e \in E$, and $|e \cup \{e\}| = |e| + 1 = k$. So S_G is k uniform.

For any pair of distinct edges $e_1 \cup \{e_1\}$ and $e_2 \cup \{e_2\}$ in S_G, we have

$$|(e_1 \cup \{e_1\}) \cap (e_2 \cup \{e_2\})| = |e_1 \cap e_2| \leq 1.$$

Thus S_G is linear.

Finally, each vertex in V has the same degree in G and S_G, while each element in E has degree 1 in S_G. Therefore, the maximum degree of S_G is the same as the maximum degree of G. □

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In the next two lemmas, we give an explicit formula for the independence polynomial Z_{S_G} of S_G and prove that it has a zero close to the origin whenever G satisfies a mild expansion property.

Lemma 2.3. *Let $G = (V, E)$ be a hypergraph. For each set of vertices $S \subset V$, let $E(S)$ denote the edges of G with at least one vertex in S, and write $e(S) = |E(S)|$. Then we have*

$$Z_{S_G}(\lambda) = \sum_{S \subset V} \lambda^{|V|-|S|}(1+\lambda)^{e(S)}.$$

Proof. Classifying the independence sets of S_G based on their intersections with $V \subset V(S_G)$, we have

$$Z_{S_G}(\lambda) = \sum_{S \subset V} \sum_{I \in \mathcal{I}(S_G) : I \cap V = V \setminus S} \lambda^{|I|}.$$

A set of vertices $I \subset V \sqcup E$ with $I \cap V = V \setminus S$ is independent in S_G if and only if $J := I \cap E$ is contained in $E(S)$. So we have

$$\sum_{I \in \mathcal{I}(S_G) : I \cap V = V \setminus S} \lambda^{|I|} = \lambda^{|V \setminus S|} \sum_{J \subset E(S)} \lambda^{|J|} = \lambda^{|V \setminus S|}(1+\lambda)^{e(S)}$$

and the lemma follows. □

Lemma 2.4. *Let $G = (V, E)$ be a hypergraph with $n \geq 3$ vertices. Assume that n is odd. Furthermore, assume that for some $\alpha \in [3 \log n, n]$, we have $e(S) \geq \alpha\, |S|$ for any $S \subset V$. Then Z_{S_G} has a negative real zero in the interval*

$$\left[-\frac{3 \log n}{\alpha}, 0 \right].$$

Proof. Set $\lambda_0 = -\frac{3 \log n}{\alpha}$. As $Z_{S_G}(0) = 1$, it suffices to show that $Z_{S_G}(\lambda_0) < 0$.

By Lemma 2.3, we have the identity

$$Z_{S_G}(\lambda_0) = \sum_{S \subset V} \lambda_0^{|V|-|S|}(1+\lambda_0)^{e(S)}.$$

We isolate the term with $S = \emptyset$ and obtain

$$Z_{S_G}(\lambda_0) = \lambda_0^{|V|} \left(1 + \sum_{S \subset V, S \neq \emptyset} \lambda_0^{-|S|}(1+\lambda_0)^{e(S)} \right)$$

Since $0 \leq 1 + \lambda_0 \leq e^{\lambda_0}$, for each $S \subset V$ we can estimate

$$\left| \lambda_0^{-|S|}(1+\lambda_0)^{e(S)} \right| \leq |\lambda_0|^{-|S|}\, e^{\lambda_0 e(S)} \leq \alpha^{|S|} e^{-3 \log n \cdot |S|}.$$

Thus we have

$$\left| \sum_{S \subset V, S \neq \emptyset} \lambda_0^{-|S|}(1+\lambda_0)^{e(S)} \right| \leq \sum_{k=1}^{n} \binom{n}{k} \alpha^k e^{-3 \log n \cdot k} \leq \sum_{k=1}^{n} \frac{1}{k!} \left(\frac{\alpha}{n^2} \right)^k.$$

where the second inequality uses the estimate $\binom{n}{k} \leq \frac{n^k}{k!}$.

We assume that $\alpha \leq n$, so $\left(\frac{\alpha}{n^2} \right)^k < \frac{1}{n}$ for each $k \geq 1$. This leads to

$$\left| \sum_{S \subset V, S \neq \emptyset} \lambda_0^{-|S|}(1+\lambda_0)^{e(S)} \right| \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k!} < \frac{e}{n} < 1.$$

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Thus we conclude that

$$1+\sum_{S\subset V, S\neq\emptyset}\lambda_0^{-|S|}(1+\lambda_0)^{e(S)}>0$$

so $Z_{S_G}(\lambda_0)<0$, as desired. □

Our main theorem is an easy corollary of this result. Indeed, for any $(k-1)$ uniform hypergraph G we can show that $e(S)\geq\frac{\delta(G)|S|}{k-1}$ for any $S\subset V$. So when $\delta(G)\geq\Delta(G)-1$, one can take $\alpha=\frac{\Delta(G)-1}{k-1}$ in Lemma 2.4. We give an explicit construction of such a hypergraph.

Lemma 2.5. *For any uniformity $k\geq 2$ and $\Delta\geq k$, there exists a k uniform, Δ regular linear hypergraph $H_{k,\Delta}$ on at most $2k\Delta$ vertices.*

Proof. We take a prime p in $[\Delta,2\Delta]$, which exists by Chebyshev's theorem. Let $H_{k,\Delta}$ be the hypergraph on the vertex set $V=[k]\times\mathbb{Z}_p$ with an edge $\{(i,a+id):i\in[k]\}$ for each pair $(a,d)\in\mathbb{Z}_p\times[\Delta]$. The number of vertices in $H_{k,\Delta}$ is $|V|=kp\leq 2\Delta k$.

Any vertex $(i,x)\in V$ is contained precisely in the edges corresponding to $(a,d)\in\mathbb{Z}_p\times[\Delta]$ with $a\equiv x-id \pmod p$. As there is exactly one $a\in\mathbb{Z}_p$ corresponding to each $d\in[\Delta]$, $H_{k,\Delta}$ is Δ regular.

The size of the intersection between two distinct edges corresponding to (a,d) and (a',d') is the number of solutions $i\in[k]$ to the linear congruence equation $a+id\equiv a'+id' \pmod p$. As $k\leq\Delta\leq p$, there is at most one solution $i\in[k]$ to this linear congruence equation, so every pair of hyperedges in $H_{k,\Delta}$ intersect in at most one vertex. Therefore, $H_{k,\Delta}$ is a linear hypergraph. □

Proof of Theorem 1.2. Let $H=H_{(k-1),\Delta}$ be the $(k-1)$ uniform linear hypergraph constructed in Lemma 2.5. If H has an even number of vertices, we remove an arbitrary vertex v of H together with any edge containing the vertex. Thus, we obtain a $(k-1)$ uniform linear hypergraph H with an odd number of vertices, maximum degree Δ, and minimum degree at least $(\Delta-1)$. By Proposition 2.2, $G=S_H$ is a k uniform linear hypergraph with maximum degree Δ.

Let $n\leq 2k\Delta$ be the number of vertices in H. For any vertex subset S of H, the $(k-1)$ uniformity of H implies that the number of edges in G with at least one vertex in S is lower bounded by

$$e(S)\geq\frac{1}{k-1}\sum_{v\in S}d(v)\geq\frac{\Delta-1}{k-1}|S|.$$

By our assumptions $\Delta>100k^2$ and $n\leq 2k\Delta$, we can check that

$$\frac{\Delta-1}{k-1}>\frac{\Delta}{k}>10\sqrt{\Delta}>10\log\Delta>3\log(2k\Delta)\geq 3\log(n).$$

Furthermore, we have $\frac{\Delta-1}{k-1}\leq\Delta-1<n$. So we can apply Lemma 2.4 with $\alpha=\frac{\Delta-1}{k-1}$. We conclude that Z_G has a negative real zero λ with

$$-\lambda\leq\frac{3\log n}{(\Delta-1)/(k-1)}\leq\frac{3k\log(2\Delta k)}{\Delta}\leq\frac{6k\log\Delta}{\Delta}$$

where the last two inequalities follow from $\Delta>100k^2$. So G satisfies the requirements of Theorem 1.2. □

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