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Preimage pressure for random transformations

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Abstract. In this paper, preimage pressure, which is based on the preimage structure of the system, is defined and studied for random transformations. We obtain analogs of many known results of preimage entropy and preimage pressure for deterministic cases in Cheng and Newhouse [Pre-image entropy. *Ergod. Th. & Dynam. Sys.* **25** (2005), 1091–1113] and Zeng *et al* [Pre-image pressure and invariant measures. *Ergod. Th. & Dynam. Sys.* **27** (2007), 1037–1052]. In particular, a variational principle is given and some applications of preimage pressure, such as the investigation of the invariant measures and the equilibrium states, are obtained.

1. *Introduction*

Entropies and pressure are important invariants in the study of dynamical systems and ergodic theory. Entropies, including topological entropy and measure-theoretic entropy, are all the measurements of the complexity of the orbit structure of the system. As a generalization of topological entropy, the pressure and its related study are the main constituent components of the thermodynamic formalism. There is a symmetry in the definitions of them, which looks only at the future behavior of the considered system. However, how can we describe the complexity of the system by using the 'inverse orbits'? In recent years, Langevin and Przytycki [**9**], Hurley [**5**], Nitecki and Przytycki [**12**] and Nitecki [**13**] formulated and studied several entropy-like invariants based on preimage structure of a mapping. However, most of them are the topological versions. In order to give a reasonable notion of measure-theoretic preimage entropy, recently, Cheng and Newhouse [**4**] introduced a new version of preimage entropies and obtained a variational principle. Lately, the related topics, such as the preimage entropies in random dynamical systems (see Zhu [**16**]) and the preimage pressure (see Zeng *et al* [**15**]), have been studied.

Our purpose of the paper is to formulate and study the notion of preimage pressure for the random transformations. More precisely, in §2, we first define the preimage pressure for the random transformations and then give some properties of it. In §3, a variational

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principle for the preimage pressure is given. In §4, some applications of preimage pressure, such as the investigation of the invariant measures and the equilibrium states, are obtained.

Throughout this paper, $(\Omega, \mathcal{F}, \mathbf{P}, \vartheta)$ denotes an abstract dynamical system, where $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and ϑ is a measure-preserving transformation. Let $(X, \mathcal{B})$ be a measurable space and $\mathbf{F}$ a set of measurable transformations on X.

Definition 1.1. A map $\phi : \mathbb{Z}^+ \times \Omega \to \mathbf{F}$ with the following properties is called a measurable *random dynamical system* (abbreviated as *RDS*) or a measurable *random transformation* over ϑ.

(1) $(\omega, x) \mapsto \phi(n, \omega)x$ is $\mathcal{F} \times \mathcal{B}$-measurable for all $n \in \mathbb{Z}^+$.

(2) $\phi(n + m, \omega) = \phi(n, \vartheta^m \omega) \circ \phi(m, \omega)$ for all $n, m \in \mathbb{Z}^+$.

If we put $\phi(\omega) := \phi(1, \omega)$ we may write by the above cocycle property (2)

$$\phi(n, \omega) := \begin{cases} \mathrm{id} & \text{if } n = 0, \\ \phi(\vartheta^{n-1}\omega) \circ \cdots \circ \phi(\vartheta\omega) \circ \phi(\omega) & \text{if } n > 0. \end{cases}$$

So an RDS could as well be defined as a map $\phi : \Omega \to \mathbf{F}$ for which $(\omega, x) \mapsto \phi(\omega)x$ is $\mathcal{F} \times \mathcal{B}$-measurable.

If X is a compact metric space with a metric d, $\mathcal{B}$ is the Borel σ-algebra of X and $\mathbf{F}$ is a set of continuous maps on X which possesses a measurable structure given by the metric

$$d_0(\phi_1, \phi_2) = \sup\{d(\phi_1(x), \phi_2(x)) \mid x \in X\}, \quad \phi_1, \phi_2 \in \mathbf{F}.$$

Then the RDS determined by the map $\phi : \Omega \to \mathbf{F}$ is called a *topological RDS*.

Definition 1.2. Let ϕ be an RDS on X over ϑ. A probability measure μ on $\mathcal{F} \times \mathcal{B}$ is said to be *ϕ-invariant* if it is invariant under induced skew-product transformation $\Theta : \Omega \times X \to \Omega \times X$, $(\omega, x) \mapsto (\vartheta\omega, \phi(\omega)x)$ and if it has marginal $\mathbf{P}$ on Ω. A ϕ-invariant measure μ is said to be *ergodic* if it is ergodic with respect to Θ.

Let ϕ be an RDS on X over ϑ and μ a ϕ-invariant measure. We always assume that μ disintegrates with respect to $\mathbf{P}$, i.e. there is a family of conditional probabilities $\{\mu_\omega\}$ such that $d\mu(\omega, x) = d\mu_\omega(x)\, d\mathbf{P}(\omega)$. By Bogenschutz [**3**, Lemma 1.1.2], if ϑ is measurably invertible then μ is ϕ-invariant if and only if $\phi(\omega)\mu_\omega = \mu_{\vartheta\omega}$ $\mathbf{P}$-a.e. ω. For a general theory of RDS we refer to [**1, 6, 8, 10**] and [**11**].

In [**4**], Cheng and Newhouse defined the *measure-theoretic preimage entropy* $h_{\mathrm{pre},\nu}(T)$ for an abstract deterministic system $(Y, \mathcal{C}, \nu, T)$ by

$$h_{\mathrm{pre},\nu}(T) = \sup_{\xi} h_{\mathrm{pre},\nu}(T, \xi),$$

where ξ ranges over all finite partitions of Y,

$$h_{\mathrm{pre},\nu}(T, \xi) = h_\nu(T \mid \mathcal{C}^-; \xi) := \lim_{n\to\infty} \frac{1}{n} H_\nu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi \,\middle|\, \mathcal{C}^-\right)$$

in which $H_\nu(\cdot|\cdot)$ is the standard conditional entropy and $\mathcal{C}^-$ is the infinite past σ-algebra $\bigcap_{n\geq 0} T^{-n}\mathcal{C}$ related to $\mathcal{C}$.

In [**16**], Zhu gave the measure-theoretic preimage entropy for RDSs. Denote by π_Ω and π_X the projections from $\Omega \times X$ onto Ω and X, respectively.

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Definition 1.3. The conditional entropy of Θ with respect to the sub-σ-algebra $\pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^- \subset \mathcal{F} \times \mathcal{B}$

$$h_\mu(\Theta \mid \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^-) := \sup_\xi \{h_\mu(\Theta \mid \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^-; \xi)\}, \tag{1.1}$$

where ξ ranges over all finite partitions of $\Omega \times X$ and

$$h_\mu(\Theta \mid \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^-, \xi) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} \Theta^{-i}\xi \;\middle|\; \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^-\right),$$

is called the *measure-theoretic* (or *metric*) *preimage entropy* of ϕ.

As in [**16**], there are alternative definitions of measure-theoretic entropy.

PROPOSITION 1.4. *Let ξ be a finite partition of $\Omega \times X$. For $\omega \in \Omega$, let*

$$\mathcal{B}_\omega^- = \bigcap_{n\geq 0} \phi^{-1}(n, \omega)\mathcal{B}$$

and denote by ξ_ω the partition of X determined by the ω-sections of members of ξ, i.e. $\xi_\omega = \{x \in X \mid (\omega, x) \in \xi\}$. Then the following hold.

(1)

$$h_{\mathrm{pre},\mu}(\phi, \xi) = \lim_{n\to\infty} \frac{1}{n} \int H_{\mu_\omega}\left(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\xi_{\vartheta^i\omega} \;\middle|\; \mathcal{B}_\omega^-\right) d\mathbf{P}(\omega). \tag{1.2}$$

(2) *If ϑ is measurably invertible then $h_{\mathrm{pre},\mu}(\phi, \xi)$ is also equal to*

$$\int \lim_{n\to\infty} \frac{1}{n} H_{\mu_\omega}\left(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\xi_{\vartheta^i\omega} \;\middle|\; \mathcal{B}_\omega^-\right) d\mathbf{P}(\omega). \tag{1.3}$$

In addition, if $\mathbf{P}$ is ergodic, then the integrations in (1.2) and (1.3) are extraneous, and the limit exists and is constant for $\mathbf{P}$-a.e. ω.

Moreover, if $\mathcal{B}$ is countably generated, then

$$h_{\mathrm{pre},\mu}(\phi) = \sup_\zeta h_{\mathrm{pre},\mu}(\phi, \pi_X^{-1}\zeta), \tag{1.4}$$

where ζ ranges over all finite partitions of X.

2. *Preimage pressure for random transformations*

In this section, we always assume that (X, d) is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra on X (so it is countably generated) and ϕ is a topological RDS on X over ϑ. Let $\mathcal{M}(\phi)$ denote all the ϕ-invariant measures, $\mu \in \mathcal{M}(\phi)$ and Θ is the induced skew product transformation acting on $\Omega \times X$.

Define the following family of metrics of X

$$d_\omega^n(x, y) = \max_{0\leq k\leq n-1} d(\phi(k, \omega)x, \phi(k, \omega)y), \quad n \in \mathbb{N}, \omega \in \Omega, x, y \in X.$$

Since $(\omega, x) \to \phi(\omega)x$ is measurable in ω and continuous in x, we conclude that $d_\omega^n(x, y)$ depends measurably on ω and continuously on (x, y).

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For $\omega \in \Omega$, $n \in \mathbb{N}$, a subset $E \subset K$ is said to be a (ω, n, ϵ)-*separated set* of K, if $x, y \in E$, $x \neq y$ implies $d_\omega^n(x, y) > \epsilon$.

Now we define the measure-theoretic preimage pressure for random transformations.

For each measurable-in-(ω, x) and continuous-in-x function f on $\Omega \times X$, set $\|f\|_1 = \int \|f(\omega)\|_\infty \, d\mathbf{P}$, where $\|f(\omega)\|_\infty = \sup_{x \in X} |f(\omega, x)|$. Denote by $L^1(\Omega, C(X))$ the space of such functions f with $\|f\|_1 < \infty$. If we identify f and g, provided $\|f - g\|_1 = 0$, then $L^1(\Omega, C(X))$ becomes a Banach space with norm $\|\cdot\|_1$.

For $f \in L^1(\Omega, C(X))$, $n \in \mathbb{N}$, we write

$$S_n f(\omega, x) := \sum_{i=0}^{n-1} f \circ \Theta^i(\omega, x) = \sum_{i=0}^{n-1} f(\vartheta^i \omega, \phi(i, \omega)x).$$

Let $\epsilon > 0$. For any $\omega \in \Omega$, $k, n \in \mathbb{N}$, $k \geq n$, $F \subset X$ and $f \in L^1(\Omega, C(X))$, we put

$$P_n(\phi, f, \omega, F) := \sum_{x \in F} \exp S_n f(\omega, x),$$

$$P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k, x) := \sup\{P_n(\phi, f, \omega, F) \mid \mathrm{F \text{ is an } } (\omega, n, \epsilon)\text{-separated set of } \phi^{-1}(k, \omega)x\}$$

and

$$P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k) := \sup_{x \in X} P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k, x).$$

LEMMA 2.1. *For any $\epsilon > 0, k, n \in \mathbb{N}$ and $f \in L^1(\Omega, C(X))$, the function $P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k)$ is measurable in ω. And for each $\delta > 0$ there exists a family of maximal (ω, n, ϵ)-separated sets $G_\omega \subset \phi^{-1}(k, \omega)x^*$ for some $x^* \in X$ satisfying*

$$P_n(\phi, f, \omega, G_\omega) \geq (1 - \delta) P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k) \tag{2.1}$$

and depending measurably on ω in the sense that

$$G = \{(\omega, x) \mid x \in G_\omega\} \in \mathcal{F} \times \mathcal{B}.$$

Proof. Given $\epsilon > 0, k, n \in \mathbb{N}$, $f \in L^1(\Omega, C(X))$ and $x \in X$, set $\mathcal{E}_k(x) = \Theta^{-k}(\Omega \times \{x\})$. Since Θ is measurable in ω and continuous in x, $\mathcal{E}_k(x) \in \mathcal{F} \times \mathcal{B}$ and the fibers $\mathcal{E}_k(x, \omega)$, $\omega \in \Omega$, are all compact subsets of X.

Following the argument of [**7**, Lemma 1.2] (we only need to replace the subset '$\mathcal{E}$' of $\Omega \times X$ there by '$\mathcal{E}_k(x)$' and the arguments hold verbatim), we can conclude that the function $P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k, x)$ is measurable in ω. And for each $\delta > 0$ there exists a family of maximal (ω, n, ϵ)-separated sets $G_\omega(x) \subset \phi^{-1}(k, \omega)x$ satisfying

$$P_n(\phi, f, \omega, G_\omega(x)) \geq (1 - \delta) P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k, x) \tag{2.2}$$

and depending measurably on ω in the sense that $G(x) = \{(\omega, y) \mid y \in G_\omega(x)\} \in \mathcal{F} \times \mathcal{B}$.

Note that the collection $\{\phi^{-1}(k, \omega)x \mid x \in \phi(k, \omega)X\}$ is an upper semi-continuous decomposition of X. From the similar reasoning in the proof of [**4**, Theorem 2.5, p. 1105], we can conclude that for any points $x, x' \in \phi(k, \omega)X$, if the distance between them is sufficiently small then the distance of the preimage trees

$$T_k(\phi, \omega, x) = \{[z_k, z_{k-1}, \ldots, z_1, z_0 = x] \mid \phi(k - j + 1, \omega)z_j = z_{j-1}, 1 \leq j \leq k\}$$

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and

$$T_k(\phi, \omega, x') = \{[z'_k, z'_{k-1}, \ldots, z'_1, z'_0 = x] \mid \phi(k-j+1, \omega)z'_j = z'_{j-1}, 1 \le j \le k\}$$

is small enough in the sense of Hausdorff metric. Therefore, if $E(x, \omega)$ is an (ω, n, ϵ)-separated set of $\phi^{-1}(k, \omega)x$ then there is an (ω, n, ϵ)-separated set $E(x', \omega)$ of $\phi^{-1}(k, \omega)x'$ such that the Hausdorff distance between $E(x, \omega)$ and $E(x', \omega)$ is also small. So, for any countable dense subset $\{x_i\}_{i\in\mathbb{N}} \subset X$, we have

$$P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k) = \sup_{i\in\mathbb{N}} P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k, x_i)$$

and hence measurable in ω.

From the definition of $P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k)$ and (2.2), we can easily choose for $\delta > 0$ a point $x^* \in X$ and a family of maximal (ω, n, ϵ)-separated set $G_\omega \subset \phi^{-1}(k, \omega)x^*$ satisfying (2.1) and depending measurably on ω in the sense that $G = \{(\omega, x) \mid x \in G_\omega\} \in \mathcal{F} \times \mathcal{B}$.

For $f \in L^1(\Omega, \mathcal{C}(X))$ and $\epsilon > 0$, put

$$P_{\mathrm{pre}}(\phi, f, \epsilon) := \limsup_{n\to\infty} \frac{1}{n} \sup_{k\ge n} \int \log P_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k)\, d\mathbf{P}(\omega).$$

Let

$$P_{\mathrm{pre}}(\phi, f) = \lim_{\epsilon\to 0} P_{\mathrm{pre}}(\phi, f, \epsilon),$$

where the limit exists since $P_{\mathrm{pre}}(\phi, f, \epsilon)$ is monotone in ϵ, and in fact, '$\lim_{\epsilon\to 0}$' above can be replaced by '$\sup_{\epsilon>0}$'.

Definition 2.2. (topological preimage pressure) The map

$$P_{\mathrm{pre}}(\phi, \cdot) : L^1(\Omega, C(X)) \to \mathbb{R} \cup \{\infty\}$$

defined above is called the *topological preimage pressure* of ϕ.

Remark 2.3. It is clear that the following hold.

(1) $P_{\mathrm{pre}}(\phi, 0) = h_{\mathrm{pre}}(\phi)$, where $h_{\mathrm{pre}}(\phi)$ is the topological preimage entropy (see [**16**]).
(2) $P_{\mathrm{pre}}(\phi, f) \le P(\phi, f)$, for every $f \in L^1(\Omega, C(X))$, where $P(\phi, f)$ is the topological pressure of f (see [**7**]).
(3) $P_{\mathrm{pre}}(\phi, f) \le \|f\|_1$.

Next we shall give an equivalent definition of the preimage pressure using open covers.

For each finite open cover α of X, $\omega \in \Omega$, $x \in X$, $f \in L^1(\Omega, C(X))$ and $k, n \in \mathbb{N}$, we write

$$p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k, x) := \inf_{\eta} \left\{ \sum_{B\in\eta} \sup_{y\in B} \exp S_n f(\omega, y) \right\},$$

where the infimum is taken over all the finite subcover η of

$$\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\alpha \Big|_{\phi^{-1}(k,\omega)x}.$$

Similarly, we write

$$p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k) := \sup_{x\in X} p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k, x).$$

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LEMMA 2.4. *For any $k, n \in \mathbb{N}$, finite open cover α of X and $f \in L^1(\Omega, C(X))$, the function $p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k)$ is measurable in ω.*

Proof. As in the proof of Lemma 2.1, given $k, n \in \mathbb{N}$, $f \in L^1(\Omega, C(X))$ and $x \in X$, set $\mathcal{E}_k(x) = \Theta^{-k}(\Omega \times \{x\})$. Following the arguments of [**7**, Proposition 1.6], we conclude that $p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k, x)$ is measurable in ω. By the similar discussion in the proof of Lemma 2.1, we have that for any countable dense subset $\{x_i\}_{i\in\mathbb{N}}$ of X,

$$p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k) = \sup_{i\in\mathbb{N}} p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k, x_i)$$

is measurable in ω. □

LEMMA 2.5. *Let α be a finite open cover of X and $f \in L^1(\Omega, C(X))$. The limit*

$$p_{\mathrm{pre}}(\phi, f, \alpha) := \lim_{n\to\infty} \frac{1}{n} \sup_{k\geq n} \int \log \sup_{x\in X} p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k)\, d\mathbf{P}(\omega)$$

exists.

Proof. Since for any $x \in X, m, n \in \mathbb{N}$ and $k \geq n + m$, if β is a finite subcover of $(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\alpha)|_{\phi^{-1}(k,\omega)x}$ and γ is a finite subcover of $(\bigvee_{i=0}^{m-1} \phi^{-1}(i, \vartheta^n\omega)\alpha)|_{\phi^{-1}(k-n,\vartheta^n\omega)x}$, then $\beta \vee \phi^{-1}(n, \omega)\gamma$ is a finite subcover of $(\bigvee_{i=0}^{n+m-1} \phi^{-1}(i, \omega)\alpha)|_{\phi^{-1}(k,\omega)x}$. This implies

$$\begin{aligned}
&\sum_{D\in\beta\vee\phi^{-1}(n,\omega)\gamma} \sup_{y\in D} \exp S_{n+m} f(\omega, y) \\
&\quad\leq \Bigg(\sum_{B\in\beta} \sup_{y\in B} \exp S_n f(\omega, y)\Bigg)\Bigg(\sum_{C\in\gamma} \sup_{y\in B} \exp S_m f(\vartheta^n\omega, y)\Bigg).
\end{aligned}$$

Hence we have

$$p_{\mathrm{pre},n+m}(\phi, f, \omega, \alpha, k) \leq p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k) \cdot p_{\mathrm{pre},m}(\phi, f, \omega, \alpha, k - n).$$

For any $n \in \mathbb{N}$, let

$$a_n = \sup_{k\geq n} \int \log \sup_{x\in X} p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k)\, d\mathbf{P}(\omega).$$

Therefore,

$$\begin{aligned}
a_{n+m} &= \sup_{k\geq n+m} \int \log p_{\mathrm{pre},n+m}(\phi, f, \omega, \alpha, k)\, d\mathbf{P}(\omega) \\
&\leq \sup_{k\geq n+m} \int \log p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k)\, d\mathbf{P}(\omega) \\
&\quad + \sup_{k\geq n+m} \int \log p_{\mathrm{pre},m}(\phi, f, \omega, \alpha, k - n)\, d\mathbf{P}(\omega) \\
&\leq \sup_{k\geq n} \int \log p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k)\, d\mathbf{P}(\omega) \\
&\quad + \sup_{l\geq m} \int \log p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, l)\, d\mathbf{P}(\omega) \\
&= a_n + a_m.
\end{aligned}$$

Hence, $\lim_{n\to\infty}(1/n)a_n$ exists.

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For any $f \in L^1(\Omega, C(X))$, we denote

$$p_{\mathrm{pre}}(\phi, f) = \lim_{\delta \to 0} \left[\sup_{\alpha} \{ p_{\mathrm{pre}}(\phi, f, \alpha) \mid \alpha \text{ is an open cover of } X \text{ with } \mathrm{diam}(\alpha) \leq \delta \} \right],$$

where the limit exists since the function in the brackets is monotone in δ.

PROPOSITION 2.6. *For any $f \in L^1(\Omega, C(X))$, we have that*

$$p_{\mathrm{pre}}(\phi, f) = P_{\mathrm{pre}}(\phi, f).$$

Proof. Let $f \in L^1(\Omega, C(X))$. Given $\epsilon > 0$ and a finite open cover α of X with $\mathrm{diam}(\alpha) \leq \epsilon$, it is clear that the inequality

$$p_{\mathrm{pre},n}(\phi, f, \omega, \epsilon, k, x) \leq p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k, x)$$

holds for any $\omega \in \Omega$, $x \in X$, and $k, n \in \mathbb{N}$. Therefore, we obtain

$$P_{\mathrm{pre}}(\phi, f) \leq p_{\mathrm{pre}}(\phi, f)$$

directly by their definitions.

Now we prove the inverse inequality. Let α be a finite open cover of X with Lebesgue number λ. Follow a standard argument, we have that the inequality

$$p_{\mathrm{pre},n}(\phi, f, \omega, \alpha, k, x) \leq \exp\left(\sum_{i=0}^{n-1} \tau^f_{\mathrm{diam}(\alpha)}(\vartheta^i \omega) \right) P_{\mathrm{pre},n}\left(\phi, f, \omega, \frac{\lambda}{2}, k, x \right)$$

holds for every $\omega \in \Omega$, $x \in X$ and $k, n \in \mathbb{N}$, where for any $\delta > 0$ we denote

$$\tau^f_\delta(\omega) = \sup\{ |f(\omega, x) - f(\omega, y)| : x, y \in X, d(x, y) \leq \delta \}.$$

Since $f \in L^1(\Omega, C(X))$ then by Birkhoff ergodic theorem,

$$\int \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \tau^f_\delta(\vartheta^i \omega)\, d\mathbf{P}(\omega) = \int \tau^f_\delta(\omega)\, d\mathbf{P}(\omega)$$

and

$$\lim_{\delta \to 0} \int \tau^f_\delta(\omega)\, d\mathbf{P}(\omega) = 0. \tag{2.3}$$

Therefore, we can conclude that

$$P_{\mathrm{pre}}(\phi, f) \geq p_{\mathrm{pre}}(\phi, f). \qquad \square$$

Given an RDS ϕ over ϑ we can define an RDS ϕ^k over ϑ^k via $\phi^k(n, \omega) := \phi(kn, \omega)$.

PROPOSITION 2.7. *Let $f \in L^1(\Omega, C(X))$ and $m \in \mathbb{N}$. Then*

$$P_{\mathrm{pre}}(\phi^m, S_m f) \leq m \cdot P_{\mathrm{pre}}(\phi, f).$$

Proof. Write $\psi = \phi^m$. Let $\epsilon > 0$, $n \in \mathbb{N}$, $k \geq n$ and $x \in X$.

It is clear that for any $\omega \in \Omega$, if E is an (ω, n, ϵ)-separated subset of $\psi^{-1}(k, \omega)x$ with respect to ψ, then E is also an (ω, mn, ϵ)-separated subset of $\phi^{-1}(mk, \omega)x$ with respect to ϕ.

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Hence we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\sup_{k\geq n}\int \log P_{\mathrm{pre},n}(\psi, S_m f, \omega, \epsilon, k)\, d\mathbf{P}(\omega)\\
&\leq \limsup_{n\to\infty}\frac{1}{n}\sup_{k\geq n}\int \log P_{\mathrm{pre},mn}(\phi, S_m f, \omega, \epsilon, km)\, d\mathbf{P}(\omega)\\
&\leq \limsup_{n\to\infty}\frac{m}{nm}\sup_{l\geq mn}\int \log P_{\mathrm{pre},mn}(\phi, S_m f, \omega, \epsilon, l)\, d\mathbf{P}(\omega)\\
&\leq m\limsup_{n\to\infty}\frac{1}{n}\sup_{t\geq n}\int \log P_{\mathrm{pre},n}(\phi, S_m f, \omega, \epsilon, t)\, d\mathbf{P}(\omega).
\end{aligned}$$

So, $P_{\mathrm{pre}}(\psi, S_m f, \epsilon) \leq m \cdot P_{\mathrm{pre}}(\phi, f, \epsilon)$. Taking $\epsilon \to 0$ gives

$$P_{\mathrm{pre}}(\psi, S_m f) \leq m \cdot P_{\mathrm{pre}}(\phi, f). \qquad \square$$

3. *The variational principle for preimage pressure*

The mathematical foundation of the thermodynamic formalism, i.e. the formalism of equilibrium statistical physics, has been led by Ruelle. Bowen, Ruelle and Sinai have used the thermodynamical approach to study ergodic properties of smooth hyperbolic dynamical systems. As the main constituent components of the thermodynamic formalism, the topological pressure and the variational principle for the topological pressure have been well studied. We can see the related study in [**2, 3, 14**] and [**7**] for deterministic case and random case respectively.

Recently, Zeng *et al* [**15**] gave a variational principle for the preimage pressure for deterministic dynamical systems. In this section we shall combine the technique in [**7**] and [**15**] to give a variational principle for preimage pressure of the random dynamical systems. Indeed, we can see that most of the proofs of the variational principles in vary cases, including that long proof for preimage entropy [**4**, Theorem 2.5], are all adaptations of the well-known argument of Misiurewicz (see, for example, of [**14**, Theorem 9.10]) though they are more complicated.

In the next sections, we always assume that (X, d) is a compact metric space and ϕ is a topological RDS on X over $(\Omega, \mathcal{F}, \mathbf{P}, \vartheta)$, where $(\Omega, \mathcal{F}, \mathbf{P})$ is a Lebesgue space, $\mathcal{F}$ is the Borel σ-algebra, ϑ is measurably invertible.

THEOREM 3.1. (The variational principle) *Let* $f \in L^1(\Omega, C(X))$. *Then*

$$P_{\mathrm{pre}}(\phi, f) = \sup\left\{h_{\mathrm{pre},\mu}(\phi) + \int f\, d\mu \;\middle|\; \mu \in \mathcal{M}(\phi)\right\},$$

where $\mathcal{M}(\phi)$ *denotes the set of all the* ϕ*-invariant measures.*

Proof. Step 1.

$$P_{\mathrm{pre}}(\phi, f) \geq h_{\mathrm{pre},\mu}(\phi) + \int f\, d\mu \tag{3.1}$$

for all $\mu \in \mathcal{M}(\phi)$.

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Let $\mu \in M(\phi)$. Let $\xi = \{A_1, A_2, \ldots, A_k\}$ be a finite partition of X and choose $\epsilon > 0$ such that $\epsilon < 1/(k \log k)$. Since $\pi_X \mu$ is regular then there exist compact sets $B_j \subset A_j$, $1 \le j \le k$, with $\int \mu_\omega(A_j \setminus B_j)\, d\mathbf{P}(\omega) < \epsilon$. Put $B_0 = X \setminus \bigcup_{j=1}^k B_j$. As in the proof of [**16**, Theorem 4] , for the partition $\eta = \{B_0, B_1 \ldots, B_k\}$, we have

$$h_{\mathrm{pre},\mu}(\phi, \pi_X^{-1}\xi) \le h_{\mathrm{pre},\mu}(\phi, \pi_X^{-1}\eta) + 1.$$

Thus, for (3.1) it is suffices to prove that there exists a $\delta > 0$, such that

$$h_{\mathrm{pre},\mu}(\phi, \pi_X^{-1}\eta) + \int f\, d\mu \le \log 2 + P_{\mathrm{pre}}(\phi, f, \delta) + \int \tau_{2\delta}^f\, d\mathbf{P}(\omega). \tag{3.2}$$

Indeed, once this is done, we have

$$h_{\mathrm{pre},\mu}(\phi, \pi_X^{-1}\xi) + \int f\, d\mu \le 1 + \log 2 + P_{\mathrm{pre}}(\phi, f, \delta) + \int \tau_{2\delta}^f\, d\mathbf{P}(\omega).$$

As this holds true for every finite partition ξ of X and corresponding δ , it follows

$$h_{\mathrm{pre},\mu}(\phi) + \int f\, d\mu \le 1 + \log 2 + P_{\mathrm{pre}}(\phi, f),$$

where we use the property (2.3) of τ_δ^f.

Hence Proposition 2.7 and the power rule of $h_{\mathrm{pre},\mu}(\phi)$ [**16**, Proposition 4] yield that

$$\begin{aligned} m h_{\mathrm{pre},\mu}(\phi) + m \int f\, d\mu &= h_{\mathrm{pre},\mu}(\phi^m) + \int S_m f\, d\mu \\ &\le P_{\mathrm{pre}}(\phi^m, S_m f) + \log 2 + 1 \\ &\le m P_{\mathrm{pre}}(\phi, f) + \log 2 + 1. \end{aligned}$$

Dividing by m and taking $m \to \infty$ yields (3.1).

For (3.2), it suffices to show that there is a $\delta > 0$ such that for any $n, k \in \mathbb{N}$, $\omega \in \Omega$,

$$\begin{aligned} &H_{\mu_\omega}\Big(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta \,\Big|\, \phi^{-1}(k, \omega)\mathcal{B}\Big) + \int S_n f\, d\mu_\omega \\ &\quad \le n \log 2 + \log \sup_{x \in X} P_{\mathrm{pre},n}(\phi, f, \omega, \delta, k) + \sum_{i=0}^{n-1} \tau_{2\delta}^f(\vartheta^i \omega). \end{aligned} \tag{3.3}$$

Choose a non-decreasing sequence of finite partitions $\beta_1 \le \beta_2 \le \cdots$ with diameters tending to zero for which $\mathcal{B} = \bigvee_{i=1}^\infty \beta_j$ up to set of measure 0, and satisfying

$$H_{\mu_\omega}\Big(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta \,\Big|\, \phi^{-1}(k, \omega)\mathcal{B}\Big) = \lim_{j \to \infty} H_{\mu_\omega}\Big(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta \,\Big|\, \phi^{-1}(k, \omega)\beta_j\Big).$$

So, for (3.3), it suffices to show that for sufficiently large $j(\omega)$,

$$\begin{aligned} &H_{\mu_\omega}\Big(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta \,\Big|\, \phi^{-1}(k, \omega)\beta_{j(\omega)}\Big) + \int S_n f\, d\mu_\omega \\ &\quad \le n \log 2 + \log \sup_{x \in X} P_{\mathrm{pre},n}(\phi, f, \omega, \delta, k) + \sum_{i=0}^{n-1} \tau_{2\delta}^f(\vartheta^i \omega). \end{aligned} \tag{3.4}$$

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Now, pick $0 < \delta < \frac{1}{8} \min_{1 \le i \ne j \le k} d(B_i, B_j)$. Given $\omega \in \Omega$. Let $\delta_1(\omega, x, k, \delta) > 0$ such that $d(x, y) < \delta_1(\omega, x, k, \delta)$ implies $d_\omega^n(x, y) < \delta$. Note that the collection $\{\phi^{-1}(k, \omega)x \mid x \in \phi(k, \omega)X\}$ is an upper semi-continuous decomposition of X. Hence for each $x \in \phi(k, \omega)X$ there is a $\delta_2(\omega, x, k, \delta) > 0$ such that $d(x, y) < \delta_2(\omega, x, k, \delta)$, $y \in \phi(k, \omega)X$ and $y_1 \in \phi^{-1}(k, \omega)y$; then there is an $x_1 \in \phi^{-1}(k, \omega)x$ such that $d(x_1, y_1) < \delta_1(\omega, x, k, \delta)$. Let $\mathcal{U}$ be the collection of open $\delta_2(\omega, x, k, \delta)$ balls in $\phi(k, \omega)X$ as x varies in $\phi(k, \omega)X$ and let $\delta_3(\omega)$ be a Lebesgue number for $\mathcal{U}$.

Since $\operatorname{diam}(\beta_j) \to 0$ as $j \to \infty$, we may choose $j_0(\omega)$ for $\omega \in \Omega$ such that if $j(\omega) \ge j_0(\omega)$ and $B \in \beta_{j(\omega)}$, then $\operatorname{diam} \overline{B} < \delta_3(\omega)$. Let $j(\omega) > j_0(\omega)$. For a set $C \in \phi^{-1}(k, \omega)\beta_{j(\omega)}$, let $\mu_{\omega,C}$ denote the conditional measure of μ_ω restricted to C, $(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta)|_C$ denote the set $\{A \cap C \mid A \in \bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta, A \cap C \ne \emptyset\}$ and

$$S_n^* f(\omega, A \cap C) := \sup\{S_n f(\omega, x) \mid x \in A \cap C\}.$$

Therefore, by the standard inequality

$$\sum_{1 \le i \le m} p_i(a_i - \log p_i) \le \log\left(\sum_{1 \le i \le m} \exp a_i\right) \tag{3.5}$$

for any probability vector $(p_1, \ldots, p_m)$ (where the equality holds if and only if $p_i = ((\exp a_i)/(\sum_{1 \le j \le m} \exp a_j))$), we have

$$\begin{aligned}
&H_{\mu_\omega}\left(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta \,\middle|\, \phi^{-1}(k, \omega)\beta_{j(\omega)}\right) + \int S_n f \, d\mu_\omega \\
&\le \sum_{C \in \phi^{-1}(k,\omega)\beta_{j(\omega)}} \left[H_{\mu_{\omega,C}}\left(\left(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta\right)\Big|_C\right) \cdot \mu_\omega(C) + \int_C S_n f \, d\mu_\omega\right] \\
&\le \sum_{C \in \phi^{-1}(k,\omega)\beta_{j(\omega)}} \mu_\omega(C) \\
&\cdot \left(\sum_{A \in \bigvee_{i=0}^{n-1} \phi^{-1}(i,\omega)\eta} \mu_{\omega,C}(A \cap C)[-\log \mu_{\omega,C}(A \cap C) + S_n^* f(\omega, A \cap C)]\right) \\
&\le \max_{C \in \phi^{-1}(k,\omega)\beta_{j(\omega)}} \log\left(\sum_{A \in \bigvee_{i=0}^{n-1} \phi^{-1}(i,\omega)\eta} \exp S_n^* f(\omega, A \cap C)\right).
\end{aligned}$$

For each

$$A \cap C \in \left(\bigvee_{i=0}^{n-1} \phi^{-1}(i, \omega)\eta\right)\Big|_C$$

choose some $x_A \in \overline{A \cap C}$ such that $S_n f(\omega, x_A) = S_n^* f(\omega, \overline{A \cap C})$. Let $B \in \beta_{j(\omega)}$ such that $C = \phi^{-1}(k, \omega)B$. Since $\phi(k, \omega)x_A \in \overline{B}$ and $\operatorname{diam}(\overline{B}) < \delta_3(\omega)$, there is $u_B \in \phi(k, \omega)X$ such that if $y \in \overline{B} \cap \phi(k, \omega)X$ then $d(u_B, y) < \delta_2(\omega, u_B, k, \delta)$. This implies $d(u_B, \phi(k, \omega)x_A) < \delta_2(\omega, u_B, k, \delta)$. Hence there is a point $\phi_1(A) \in \phi^{-1}(k, \omega)u_B$ such that $d_\omega^n(x_A, \phi_1(A)) < \delta$. Let E_C be a maximal (ω, n, δ)-separated set in $\phi^{-1}(k, \omega)u_B$.

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Since E_C spans $\phi^{-1}(k,\omega)u_B$, there is a point $\phi_2(A)\in E_C$ such that $d^n_\omega(\phi_1(A),\phi_2(A))\le\delta$. Hence $d^n_\omega(x_A,\phi_2(A))<2\delta$, and then

$$S^*_n f(\omega,\, A\cap C)\le S_n f(\omega,\,\phi_2(A))+\sum_{i=0}^{n-1}\tau^f_{2\delta}(\vartheta^i\omega).$$

By the choice of δ, we have that if $y\in E_C$ then

$$\operatorname{card}\left\{A\cap C\in\left(\bigvee_{i=0}^{n-1}\phi^{-1}(i,\,\omega)\eta\right)\Big|_C : \phi_2(A)=y\right\}\le 2^n.$$

Thus for sufficiently large $j(\omega)(>j_0(\omega))$ and $C\in\phi^{-1}(i,\,\omega)\beta_{j(\omega)}$, we have

$$\begin{aligned}&\sum_{A\in\bigvee_{i=0}^{n-1}\phi^{-1}(i,\omega)\eta}\exp\left[S^*_n f(\omega,\,A\cap C)-\sum_{i=0}^{n-1}\tau^f_{2\delta}(\vartheta^i\omega)\right]\\&\quad\le\sum_{A\in\bigvee_{i=0}^{n-1}\phi^{-1}(i,\omega)\eta}\exp S_n f(\omega,\,\phi_2(A))\\&\quad\le 2^n\sum_{z\in E_C}\exp S_n f(\omega,\,z).\end{aligned}$$

This proves (3.4) and finishes the proof of step 1.

Step 2.

$$P_{\mathrm{pre}}(\phi,\,f)\le\sup\left\{h_{\mathrm{pre},\mu}(\phi)+\int f\,d\mu\;\Big|\;\mu\in\mathcal{M}(\phi)\right\}.$$

Given $\epsilon>0$, we will produce a ϕ-invariant measure μ such that

$$P_{\mathrm{pre}}(\phi,\,f,\,\epsilon)\le h_{\mathrm{pre},\mu}(\phi)+\int f\,d\mu.$$

Choose sequences $n_i\to\infty$, $k_i\ge n_i$ and points $x^{(i)}_\omega\in\pi^{-1}_\Omega(\vartheta^{k_i}\omega)$ for $\omega\in\Omega$ such that

$$P_{\mathrm{pre}}(\phi,\,f,\,\epsilon)=\lim_{i\to\infty}\frac{1}{n_i}\int\log P_{\mathrm{pre},n_i}(\phi,\,f,\,\omega,\,\epsilon,\,k_i,\,x^{(i)}_\omega)\,d\mathbf{P}(\omega).$$

For any $\omega\in\Omega$, let $E^{(i)}_\omega$ denote a maximal (ω,n_i,ϵ)-separated set in $\phi^{-1}(k_i,\,\omega)x^{(i)}_\omega$. By Lemma 2.1, we can assume that $E^{(i)}_\omega$ satisfies $\{(\omega,\,x)\mid x\in E^{(i)}_\omega\}\in\mathcal{F}\times\mathcal{B}$ and

$$\sum_{y\in E^{(i)}_\omega}\exp S_{n_i}f(\omega,\,y)\ge\frac{1}{e}P_{\mathrm{pre},n_i}(\phi,\,f,\,\omega,\,\epsilon,\,k_i,\,x^{(i)}_\omega).\tag{3.6}$$

Now define probability measures $\nu^{(i)}$ on $\Omega\times X$ via their measurable disintegrations

$$\nu^{(i)}_\omega=\frac{\sum_{y\in E^{(i)}_\omega}\exp S_{n_i}f(\omega,\,y)\delta_y}{\sum_{z\in E^{(i)}_\omega}\exp S_{n_i}f(\omega,\,z)},$$

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where δ_y denotes the point mass at the point y. Let

$$\mu^{(i)} = \frac{1}{n_i} \sum_{j=0}^{n_i-1} \Theta^j \nu^{(i)}$$

and

$$\mu = \lim_{i\to\infty} \mu^{(i)}.$$

By [**7**, Lemma 2.1], the last limit of measures exists and belongs to $\mathcal{M}(\phi)$.

Next, choose a partition ξ of X with diam $\xi < \varepsilon$ and such that $\pi_X\mu(\partial A) = 0$ for every $A \in \xi$. Since for any $A \in \xi$, $\pi_X\mu(\partial A) = \int \mu_\omega(\partial A)\, d\mathbf{P}(\omega)$ then $\mu_\omega(\partial A) = 0$ $\mathbf{P}$-a.e. ω.

In the following, we will show that, for every positive integer m,

$$\begin{aligned} &H_\mu\Big(\bigvee_{t=0}^{m-1} \Theta^{-t}(\pi_X^{-1}\xi) \,\Big|\, \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F}\times\mathcal{B})^-\Big) + m\int f\, d\mu \\ &\quad \geq m \lim_{i\to\infty} \frac{1}{n_i} \int \log P_{\mathrm{pre},n_i}(\phi,\, f,\, \omega,\, \epsilon,\, k_i,\, x_\omega^{(i)})\, d\mathbf{P}(\omega). \end{aligned} \tag{3.7}$$

Indeed, from this we have

$$\begin{aligned} &h_{\mathrm{pre},\mu}(\phi) + \int f\, d\mu \\ &\quad \geq h_{\mathrm{pre},\mu}(\phi, \pi_X^{-1}\xi) + \int f\, d\mu \\ &\quad = \lim_{m\to\infty} \frac{1}{m} H_\mu\Big(\bigvee_{t=0}^{m-1} \Theta^{-t}(\pi_X^{-1}\xi) \,\Big|\, \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F}\times\mathcal{B})^-\Big) \\ &\qquad + \lim_{m\to\infty} \frac{1}{m} \int S_m(\omega,\, f) y\, d\mu \\ &\quad \geq \lim_{i\to\infty} \frac{1}{n_i} \int \log P_{\mathrm{pre},n_i}(\phi,\, f,\, \omega,\, \epsilon,\, k_i,\, x_\omega^{(i)})\, d\mathbf{P}(\omega) \\ &\quad = P_{\mathrm{pre}}(\phi,\, f,\, \epsilon) \end{aligned}$$

as required.

As in [**16**], to prove (3.7), we will make use of the Rokhlin theory of the measurable partitions. This theory concerns itself with complete measure spaces.

For $\omega \in \Omega$, let $\mathcal{C}_\omega$ denote the subcollection of $\mathcal{B}_\omega^-$ consisting of μ_ω-null sets. For any σ-algebra $\mathcal{A}$ of subsets of X, there is an enlarged σ-algebra $\mathcal{A}_{\mathcal{C}_\omega}$ defined by $A \in \mathcal{A}_{\mathcal{C}_\omega}$ if and only if there are sets B, M, N such that $A = B \cup M$, $B \in \mathcal{A}$, $N \in \mathcal{C}_\omega$ and $M \subset N$. The σ-algebra $\mathcal{B}_{\mathcal{C}_\omega}^-$ is simply the standard μ_ω-completion of $\mathcal{B}_\omega^-$. We will also consider the σ-algebras $\mathcal{B}_\omega^k = (\phi^{-1}(k, \omega)\mathcal{B})_{\mathcal{C}_\omega}$ for all $k \geq 1$. Since $\phi^{-1}(1, \omega)\mathcal{C}_{\vartheta\omega} \subset \mathcal{C}_\omega$ for any $\omega \in \Omega$, we have that $\mathcal{B}_\omega^1 \supset \mathcal{B}_\omega^2 \supset \cdots$. Let $\mathcal{B}_\omega^\infty = \bigcap_{k\geq 1} \mathcal{B}_\omega^k$, then $\mathcal{B}_\omega^- \subset \mathcal{B}_{\mathcal{C}_\omega}^- \subset \mathcal{B}_\omega^\infty$ and $\phi^{-1}(l, \omega)\mathcal{B}_{\vartheta^l\omega}^k \subset \mathcal{B}_\omega^{l+k}$ for all $l \geq 1$.

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Let $\mathcal{C}$ denote the subcollection of $(\mathcal{F}\times\mathcal{B})^-$ consisting of μ-null sets. Define $(\mathcal{F}\times\mathcal{B})^k$, $(\mathcal{F}\times\mathcal{B})^\infty$ using the similar method as above. It is clear that

$$(\mathcal{F}\times\mathcal{B})^1 \supset (\mathcal{F}\times\mathcal{B})^2 \supset \cdots, (\mathcal{F}\times\mathcal{B})^- \subset (\mathcal{F}\times\mathcal{B})^\infty \subset (\mathcal{F}\times\mathcal{B})^k$$

and

$$\Theta^{-l}(\pi_\Omega^{-1}\mathcal{F}\vee(\mathcal{F}\times\mathcal{B})^k)\subset\pi_\Omega^{-1}\mathcal{F}\vee(\mathcal{F}\times\mathcal{B})^{l+k}\quad\text{for all } l\geq 1.$$

As in the proof of [**16**, Proposition 2], for any finite partition η of $\Omega\times X$, we have,

$$H_\mu(\eta\mid\pi_\Omega^{-1}\mathcal{F}\vee(\mathcal{F}\times\mathcal{B})^k)=\int H_{\mu_\omega}(\eta_\omega\mid\mathcal{B}_\omega^k)\,d\mathbf{P}(\omega).$$

By the claim in the proof of [**16**, Theorem 4], we have

$$\begin{aligned}&H_\mu\left(\bigvee_{t=0}^{m-1}\Theta^{-t}(\pi_X^{-1}\xi)\;\middle|\;\pi_\Omega^{-1}\mathcal{F}\vee(\mathcal{F}\times\mathcal{B})^-\right)\\&\quad\geq\limsup_{i\to\infty}H_{\mu^{(i)}}\left(\bigvee_{t=0}^{m-1}\Theta^{-t}(\pi_X^{-1}\xi)\;\middle|\;\pi_\Omega^{-1}\mathcal{F}\vee(\mathcal{F}\times\mathcal{B})^\infty\right)\end{aligned}\tag{3.8}$$

and for any $\omega\in\Omega$,

$$\begin{aligned}&H_{\mu_\omega^{(i)}}\left(\bigvee_{t=0}^{m-1}\phi^{-1}(t,\omega)\xi\;\middle|\;\mathcal{B}_\omega^{k_i}\right)\\&\quad\geq\frac{1}{n_i}\sum_{l=0}^{n_i-1}H_{\nu_{\vartheta^{-l}\omega}^{(i)}}\left(\phi^{-1}(l,\vartheta^{-l}\omega)\bigvee_{t=0}^{m-1}\phi^{-1}(t,\omega)\xi\;\middle|\;\mathcal{B}_{\vartheta^{-l}\omega}^{k_i}\right).\end{aligned}\tag{3.9}$$

Proof of (3.7). Note that $\nu_\omega^{(i)}$ is supported on $\phi^{-1}(k_i,\omega)x_\omega^{(i)}$, the canonical system of conditional measures induced by $\nu_\omega^{(i)}$ on the measurable partitions $\{\phi^{-1}(k_i,\omega)x\mid x\in X\}$ reduces to a single measure on the set $\phi^{-1}(k_i,\omega)x_\omega^{(i)}$ which we may identify with $\nu_\omega^{(i)}$. Now, each element $A\in\mathcal{B}_\omega^{k_i}$ can be expressed as the disjoint union $A=B\cup C$ with $B\in\phi^{-1}(k_i,\omega)\mathcal{B}$ and $C\in\mathcal{C}_\omega$. Since $\nu_\omega^{(i)}$ is supported on elements of $\phi^{-1}(k_i,\omega)\mathcal{B}$, we have $\nu_\omega^{(i)}(C)=0$.

Hence for any finite partition γ, we have

$$H_{\nu_\omega^{(i)}}(\gamma\mid\mathcal{B}_\omega^{k_i})=H_{\nu_\omega^{(i)}}(\gamma|_{\phi^{-1}(k_i,\omega)x_\omega^{(i)}}).$$

Since each element of $\bigvee_{l=0}^{n_i}\phi^{-1}(l,\omega)\xi$ has at most one point of $E_\omega^{(i)}$, then we have

$$\begin{aligned}&H_{\nu_\omega^{(i)}}\left(\left(\bigvee_{l=0}^{n_i-1}\phi^{-1}(l,\omega)\xi\right)\;\middle|\;\mathcal{B}_\omega^{k_i}\right)+\int_X S_{n_i}f(\omega,y)\,d\nu_\omega^{(i)}(y)\\&=H_{\nu_\omega^{(i)}}\left(\left(\bigvee_{l=0}^{n_i-1}\phi^{-1}(l,\omega)\xi\right)\Big|_{\phi^{-1}(k_i,\omega)x_\omega^{(i)}}\right)+\int_X S_{n_i}f(\omega,y)\,d\nu_\omega^{(i)}(y)\\&=\sum_{y\in E_\omega^{(i)}}\nu_\omega^{(i)}(\{y\})(S_{n_i}f(\omega,y)-\log\nu_\omega^{(i)}(\{y\}))\\&=\log\sum_{y\in E_\omega^{(i)}}\exp S_{n_i}f(\omega,y)(\text{by the definition of }\nu_\omega^{(i)}\text{ and (3.5)}).\end{aligned}$$

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Consider n_i with $n_i > m$ and let $a(j)$ denote the integer part of $(n_i - j)/m$ for $1 \le j \le m-1$. Then, clearly for any $\omega \in \Omega$, we have

$$\bigvee_{l=0}^{n_i-1} \phi^{-1}(l,\omega)\xi = \bigvee_{r=0}^{a(j)-1} \left(\phi^{-1}(rm+j,\omega) \bigvee_{t=0}^{m-1} \phi^{-1}(t, \vartheta^{rm+j}\omega)\xi \right) \bigvee_{l\in S} \phi^{-1}(t,\omega)\xi,$$

where card $S \le 2m$. Hence,

$$\begin{aligned}
&\log \sum_{y\in E_\omega^{(i)}} \exp S_{n_i} f(\omega, y) \\
&= H_{\nu_\omega^{(i)}}\left(\left(\bigvee_{l=0}^{n_i-1} \phi^{-1}(l,\omega)\xi \right)\Big|_{\phi^{-1}(k_i,\omega)x_\omega^{(i)}} \right) + \int_X S_{n_i} f(\omega, y)\, d\nu_\omega^{(i)}(y) \\
&\le H_{\nu_\omega^{(i)}}\left(\left(\bigvee_{r=0}^{a(j)-1} \left(\phi^{-1}(rm+j,\omega) \bigvee_{t=0}^{m-1} \phi^{-1}(t,\vartheta^{rm+j}\omega)\xi \right) \right)\Big|_{\phi^{-1}(k_i,\omega)x_\omega^{(i)}} \right) \\
&\quad + H_{\nu_\omega^{(i)}}\left(\left(\bigvee_{l\in S} \phi^{-1}(l,\omega)\xi \right)\Big|_{\phi^{-1}(k_i,\omega)x_\omega^{(i)}} \right) + \int_X S_{n_i} f(\omega, y)\, d\nu_\omega^{(i)}(y) \\
&\le H_{\nu_\omega^{(i)}}\left(\left(\bigvee_{r=0}^{a(j)-1} \left(\phi^{-1}(rm+j,\omega) \bigvee_{t=0}^{m-1} \phi^{-1}(t,\vartheta^{rm+j}\omega)\xi \right) \right)\Big|_{\phi^{-1}(k_i,\omega)x_\omega^{(i)}} \right) \\
&\quad + 2m \log \operatorname{card} \xi + \int_X S_{n_i} f(\omega, y)\, d\nu_\omega^{(i)}(y).
\end{aligned}$$

Notice that

$$\int_\Omega \int_X S_{n_i} f(\omega, y)\, d\nu_\omega^{(i)}(y)\, d\mathbf{P}(\omega) = \int S_{n_i} f(\omega, y)\, d\nu^{(i)} = n_i \int f\, d\mu^{(i)}.$$

Sum the above inequality over j from 0 to $m-1$ and integrate it against $\mathbf{P}$ to get

$$\begin{aligned}
&m \int \log \sum_{y\in E_\omega^{(i)}} \exp S_{n_i} f(\omega, y)\, d\mathbf{P}(\omega) - 2m^2 \log \operatorname{card} \xi \\
&\le \int \sum_{j=0}^{m-1} H_{\nu_\omega^{(i)}}\left(\left(\bigvee_{r=0}^{a(j)-1} \phi^{-1}(rm+j,\omega) \right.\right. \\
&\qquad \left.\left. \times \bigvee_{t=0}^{m-1} \phi^{-1}(t,\vartheta^{rm+j}\omega)\xi \right)\Big|_{\phi^{-1}(k_i,\omega)x_\omega^{(i)}} \right) d\mathbf{P}(\omega) \\
&\quad + m \int_\Omega \int_X S_{n_i} f(\omega, y)\, d\nu_\omega^{(i)}(y)\, d\mathbf{P}(\omega) \\
&\le \int \sum_{j=0}^{m-1} H_{\nu^{(i)}_{\vartheta^{-(rm+j)}\omega}}\left(\left(\bigvee_{r=0}^{a(j)-1} \phi^{-1}(rm+j, \vartheta^{-(rm+j)}\omega) \right.\right. \\
&\qquad \left.\left. \times \bigvee_{t=0}^{m-1} \phi^{-1}(t,\omega)\xi \right)\Big|_{\phi^{-1}(k_i,\omega)x^{(i)}_{\vartheta^{-(rm+j)}\omega}} \right) d\mathbf{P}(\omega)
\end{aligned}$$

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$$
\begin{aligned}
&+ m \int_\Omega \int_X S_{n_i} f(\omega, y)\, d\nu_\omega^{(i)}(y)\, d\mathbf{P}(\omega) \\
&\le \int \sum_{l=0}^{n_i-1} H_{\nu^{(i)}_{\vartheta^{-l}\omega}}\left(\phi^{-1}(l, \vartheta^{-l}\omega) \bigvee_{t=0}^{m-1} \phi^{-1}(t, \omega)\xi \;\middle|\; \mathcal{B}^{k_i}_{\vartheta^{-l}\omega}\right) d\mathbf{P}(\omega) \\
&\quad + mn_i \int f\, d\mu^{(i)} \\
&\le n_i \int H_{\mu^{(i)}_\omega}\left(\bigvee_{t=0}^{m-1} \phi^{-1}(t, \omega)\xi \;\middle|\; \mathcal{B}^{k_i}_\omega\right) d\mathbf{P}(\omega) + mn_i \int f\, d\mu^{(i)} \quad \text{(by (3.9))}.
\end{aligned}
$$

Therefore, from (3.6), we have

$$
\begin{aligned}
&m \int \log P_{\mathrm{pre},n_i}(\phi, f, \omega, \epsilon, k_i, x_\omega^{(i)})\, d\mathbf{P}(\omega) - 2m^2 \log \operatorname{card} \xi - m \\
&\quad \le m \int \log \sum_{y \in E_\omega^{(i)}} \exp S_{n_i} f(\omega, y)\, d\mathbf{P}(\omega) - 2m^2 \log \operatorname{card} \xi \\
&\quad \le n_i \int H_{\mu^{(i)}_\omega}\left(\bigvee_{t=0}^{m-1} \phi^{-1}(t, \omega)\xi \;\middle|\; \mathcal{B}^{k_i}_\omega\right) d\mathbf{P}(\omega) + mn_i \int f\, d\mu^{(i)}.
\end{aligned}
$$

Now divide by mn_i to obtain

$$
\begin{aligned}
&\frac{1}{n_i} \int \log P_{\mathrm{pre},n_i}(\phi, f, \omega, \epsilon, k_i, x_\omega^{(i)})\, d\mathbf{P}(\omega) - \frac{2m}{n_i} \log \operatorname{card} \xi - \frac{1}{n_i} \\
&\quad \le \frac{1}{m} \int H_{\mu^{(i)}_\omega}\left(\bigvee_{t=0}^{m-1} \phi^{-1}(t, \omega)\xi \;\middle|\; \mathcal{B}^{k_i}_\omega\right) d\mathbf{P}(\omega) + \int f\, d\mu^{(i)} \\
&\quad = \frac{1}{m} H_{\mu^{(i)}}\left(\bigvee_{t=0}^{m-1} \Theta^{-t}(\pi_X^{-1}\xi) \;\middle|\; \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^{k_i}\right) + \int f\, d\mu^{(i)} \\
&\quad \le \frac{1}{m} H_{\mu^{(i)}}\left(\bigvee_{t=0}^{m-1} \Theta^{-t}(\pi_X^{-1}\xi) \;\middle|\; \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^{\infty}\right) + \int f\, d\mu^{(i)}.
\end{aligned}
$$

From (3.8) and the above inequality, letting $i \to \infty$, we get that

$$
\begin{aligned}
&\lim_{i\to\infty} \frac{1}{n_i} \int \log P_{\mathrm{pre},n_i}(\phi, f, \omega, \epsilon, k_i, x_\omega^{(i)})\, d\mathbf{P}(\omega) \\
&\quad \le \limsup_{i\to\infty} \left(\frac{1}{m} H_{\mu^{(i)}}\left(\bigvee_{t=0}^{m-1} \Theta^{-t}(\pi_X^{-1}\xi) \;\middle|\; \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^{\infty}\right) + \int f\, d\mu^{(i)}\right) \\
&\quad \le \frac{1}{m} H_\mu\left(\bigvee_{t=0}^{m-1} \Theta^{-t}(\pi_X^{-1}\xi) \;\middle|\; \pi_\Omega^{-1}\mathcal{F} \vee (\mathcal{F} \times \mathcal{B})^{-}\right) + \int f\, d\mu.
\end{aligned}
$$

This proves (3.7). □

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4. *Invariant measures and equilibrium states*

It is well known that the pressure can determine the invariant measures and the variational principle gives a natural way to pick up some members which are called equilibrium states. These properties had been studied in [**14**] and [**3**] for deterministic and random cases respectively. In [**15**], the similar results are all obtained for the preimage pressure in the deterministic case. In fact we can see that these results in [**15**] and [**3**] are all obtained along the lines of that in [**14**]. Combining the techniques in [**15**] and [**3**], we shall give some corresponding results for the preimage pressure in the random case under the same assumptions for the RDS ϕ in §3 and only prove some of them when necessary.

By the definition of preimage pressure, we have the following properties of it.

PROPOSITION 4.1. *Let ϕ be a topological RDS on X over $(\Omega, \mathcal{F}, \mathbf{P}, \vartheta)$. If f, $g \in L^1(\Omega, C(X))$ and $c \in L^1(\Omega, \mathbf{P})$, then the following are true.*

(1) $f \le g$, $\mathbf{P}$*-a.s. implies* $P_{\mathrm{pre}}(\phi, f) \le P_{\mathrm{pre}}(\phi, g)$.
(2) $P_{\mathrm{pre}}(\phi, f + c) = P_{\mathrm{pre}}(\phi, f) + \int c\, d\mathbf{P}(\omega)$.
(3) *If* $h_{\mathrm{pre}}(\phi) < \infty$, *then* $|P_{\mathrm{pre}}(\phi, f) - P_{\mathrm{pre}}(\phi, g)| \le \|f - g\|_1$.
(4) *If* $h_{\mathrm{pre}}(\phi) < \infty$, *then the map* $P_{\mathrm{pre}}(\phi, \cdot) : L^1(\Omega, C(X)) \to \mathbb{R} \cup \{\infty\}$ *is convex.*
(5) $P_{\mathrm{pre}}(\phi, f + g \circ \Theta - g) = P_{\mathrm{pre}}(\phi, f)$.
(6) $P_{\mathrm{pre}}(\phi, f + g) \le P_{\mathrm{pre}}(\phi, f) + P_{\mathrm{pre}}(\phi, g)$.

As the treatment in [**3**, Theorem 3.1.4], we can apply (2) and (5) of Proposition 4.1 and the variational principle (Theorem 3.1) to determine the member of $\mathcal{M}(\phi)$.

PROPOSITION 4.2. *Let $\mu : \mathcal{F} \times \mathcal{B} \to \mathbb{R}$ be a finite signed measure on $\Omega \times X$ with $\pi_\Omega \mu = \mathbf{P}$. If $h_{\mathrm{pre}}(\phi) < \infty$, then $\mu \in \mathcal{M}(\phi)$ if and only if $\int f\, d\mu \le P_{\mathrm{pre}}(\phi, f)$, for all $f \in L^1(\Omega, \mathcal{C}(X))$.*

Proof. The proof follows the idea of the proof of [**3**, Theorem 3.1.4] and is omitted.

To derive further properties of $P_{\mathrm{pre}}(\phi, \cdot)$, it is useful to consider the preimage entropy map of ϕ , i.e.

$$h_{\mathrm{pre},\cdot}(\phi) : \mathcal{M}(\phi) \to \mathbb{R}^+ \cup \{\infty\},$$
$$\mu \mapsto h_{\mathrm{pre},\mu}(\phi).$$

It is said that $h_{\mathrm{pre},\cdot}(\phi)$ is *upper semi-continuous* at $\mu_0 \in \mathcal{M}(\phi)$ if

$$\limsup_{\mu \to \mu_0} h_{\mathrm{pre},\mu}(\phi) \le h_{\mathrm{pre},\mu_0}(\phi),$$

i.e. for $\epsilon > 0$, there is a neighborhood $\mathcal{U}$ of μ_0 in $\mathcal{M}(\phi)$ such that $\mu \in \mathcal{U}$ implies $h_{\mathrm{pre},\mu}(\phi) \le h_{\mathrm{pre},\mu_0}(\phi) + \epsilon$.

LEMMA 4.3. *The preimage entropy map of ϕ is affine, i.e., for any $\mu, \nu \in \mathcal{M}(\phi)$, $p \in [0, 1]$, we have*

$$h_{\mathrm{pre}, p\mu + (1-p)\nu}(\phi) = p h_{\mathrm{pre},\mu}(\phi) + (1 - p) h_{\mathrm{pre},\nu}(\phi).$$

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Proof. By Proposition 1.4, it is suffices to prove that for any $\omega \in \Omega$ and finite partition ξ of X,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}H_{(p\mu+(1-p)\nu)_\omega}\left(\bigvee_{i=0}^{n-1}\phi^{-1}(i,\omega)\xi \;\middle|\; \mathcal{B}_\omega^-\right)&= p\lim_{n\to\infty}\frac{1}{n}H_{\mu_\omega}\left(\bigvee_{i=0}^{n-1}\phi^{-1}(i,\omega)\xi \middle| \mathcal{B}_\omega^-\right)\\&\quad+(1-p)\lim_{n\to\infty}\frac{1}{n}H_{\nu_\omega}\left(\bigvee_{i=0}^{n-1}\phi^{-1}(i,\omega)\xi \;\middle|\; \mathcal{B}_\omega^-\right).\end{aligned}\tag{4.1}$$

Along the line of the proof of [**4**, Theorem 2.3], we can prove (4.1) easily, so we omit the details. □

By Lemma 4.3 and Theorem 3.1, we can get the following proposition along the line of the proof of [**3**, Theorem 3.1.6].

PROPOSITION 4.4. *If $h_{\mathrm{pre}}(\phi) < \infty$ and $\mu_0 \in \mathcal{M}(\phi)$, then*

$$h_{\mathrm{pre},\mu_0}(\phi) = \inf\left\{P_{\mathrm{pre}}(\phi, f) - \int f\,d\mu_0 \;\middle|\; f \in L^1(\Omega, C(X))\right\}$$

if and only if $h_{\mathrm{pre},\cdot}(\phi)$ is upper semi-continuous at $\mu_0 \in \mathcal{M}(\phi)$.

In the following, we give some applications of preimage pressure $P_{\mathrm{pre}}(\phi, \cdot)$ to equilibrium states.

$\mu \in \mathcal{M}(\phi)$ is called *an equilibrium state for f* if

$$P_{\mathrm{pre}}(\phi, f) = h_{\mathrm{pre},\mu}(\phi) + \int f\,d\mu.$$

Let $\mathcal{M}(\phi, f)$ denote the collection of all equilibrium states for f.

PROPOSITION 4.5. *Let $f \in L^1(\Omega, C(X))$. Then the following hold.*

(1) *$\mathcal{M}(\phi, f)$ is convex.*
(2) *The extreme points of $\mathcal{M}(\phi, f)$ are precisely the ergodic members of $\mathcal{M}(\phi, f)$.*
(3) *If the preimage entropy map is upper semi-continuous then $\mathcal{M}(\phi, f)$ is compact and has an ergodic equilibrium state.*
(4) *Assume $f, g \in L^1(\Omega, C(X))$ are cohomologeous, i.e. $f = g + u - u\circ\Theta - c$ for some $c \in L^1(\Omega, \mathbf{P})$ and $u \in L^1(\Omega, C(X))$. Then f and g have the same equilibrium states, and*

$$P_{\mathrm{pre}}(\phi, f) = P_{\mathrm{pre}}(\phi, g) - \int c\,d\mathbf{P}.$$

Proof. For each $\nu \in \mathcal{M}(\phi)$, $f \in L^1(\Omega, C(X))$ define $L(\phi, \cdot) : \mathcal{M}(\phi) \to \mathbb{R}$ by

$$L(\phi, \nu) = h_{\mathrm{pre},\nu}(\phi) + \int f\,d\nu.$$

By Lemma 4.3, $L(\phi, \cdot)$ is affine. Moreover, if the preimage entropy map is upper semi-continuous, then so is $L(\phi, \cdot)$.

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(1) By Lemma 4.3, the preimage entropy map is affine, so $\mathcal{M}(\phi, f)$ is convex.

(2) If $\mu \in \mathcal{M}(\phi, f)$ is ergodic then it is an extreme point of $\mathcal{M}(\phi)$ [**3**, p. 19] and hence of $\mathcal{M}(\phi, f)$.

Now suppose $\mu \in \mathcal{M}(\phi, f)$ to be an extreme point of $\mathcal{M}(\phi, f)$ and $\mu = p\mu_1 + (1-p)\mu_2$ for some $\mu_1, \mu_2 \in \mathcal{M}(\phi)$, $p \in [0, 1]$. Then since

$$pL(\phi, \mu_1) + (1-p)L(\phi, \mu_2) = L(\phi, \mu) = P_{\text{pre}}(\phi, f)$$

(Lemma 4.3) and $L(\phi, \mu_1)$, $L(\phi, \mu_2) \le P_{\text{pre}}(\phi, f)$ (Theorem 3.1), we must have $\mu_1, \mu_2 \in \mathcal{M}(\phi, f)$. Hence $\mu_1 = \mu_2 = \mu$ and μ is an extreme point of $\mathcal{M}(\phi)$ and hence ergodic.

(3) By the upper semi-continuity of the preimage entropy map, $\mathcal{M}(\phi, f)$ is not empty and compact. Moreover, since $\mathcal{M}(\phi)$ is compact, $L(\phi, \cdot)$ attains its supremum at some ν. Let $\nu = \int_{\mathcal{E}_P(\phi)} m \, d\tau(m)$ be the ergodic decomposition of ν [**3**, Theorem 1.2.9]; we have

$$L(\phi, \nu) = \int_{\mathcal{E}_P(\phi)} L(\phi, m) \, d\tau(m),$$

because any affine upper semi-continuous functions is the limit of a decreasing sequence of affine continuous functions. Since $L(\phi, m) \le L(\phi, \nu)$ we must have $L(\phi, m) = L(\phi, \nu)$ for τ-almost all m. This proves the existence of an ergodic equilibrium state for f.

(4) This follows because for each $\mu \in \mathcal{M}(\phi)$ we have

$$h_{\text{pre},\mu}(\phi) + \int f \, d\mu = h_{\text{pre},\mu}(\phi) + \int g \, d\mu - \int c \, d\mathbf{P}. \qquad \square$$

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