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A convolution involving Bell polynomials

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Abstract. A convolution formula is established for Bell polynomials. This is expressed in seven equivalent ways and used to derive further properties of these polynomials. The application of these results to some twenty-seven special polynomial sets is shown and illustrated in the case of binomial, Hermite, Gegenbauer and generalized Bernoulli sets.

1. *The Bell polynomials.* In 1934 Bell(1) introduced the sequence of multi-variable polynomials $Y_0, Y_1(\alpha_1), Y_2(\alpha_1, \alpha_2)$, etc., where $\alpha_1, \alpha_2, \ldots$ are complex variables, $Y_0 = 1$ and $Y_s(\alpha_1, \ldots, \alpha_s)/s!$, $s = 1, 2, \ldots$, is the coefficient of z^s in the formal power series expansion of $\exp(\alpha_1 z + \alpha_2 z^2/2! + \ldots)$. In (2), 19.11(12) to (22), there is a brief discussion of a very closely related sequence, denoted here by $\{X_s(\alpha_1, \ldots, \alpha_s)\}$ (see section 7) and defined by (2·1) or (2·2) or (2·3) below. The connexion between the $\{X_s\}$ and $\{Y_s\}$ is given by 1·0.

If $\alpha_1, \alpha_2, \ldots$ is any sequence of complex variables, $n = 0, 1, \ldots$ and $0! = 1$, then

$$n! X_n(\alpha_1, \ldots, \alpha_n) = Y_n(0!\,\alpha_1, 1!\,\alpha_2, \ldots, (n-1)!\,\alpha_n). \tag{1·0}$$

Both the $\{X_s\}$ and $\{Y_s\}$ occur in probability theory, statistics and combinatorial analysis. This is exemplified by the fact that tables of both Y_s and $s!X_s$ (denoted by C_s) are given by Riordan (3) (p. 49 and p. 69) and tables of what is in fact Y_s are given by Kendall and Stuart (4) and (5). They occur in the algebra of field elements and in more abstract algebra ($s!X_s$ is the cycle indicator of the symmetric group). The X_s are a special case of Schurr functions (6). They have application to the construction of algorithms for computing coefficients in power series (products and inverse functions). They have some relevance to the theory of analytic functions and to the theory of numbers. Most of this work is hitherto unpublished.

It is the purpose of this paper to show that general properties of Bell polynomials exist such that, on specialization, properties of very many polynomial sets which have been introduced into mathematics are obtained, many of them being new properties.

On the whole the X_s seem to occur more frequently than the Y_s and so they are employed here. It is probably best to retain both forms.

2. *Elementary properties required and Bell's vector notation.* The sequence

$$\{X_s(\alpha_1, \ldots, \alpha_s)\},\ s = 1, 2, \ldots \quad \text{with} \quad X_0 = 1$$

may be defined in the following way. If $\alpha_1, \alpha_2, \ldots$ is any sequence of complex variables then

$$\exp\left\{\sum_{s=1}^{m} \frac{\alpha_s}{s} z^s\right\} = X_0 + \sum_{s=1}^{m} X_s(\alpha_1, \ldots, \alpha_s)\, z^s + R_m(z), \tag{2·1}$$

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where $m = 1, 2, \ldots$ and the remainder $R_m(z)$ is an entire function of z which is $O(z^{m+1})$ as $z \to 0$. This definition, involving $R_m(z)$, is one way of avoiding difficulty if $\Sigma\,(\alpha_s/s)\,z^s$ does not converge. It enables analytical methods to be used. The main results of this paper were found analytically by expressing X_n as a Cauchy integral using (2·1). But the proofs given here are algebraic and no longer.

The expression of many properties of the X_s and Y_s is greatly shortened by a notation (applicable to them both) introduced by Bell (1). The letter α without a suffix denotes a sequence $\{\alpha_s\}, s = 1, 2, \ldots$ of complex numbers and this holds similarly for the other Greek letters. The polynomial $X_s(\alpha_1, \ldots, \alpha_s)$ is then written as $X_s(\alpha)$ even though it is independent of α_t for $t > s$. When $s = 0$, $X_0(\alpha)$ may be written for X_0 (both are unity). Furthermore, if x is any complex number, $x\alpha$ and $\alpha + \beta$ denote the sequences $\{x\alpha_i\}$ and $\{\alpha_i + \beta_i\}, i = 1, 2, \ldots$ respectively and this notation is extended in an obvious way to any linear form in Greek symbols without suffices.

If both sides of (2·1) be differentiated with respect to z, the exponential function remaining then expanded by (2·1), the two series on the left multiplied by Cauchy's rule and coefficients of like powers identified, it follows, since m can be arbitrarily large, that

$$X_0(\alpha) = 1 \quad \text{and} \quad nX_n(\alpha) = \sum_{s=1}^{n} \alpha_s X_{n-s}(\alpha) \quad (n = 1, 2, \ldots). \tag{2·2}$$

Further, if (2·2) is taken as the definition of the X_s, any set of polynomials (X'_s say) defined by (2·1) must satisfy (2·2) and are easily shown by induction to be the X_s. Thus (2·2) implies (2·1).

If the left side of (2·1) is expressed as the product of m exponential functions and if these be expanded and the product formed by Cauchy's rule and coefficients of like powers identified then, since m can be arbitrarily large,

$$X_n(\alpha) = \Sigma \prod_{i=1}^{n}{}' \{(\alpha_i/i)^{s_i}/s_i!\} \quad (n = 0, 1, 2, \ldots), \tag{2·3}$$

where the sum is taken over all non-negative integers s_i satisfying $\sum_{i=1}^{n} i s_i = n$ and where the prime denotes that the product is unity when $n = 0$. An algebraic proof that (2·2) and (2·3) are equivalent is not difficult and any one of (2·1), (2·2) or (2·3) may be taken as the definition of Bell polynomials.

Expansion of the exponential functions by (2·1) in the identity

$$\exp\left\{\sum_{s=1}^{m} \frac{\alpha_s}{s} z^s\right\} \exp\left\{\sum_{s=1}^{m} \frac{\beta_s}{s} z^s\right\} = \exp\left\{\sum_{s=1}^{m} \frac{\alpha_s + \beta_s}{s} z^s\right\}$$

multiplication by Cauchy's rule and identification of coefficients gives the main addition theorem, since m can be arbitrarily large,

$$\sum_{s=0}^{n} X_s(\alpha)\, X_{n-s}(\beta) = X_n(\alpha + \beta) \quad (n = 0, 1, 2, \ldots). \tag{2·4}$$

These results were all found by Bell for his Y_s.

From (2·2) or (2·3) it follows that

$$X_n(x\alpha) = x\alpha_n/n + O(x^2) \quad \text{as} \quad x \to 0 \quad (n > 0) \tag{2·5}$$

and

$$X_n(\epsilon) = \delta_{n0} \quad (\text{Kronecker } \delta), \tag{2·6}$$

where ϵ is the null sequence ($\epsilon_s = 0, s = 1, 2, \ldots$).

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3. *Some equivalent convolution theorems.*

THEOREM 1. *If n is any integer, x any complex number such that the denominators in* (3·1) *do not vanish and γ is any sequence of complex numbers then*

$$\left.\begin{aligned}\sum_{s=0}^{n}\frac{X_s((s-x)\gamma)\,X_{n-s}((x-s)\gamma)}{x-s} &= 0 \quad \text{if} \quad n>0,\\ &= 1/x \quad \text{if} \quad n=0.\end{aligned}\right\} \tag{3·1}$$

The restriction on x may be dropped for $n > 0$ if the denominator is divided out algebraically. This is always possible by 2·5 for $n > 0$.

Proof. When $n = 0$ the left side of (3·1) is $1/x$ by (2·6).

When $n > 0$ let $t_1, \ldots, t_n$ be non-negative integers satisfying

$$\sum_{i=1}^{n} it_i = n. \tag{3·11}$$

Define $S(x, t_1, \ldots, t_n)$ by

$$S(x,t_1,\ldots,t_n) = \Sigma\left\{\prod_{i=1}^{n}(-1)^{r_i}\binom{t_i}{r_i}\right\}\left[x-\sum_{j=1}^{n}jr_j\right]^{t_1+\ldots+t_n-1}, \tag{3·12}$$

where the sum is taken over all integers $r_1, \ldots, r_n$ satisfying $0 \leqslant r_i \leqslant t_i$. By (3·11) the index in the last term is never negative and the term may therefore be expanded as the sum of terms in which the r_j only occur as factors of the form $r_1^{a_1} \ldots r_n^{a_n}$. The a_j are non-negative integers depending on which term is being considered but they always satisfy

$$a_1 + \ldots + a_n \leqslant t_1 + \ldots + t_n - 1$$

and hence for each term there is always an integer k such that $a_k < t_k$. Thus S itself consists of the sum of terms which each contain a factor of the form

$$\sum_{0\leqslant r_k\leqslant t_k}(-1)^{r_k}\binom{t_k}{r_k}r_k^{a_k}$$

and as is well known, and can easily be proved by expressing $r_k^{a_k}$ as a form in the factorials $(r_k)_{a_k-j}$, $j = 0, \ldots, a_k$ this vanishes when $a_k < t_k$. Thus

$$S(x, t_1, \ldots, t_n) = 0. \tag{3·13}$$

If both sides of (3·13) be multiplied by $(\gamma_1/_1)^{t_1} \ldots (\gamma_n/n)^{t_n}/t_1! \ldots t_n!$ and summed over all non-negative t_i satisfying (3·11) then

$$\sum_{3\cdot11}\sum_{0\leqslant r_i\leqslant t_i}\left\{\prod_{i=1}^{n}\frac{\left(\left(\sum_{j=1}^{n}jr_j-x\right)\gamma_i/i\right)^{r_i}\left(\left(x-\sum_{j=1}^{n}jr_j\right)\gamma_i/i\right)^{t_i-r_i}}{r_i!\,(t_i-r_i)!}\right\}\left(x-\sum_{j=1}^{n}jr_j\right)^{-1} = 0, \tag{3·14}$$

where the notation indicates the range of summation.

If new variables p_i are introduced given by $t_i = r_i + p_i$ then the summation in (3·14) becomes summation over all integers r_i and p_i such that $r_i \geqslant 0$ and $p_i \geqslant 0$ and

$$\Sigma i(r_i + p_i) = n.$$

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Furthermore, if the summands in (3·14) are first grouped and summed over those summands for which Σjr_j has the same value, s say, (3·14) may be written

$$\sum_{s=0}\sum_{A_s}\sum_{B_s}\left\{\prod_{i=1}^{n}\frac{((s-x)\gamma_i/i)^{r_i}}{r_i!}\frac{((x-s)\gamma_i/i)^{p_i}}{p_i!}\right\}(x-s)^{-1}=0, \tag{3·15}$$

where A_s and B_s denote summation over all r_i and p_i satisfying

$$\Sigma ir_i = s \quad \text{and} \quad \Sigma ip_i = n-s$$

respectively. The inner double sum can be factorized to give

$$\sum_{s=0}^{n}\left\{\sum_{A_s}\prod_{i=1}^{n}{}'\frac{((s-x)\gamma_i/i)^{r_i}}{r_i!}\right\}\left\{\sum_{B_s}\prod_{j=1}^{n}{}'\frac{((x-s)\gamma_j/j)^{p_j}}{p_j!}\right\}(x-s)^{-1}=0.$$

By (A_s), γ_i for $i > s$ does not occur in the first product and so the n above the first symbol may be replaced by s and similarly (by (B_s)) by $n-s$ in the second product. Application of the explicit form (2·3) then completes the proof of Theorem 1.

THEOREM 2. *If n is any non-negative integer, x any complex number such that the denominators in* (3·2) *do not vanish and β and γ are any sequences of complex numbers then*

$$\sum_{s=0}^{n}\frac{X_s((s-x)\gamma)\,X_{n-s}(\beta-s\gamma)}{x-s}=\frac{X_n(\beta-x\gamma)}{x}. \tag{3·2}$$

Proof. Replace n by m in (3·1), multiply both sides by $X_{n-m}(\beta-x\gamma)$ and sum from $m=0$ to $m=n$. The right side becomes the right side of (3·2). If $a_s = X_s((s-x)\gamma)/x-s$, $b_{m-s} = X_{m-s}((x-s)\gamma)$ and $c_{n-m} = X_{n-m}(\beta-x\gamma)$ then the left side in (3·2) can be found by substitution in the identity

$$\sum_{m=0}^{n}c_{n-m}\sum_{s=0}^{m}a_s b_{m-s}=\sum_{s=0}^{n}a_s\left\{\sum_{r=0}^{n-s}b_r c_{n-s-r}\right\} \tag{3·21}$$

and application of (2·4) to the sum in braces. This establishes Theorem 2.

Theorem 1 is recovered from Theorem 2 by taking β as $x\gamma$ and using (2·6).

THEOREM 3. *Under the same conditions as Theorem 2 and if α is any sequence of complex numbers*

$$\sum_{s=0}^{n}\frac{X_s(\alpha+s\gamma)\,X_{n-s}(\beta-s\gamma)}{x-s}=\sum_{s=0}^{n}\frac{X_s(\alpha+x\gamma)\,X_{n-s}(\beta-x\gamma)}{x-s}. \tag{3·3}$$

Proof. By a change of notation, s for $n-s$, $n-x$ for x, α for $\beta-n\gamma$, (3·2) becomes

$$\sum_{s=0}^{n}\frac{X_s(\alpha+s\gamma)\,X_{n-s}((x-s)\gamma)}{x-s}=\frac{X_n(\alpha+x\gamma)}{x-n}. \tag{3·31}$$

The same method as used in proving Theorem 2 can now be used. Replace n by m in (3·31), multiply both sides by $X_{n-m}(\beta-x\gamma)$ and sum from $m=0$ to n. The right side becomes the right side of (3·3). If a_s be taken as $X_s(\alpha+s\gamma)/x-s$ and b_{m-s} and c_{n-m} as before then the left side of (3·3) can be found by substitution in (3·21) and applying (2·4) to the sum in braces. This establishes Theorem 3. Theorem 2 is recovered from Theorem 3 by taking α as $-x\gamma$ and using (2·6).

It will be noted that (3·31) can be found from (3·3) by taking β as $x\gamma$ and using (2·6).

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THEOREM 4. *If α, β and γ are any sequences of complex numbers and n any non-negative integer, all independent of x, then*

$$\lim_{x\to\infty} \sum_{s=0}^{n} \frac{X_s(\alpha+x\gamma)\, X_{n-s}(\beta-x\gamma)}{x-s} = 0. \tag{3·4}$$

This is an obvious consequence of Theorem 3 but Theorem 3 may also be recovered from it in the following way. It is easy to see from (2·2) or (2·3) that $X_s(\alpha+x\gamma)$ and $X_{n-s}(\beta-x\gamma)$ are both polynomials in x. By division it therefore follows that, for $s = 0, 1, \ldots, n$

$$\frac{X_s(\alpha+x\gamma)\, X_{n-s}(\beta-x\gamma)}{x-s} = \frac{X_s(\alpha+s\gamma)\, X_{n-s}(\beta-s\gamma)}{x-s} + \psi_s(x)$$

where, by the remainder theorem, $\psi_s(x)$ is a polynomial in x. Summation gives

$$\sum_{s=0}^{n} \frac{X_s(\alpha+x\gamma)\, X_{n-s}(\beta-x\gamma)}{x-s} = \sum_{s=0}^{n} \frac{X_s(\alpha+s\gamma)\, X_{n-s}(\beta-s\gamma)}{x-s} + \psi(x), \tag{3·41}$$

where $\psi(x)$ is a polynomial in x. If α, β, γ and n are taken independent of x and 3·4 is assumed to be true then when x tends to infinity in (3·41) the left side exists (by 3·4) and is zero. The limit of the summation term on the right is plainly zero and so

$$\lim \psi(x) = 0.$$

Since $\psi(x)$ is a polynomial it must be identically zero and (3·41) reduces to (3·3). The condition that α, β, γ and n be independent of x may be dropped since (3·3) is an algebraic relation. Thus theorem 3 can be recovered from Theorem 4. Theorem 3 can be used to derive limit forms other than (3·4). For example

$$\lim_{x\to\infty} \sum_{s=0}^{n} \frac{x}{x-s} X_s(\alpha+(s-x)\gamma)\, X_{n-s}(\beta-(s-x)\gamma) = X_n(\alpha+\beta), \tag{3·5}$$

where n, α, β and γ are independent of x.

This follows by applying (3·3) to the sum on the left. The limit is then easily taken and (2·4) gives the right side.

Similarly, if n, α, β and γ' are independent of y,

$$\lim_{y\to\infty} \sum_{s=1}^{n} \frac{X_s(\alpha+sy\gamma')\, X_{n-s}(\beta-sy\gamma')}{sy-1} = X_n(\beta) - X_n(\beta-\gamma'). \tag{3·6}$$

This follows by replacing x by $1/y$ and γ by $y\gamma'$ in (3·3) to give

$$\sum_{s=1}^{n} \frac{X_s(\alpha+sy\gamma')\, X_{n-s}(\beta-sy\gamma')}{1-sy} = -X_n(\beta) + X_n(\beta-\gamma') + \sum_{s=1}^{n} \frac{X_s(\alpha+\gamma')\, X_{n-s}(\beta-\gamma')}{1-sy}, \tag{3·61}$$

where the two summands corresponding to $s = 0$ have been detached and put on the right side. When $y \to \infty$ (3·61) gives (3·6).

As in the case of Theorem 4, Theorem 3 can be recovered from (3·5) or (3·6) by using the remainder theorem.

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Theorems 1, 2, 3 and 4 and (3·31), (3·5) and (3·6) are thus convertible into one another using the addition Theorem (2·4) where necessary. All these results and those of the next section may be expressed, by (1·0), in terms of Y's instead of X's.

4. *Some applications to Bell polynomials.* Theorem 3 can be used to derive several properties of the X_s themselves. Thus if γ is taken as $-(1/x)\alpha$ in (3·3), both sides multiplied by x, which is then made to tend to ∞, then (2·4) arises which is thus seen as a limiting case.

A more novel application is the following result.

THEOREM 5. *If n is any non-negative integer, x any complex number such that*

$$x(x+n) \neq 0,$$

γ any sequence and δ the sequence such that $\delta_s = X_s(s\gamma)$, $s = 1, 2, \ldots$ then

$$\frac{X_n((x+n)\gamma)}{x+n} = \frac{X_n(x\delta)}{x}. \tag{4·1}$$

Proof. An application of Theorem 3 shows that

$$n\left\{x\frac{X_n((x+n)\gamma)}{x+n}\right\} = \sum_{s=1}^{n} xX_s(s\gamma)\left\{x\frac{X_{n-s}((x+n-s)\gamma)}{x+n-s}\right\}, \tag{4·11}$$

where the term corresponding to $s = 0$ has been incorporated in the left side. If (4·11) is compared with the recurrence relation, (2·2) (which is one definition of the Bell polynomials), and it is noted that the function in braces is unity when the suffix of X is zero, it will be observed that this function must be a Bell polynomial and (4·1) follows.

So much can be derived from Theorem 5 that a separate paper will be devoted to it.

Thirdly, if $n > 0$ and β is taken null, the term corresponding to $s = 0$ on the left in (3·3) is zero by (2·6) and (3·3) gives

$$\sum_{s=1}^{n} \frac{X_s(\alpha+s\gamma)\,X_{n-s}(-s\gamma)}{s-x} = \frac{X_n(-x\gamma)}{-x} + \frac{X_n(\alpha+x\gamma)}{n-x} + \sum_{s=1}^{n-1} \frac{X_s(\alpha+x\gamma)\,X_{n-s}(-x\gamma)}{s-x}$$

$$= \frac{X_n(-x\gamma)}{-x} + \frac{X_n(\alpha+x\gamma)}{n-x} + O(x)$$

as $x \to 0$, by (2·5). Use of (2·5) again and putting $x = 0$ gives

$$\sum_{s=1}^{n} \frac{X_s(\alpha+s\gamma)\,X_{n-s}(-s\gamma)}{s} = \frac{X_n(\alpha)+\gamma_n}{n}. \tag{4·2}$$

5. *Applications to some special polynomial sets which are special Bell polynomials.* Suppose k and r are integers, $k \leqslant r$ and $\{p_s(y_1, \ldots, y_r)\}, s = 1, 2, \ldots$, is a sequence of polynomials in the variables or parameters $y_1, \ldots, y_r$. Very many polynomial sets have been introduced with generating functions of the form

$$\phi_0(z) \prod_{i=1}^{k} [\phi_i(z)]^{y_i} = \sum_{s=0}^{\infty} p_s(y_1, \ldots, y_r)\, z^s \tag{5·0}$$

in which the $\phi_i(z)$ are functions which may depend on $y_{k+1}, \ldots, y_r$ but are independent of $y_1, \ldots, y_k$. Furthermore, the $\phi_i(z)$ are analytic functions of x in some neighbourhood

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of $z = 0$ and satisfy $\phi_0(0) = \ldots = \phi_k(0) = 1$. These last conditions imply that, since $d/dz \log \phi_i(z)$ exists everywhere in some neighbourhood of $z = 0$, $\log \phi_i(z)$ is analytic in this region. Hence there exist $k+1$ sequences $\xi^1, \ldots, \xi^k$ and ζ independent of $y_1, \ldots, y_k$ and such that

$$\left.\begin{aligned} \log \phi_0(z) &= \sum_{s=1}^{\infty} \zeta_s z^s/s, \\ \log \phi_i(z) &= \sum_{s=0}^{\infty} \xi_s^i z^s/s \end{aligned}\right\} \tag{5·01}$$

for $i = 1, \ldots, k$, where the radii of convergence are not zero.

From (5·0) and (5·01) it follows that

$$\exp\left\{\sum_{s=1}^{\infty} (\zeta_s + y_1\xi_s^1 + \ldots + y_k\xi_s^k) z^s/s\right\} = \sum_{s=0}^{\infty} p_s(y_1, \ldots, y_r) z^s, \tag{5·02}$$

where the series converge for sufficiently small $|z| > 0$.

If the infinite series $\Sigma\alpha_s z^s/s$ has a finite radius of convergence (2·1) may be replaced by

$$\exp\left\{\sum_{s=0}^{\infty} \frac{\alpha_s}{s} z^s\right\} = \sum_{s=0}^{\infty} X_s(\alpha) z^s \tag{5·03}$$

for sufficiently small $|z| > 0$.

If α_t is taken as $\zeta_t + y_1\xi_t^1 + \ldots + y_k\xi_t^k$ for $t = 1, 2, \ldots$ then identification of coefficients of like powers of s in (5·02) and (5·03) gives

$$p_s(y_1, \ldots, y_r) = X_s(\zeta + y_1\xi^1 + \ldots + y_k\xi^k). \tag{5·1}$$

Thus all polynomial sets p_s with generating functions given by the left side of (5·0), or the equivalent (5·02), are Bell polynomials in a sequence of linear functions of $y_1, \ldots, y_k$.

Some well known polynomial sets are so defined that p_s must be replaced by $p_s/s!$ in (5·0), (5·02) and (5·1).

When $p_s(y_1, \ldots, y_r)$ is such that ζ is the null sequence then Theorems 1, 2, 3 and 4 and (3·31), (3·5), (3·6) and (4·2) can all be specialized to give a property of p_s.

The simplest cases are when $r = k = 1$ and ζ is null, so that $p_s(y_1) = X_s(y_1\xi^1)$. Some polynomials of this type are $y_1^s/s!$, $\binom{y_1}{s}$ (binomial), $y_1(y_1 - s)^{s-1}/s!$ (Abel), Mittag–Leffler polynomials (2), 19·6(22), Mahler polynomials (2), 19·7(40), and some polynomials used by Mott in the theory of electrons (2), 19·7(14). The Gegenbauer polynomials $C_s^{y_1}(y_2)$ are also of the type $X_s(y_1\xi^1)$ but ξ^1 depends on y_2.

As an example, let ξ be that sequence which specializes $X_s(y_1\xi)$ into $y_1^s/s!$ (it is only necessary to know that ξ exists, not that $\xi_1 = 1$ and $\xi_i = 0, i > 1$) and γ is taken as $-\xi$ and β as $y\xi$ in Theorem 2 (3·2). It follows that

$$\sum_{s=0}^{n} \binom{n}{s} (x-s)^{s-1}(s-y)^{n-s} = (x-y)^n/x, \tag{5·2}$$

which is a theorem of Abel of which the binomial theorem is a limiting form.

The Gegenbauer polynomials have the generating function

$$(1 - 2\cos\theta z + z^2)^{-y_1} = \exp\left(2y_1 \sum_{s=1}^{\infty} \cos(s\theta) z^2/s\right) = \sum_{s=0}^{\infty} C_s^{y_1}(\cos\theta) z^s,$$

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and so $$C_s^{y_1}(y_2) = X_s(2y_1T_1(y_2), \ldots, 2y_1T_s(y_2)), \tag{5·21}$$

where the T_s are Chebyshev polynomials. Thus if ξ^1 is taken as $\{2T_i(x)\}$ and α and γ taken as $u\xi^1$ and $v\xi^1$ respectively in (4·2) then, for $n = 1, 2, \ldots$ and all complex u, v and x

$$\sum_{s=1}^{n} C_s^{u+sv}(x)\, C_{n-s}^{-sv}(x)/s = [C_n^u(x) + 2vT_n(x)]/n. \tag{5·3}$$

Another class of sets of special polynomials are those when ζ is null and $r = k = 2$ in (5·1) and are thus of the form $X_s(y_1\xi^1 + y_2\xi^2)$. These include $y_2^{\frac{1}{2}s}H_s(y_1/\sqrt{y_2})/s!$ [H_s is the Hermite polynomial. The generating function has been put in the form

$$\exp(2y_1z - y_2z^2)], \quad L_s^{y_1-1}(y_2)$$

(generalized Laguerre), $B_s^{y_1}(y_2)/s!$ and $E_s^{y_1}(y_2)/s!$ (generalized Bernoulli and Euler) (2), (19·7) (30) et seq. and polynomials introduced by Toscano (2) 19·7(41) and Tricomi (2) 19·7(53), In addition Meixner (2), 10·24(13), and Pollaczek (2), 10·22(19), have introduced polynomials which may be expressed in the form $X_s(y_1\xi^1 + y_2\xi^2)$ where the ξ's depend on a third variable. As in the previous class, when only one ξ was present, Theorems 1, 2, 3 and 4 and (3·31), (3·5), (3·6) and (4·2) can all be specialized to give properties of these eight polynomial sets. They may when required, as in Theorem 1, be reduced to the previous form $X_s(y\xi)$ by putting $y_1 = yu$ and $y_2 = yv$. Thus, for example, Theorem 1 gives

$$\sum_{s=0}^{n} \binom{n}{s} B_s^{(s-x)u}((s-x)v)\, B_{n-s}^{(x-s)u}((x-s)v)/(x-s) = 0, \tag{5·4}$$

for the generalized Bernoulli polynomials, where $n > 0$, u, v and x are any complex numbers such that the $x-s$ do not vanish (unless they are first divided into the numerators algebraically).

A third class of sets of special polynomials are those for which ζ is not the null sequence. Some of the results given here are not applicable in this case. If $p_s(y_1)$ is a polynomial of form (5·1) with ζ not null then (2·4) does not give $\Sigma p_s(u)\, p_{n-s}(v)$ as a polynomial of the same set since ζ is doubled on the right side of (2·4). However, Theorems 3 and 4 and (3·6) can still be applied.

Polynomial sets in this class are the Bernoulli and Euler, Bernoulli and Euler of higher order (2), 1·13·1(30) and 1·14·1(22), Jacobi, simple Laguerre, $H_s(y_1)/s!$ (where H_s are the Hermite polynomials), $H_s(\sqrt{y_1}u)\, H_s(\sqrt{y_2}, v)/s!$, (7), 111(2), Appell (2), 19·3(2), generalized Appell (2) 19·3(13), $p_s(y_1)/(y_2^s s!)^{\frac{1}{2}}$ (where p_s are the Poisson–Charlier polynomials (2) 19·3(2)) and the Bessel polynomials (2), 19·7(18).

As an application of the present results to a polynomial set of this class consider the Hermite polynomials. The usual generating function shows $H_s(y_1)/s! = X_s(y_1\xi + \zeta)$ where $\xi_s = 2\delta_{1s}$ and $\zeta_s = -\delta_{2s}$ (Kronecker δ). If α is taken as the sequence $u\xi + \zeta$, β as $w\xi + \zeta$ and γ as $v\xi$ then Theorem 4 gives

$$\lim_{x\to\infty} \sum_{s=0}^{n} \binom{n}{s} \frac{H_s(u+vx)\, H_{n-s}(w-vx)}{x-s} = 0, \tag{5·5}$$

which holds for $n = 0, 1, \ldots$ and complex u, v and w independent of x.

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The Legendre $\{P_s(x)\}$ and Chebyshev polynomials $\{T_s(x)\}$ and $\{U_s(x)\}$, in the usual notation, are not of the required form (5·1) except in a vacuous sense with the ξ's null and ζ not null and the theorems do not apply. However it seems worth noting here that

$$P_n(\cos\theta) = X_n(\cos\theta, \cos 2\theta, \ldots, \cos n\theta) \tag{5·6}$$

follows from (5·21) with $y_1 = \frac{1}{2}$ and $y_2 = \cos\theta$ and

$$U_n(x) = X_n(2T_1(x), 2T_2(x), \ldots, 2T_n(x)) \tag{5·7}$$

is easily derived from the usual generating functions and (2·1), (2·2) and (2·3) do give properties of P_n and U_n.

6. *Some particular applications.* The identities of section 5 (5·2), (5·3), (5·4) and (5·5), are intended to be merely typical of the application of the formulae of sections 3 and 4 to special polynomial sets. It is often possible, however, to make use of some special property of a particular set to sum the series on the right in (3·3) or simplify one of the other formulae.

For example the binomial polynomials $\binom{c}{s}$ vanish when c is a positive integer and $s > c$. Thus if a and b are any non-negative integers such that $n > a+b$, u and v any complex numbers such that $u+sv \neq 0$ for $s = 0, \ldots, n$ and α, β and γ are the sequences $(a+u)\,\delta$, $(b-u)\,\delta$ and $v\delta$ respectively where $\delta_r = (-1)^{r+1}$, $r = 1, 2, \ldots$ then the X_s in (3·3) become binomial polynomials and the right side vanishes because one binomial polynomial in each term must be zero (since $n > a+b$). It follows that

$$\sum_{s=0}^{n} \frac{1}{u+sv}\binom{a+u+sv}{s}\binom{b-u-sv}{n-s} = 0 \tag{6·1}$$

for $n > a+b$.

Another property of the $\binom{c}{s}$ is that

$$\frac{1}{s+1}\binom{c}{s} = \frac{1}{c+1}\binom{c+1}{s+1}. \tag{6·11}$$

Thus, if u, v and w are any complex numbers such that $u+1 \neq v$, $x = -1$ and α, β and γ are the sequences $u\delta$, $v\delta$ and $w\delta$ respectively, the right side of (3·3) can be summed using (6·11) and Vandermonde's convolution (which is a special case of (2·4)) to give

$$\sum_{s=0}^{n}\binom{u+sv}{s}\binom{w-sv}{n-s}(s+1)^{-1} = \left[\binom{u+w+1}{n+1} - \binom{w+v}{n+1}\right](u-v+1)^{-1}. \tag{6·2}$$

If α is any sequence in which all the odd terms are zero it follows from (2·1) or (2·2) that $X_s(\alpha)$ vanishes for odd s. Thus if α, β, γ and x are such that all odd terms in the sequences $\alpha + x\gamma$ and $\beta - x\gamma$ vanish then the right side of (3·3) vanishes for all odd n. An example of this is that for all complex u and v such that $u - sv \neq 0$ for $s = 0, 1, \ldots, n$ and all odd n,

$$\sum_{s=0}^{n}\binom{n}{s}\frac{1}{u-sv}H_s(u-sv)\,H_{n-s}(sv-u) = 0, \tag{6·3}$$

by Theorem 3.

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7. *Choice of notation.* Although the polynomials X_s studied here are exactly the same as the B_s of the Bateman project authors, (2) 19·11(15) and (16), their symbol B (doubtless adopted in honour of E. T. Bell) has been abandoned. This is simply to avoid confusion with Bernoulli numbers and polynomials, generalized Bernoulli numbers and polynomials (2), 19·7 (30) et seq., and Bernoulli numbers and polynomials of higher order (2), 1·13·1(30) which all involve the symbol B with a suffix. These have numerous relationships with the X_s. In addition to such identities as (5·4), probably the simplest way of computing higher Bernoulli numbers and polynomials is to regard them as Bell polynomials and use (2·2). Furthermore very many functions involving circular functions ($\sqrt{(1-\cos z)}\cot z$ for example) have power series expansions in which the coefficients are Bell polynomials of multiples of Bernoulli numbers.

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