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# Differential equation for Hermite–Bell polynomials

BY M. W. KALINOWSKI AND M. SEWERYŃSKI
*Institute of Philosophy and Sociology, Polish Academy of Sciences, Warsaw and Institute of Nuclear Research, Warsaw*

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*Abstract*

An ordinary differential equation for Hermite–Bell polynomials is derived. Bell's error concerning orthogonality of his polynomials is corrected.

## 1. *Introduction*

In 1934 E. T. Bell (1) introduced generalized Hermite polynomials $H_{n,N}(p,z)$(observe that $H_{n,N}(-p,z) = (-1)^n \xi_n(p,z;N)$ in Bell's notation):

$$H_{n,N}(p,z) = (-1)^n \exp(pz^N) D^n \exp(-pz^N) \quad n = 0,1,2,\ldots, \quad N = 2,3,4,\ldots, \quad (1{\cdot}1)$$

where $D^n = \partial_z^n$. We will call the $H_{n,N}(p,z)$ Hermite–Bell polynomials (HB polynomials) hereafter. Curiously enough, HB polynomials remained almost unnoticed until Riordan demonstrated their usefulness in some problems of Combinatorial Analysis, Number theory and Mathematical Statistics (see e.g. (5–7)). Bell has proven that his polynomials satisfy no linear differential equation in $z$ of constant order, the coefficients of which are polynomials in $N$, $z$, $p$ and $n$, provided $N > 2$ (Truesdell observed that $H_{n,N}(-p,z)\exp(pz^N)$ is a solution of the $F$-equation ((10), p. 23). However, the Bell statement does not settle a question whether his polynomials $H_{n,N}$ obey a differential equation of order $N$. And the aim of this note is to derive an ordinary differential equation in $z$ of order $N$ which $H_{n,N}(p,z)$'s obey. Since the order of such an equation is not constant (it depends on $N$), our results do not contradict the relevant Bell statement. We will demonstrate also that HB polynomials are not orthogonal with weight $\exp(-pz^N)$, $N$ even, at the interval $-\infty < z < +\infty$ contrary to Bell's claim ((1), p. 262).

The following basic notation is used throughout the paper. The set of all real numbers is denoted by $R^1$; the set of all non-negative integers is denoted by $N_0^\infty$; a set $\{N_0 - \{0,1,\ldots,k-1\}\}$ is denoted by $N_k^\infty$. The set of all complex numbers is denoted as usually by $\mathbb{C}$. We often employ the contracted notation for generalized hypergeometrical function and write it in either of the forms

$$ {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\middle| z\right) = {}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} z^n/(1)_n, \quad (1{\cdot}2)$$

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where $(v)_c = \Gamma(v+c)/\Gamma(v)$ and the $\Gamma$-function is defined as usual:

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_1^{\infty} dt\, t^{z-1} e^{-t}. \tag{1·3}$$

A binomial coefficient for complex $(a, b)$ is defined as

$$\binom{a}{b} = \Gamma(a+1)/\Gamma(a+1-b)\,\Gamma(b+1). \tag{1·4}$$

When no confusion can arise, we simply refer to (1·2) as ${}_pF_q$. Similar notation is employed also for the Meijer $G$-function $G_{p,q}^{m,n}$ and we follow Luke (2) as closely as possible.

## 2. *Differential equation*

We define a function

$$H_{\alpha,N}(p,z) = e^{i\pi\alpha}[\exp(pz^N)]\, D_z^{\alpha} \exp(-pz^N) \quad \operatorname{Re}\alpha \geqslant 0,\ N \in N_1^{\infty}, \tag{2·1}$$

where $D_z^{\alpha}$ is a derivative of complex order $\alpha$ with respect to $z$ (see for example (3, 4, 8)).

Our derivation of a differential equation for $H_{n,N}(p,z)$ is based on an observation that $H_{\alpha,N}(p,z)$ may be expressed in terms of a generalized hypergeometrical function ${}_NF_N$ and some elementary functions. Namely,

$$H_{\alpha,N}(p,z) = e^{i\pi\alpha} z^{-\alpha} \frac{\exp(pz^N)}{\Gamma(1-\alpha)}\, {}_NF_N\left(\begin{matrix} a_N \\ b_N \end{matrix}\middle| -pz^N\right) \quad \operatorname{Re}\alpha \geqslant 0, \tag{2·2}$$

where

$$a_k = k/N, \quad b_k = (k-\alpha)/N, \quad k = 1, 2, \ldots, N.$$

Equation (2·2) follows from (2·1) and the equality

$$\begin{aligned} D_z^{\alpha} \exp(-pz^N) &= D_z^{\alpha} \sum_{n=0}^{\infty} (-p)^n z^{nN}/(1)_n = \sum_{n=0}^{\infty} (-p)^n D_z^{\alpha} z^{nN}/(1)_n \\ &= \sum_{n=0}^{\infty} (-p)^n \frac{\Gamma(nN+1)}{\Gamma(nN+1-\alpha)} z^{nN-\alpha}/(1)_n \\ &= N^{\alpha} z^{-\alpha} \left[\prod_{r=1}^{N} \Gamma(r/N)\, \Gamma\left(\frac{r-\alpha}{N}\right)\right] \cdot {}_NF_N\left(\begin{matrix} a^N \\ b_N \end{matrix}\middle| -pz^N\right) \\ &= \frac{z^{-\alpha}}{\Gamma(1-\alpha)}\, {}_NF_N\left(\begin{matrix} a_N \\ b_N \end{matrix}\middle| -pz^N\right) \end{aligned} \tag{2·3}$$

which may be obtained with help of the Gauss–Legendre multiplication formula for the $\Gamma$-function (see, for example, (2), 2·3, (1)). It follows also from (2·2) that

$$H_{\alpha,N}(p,z) = e^{i\pi\alpha} N^{\alpha} z^{-\alpha} [\exp(pz^N)]\, G_{N,N+1}^{1,N}\left(pz^N \middle| \begin{matrix} 1-a_N \\ 0, 1-b_N \end{matrix}\right). \tag{2·4}$$

Now we will prove that the functions $h_{\alpha,N}(p,z) \equiv \exp(-pz^N) H_{\alpha,N}(p,z)$ obey the differential equation

$$z^{N+\alpha}\left[D_z^N h_{\alpha,N} + pN\,\Gamma(N+\alpha) \sum_{n=0}^{N-1} \binom{N-1}{n} \frac{z^n}{\Gamma(\alpha+n+1)} D_z^n h_{\alpha,N}\right] = e^{i\pi\alpha}/\Gamma(1-\alpha-N). \tag{2·5}$$

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*Proof.* Let $\mathscr{E} < \operatorname{Re}\alpha < 1-\eta$, $\mathscr{E} > 0$, $0 < \eta < 1$ then the function

$$\Gamma^{-1}(1-\alpha)\,{}_NF_N\left(\begin{matrix}a_N\\ b_N\end{matrix}\middle| -pz^N\right),$$

is an analytic function of the complex variable $\alpha$ and obeys the differential equation

$$\Gamma^{-1}(1-\alpha)\left\{\prod_{k=1}^{N}\left[z^N\frac{d}{d(z^N)}+b_k-1\right]+pz^N\prod_{k=1}^{N-1}\left[z^N\frac{d}{d(z^N)}+a_k\right]\right\}$$
$$\times\,{}_NF_N\left(\begin{matrix}1, a_1, \ldots, a_{N-1}\\ b_1, \ldots, \ldots, b_N\end{matrix}\middle| -pz^N\right) = \Gamma^{-1}(1-\alpha)\prod_{k=1}^{N}(b_k-1) \qquad (2\cdot6)$$

which may be deduced from the general differential equation for ${}_pF_q$

$$\left[zD\prod_{k=1}^{q}(zD+b_k-1)-z\prod_{k=1}^{p}(zD+a_k)\right]{}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\middle| z\right) = 0, \qquad (2\cdot7)$$

when a numerator parameter is unity by integrating equation (2·7) and then evaluating the constant of integration by putting $z = 0$ (2). Taking into account that

$$z^N d/d(z^N) = (z/N)\,d/dz \quad \text{and} \quad \Gamma^{-1}(1-\alpha)\,{}_NF_N\left(\begin{matrix}a_N\\ b_N\end{matrix}\middle| -pz^N\right) = e^{-i\pi\alpha}z^\alpha h_{\alpha,N}(p,z)$$

(see (2·2)) one may rewrite (2·6)

$$\left[\prod_{k=1}^{N}(zD+k-N-\alpha)+pNz^N\prod_{k=2}^{N}(zD+k-1)\right][z^\alpha h_{\alpha,N}(p,z)] = e^{i\pi\alpha}/\Gamma(1-N-\alpha). \qquad (2\cdot8)$$

For

$$\prod_{r=1}^{n}(zD+a+r-1)f(z) = z^{1-a}D^n z^{a+n-1}f(z)$$

(see, for example, (2), 2·9, (4)) then equation (2.8) may be rewritten in the form (2·5). Since the right-hand side of (2·5) is a holomorphic function of $\alpha$, equation (2·5) can be analytically continued to the complex half-plane $\operatorname{Re}\alpha \geqslant 0$. ∎

One easily gets from (2·5) a recurrence relation

$$H_{\alpha+N,N} = z^{-(\alpha+N)}\frac{\exp[i\pi(\alpha+N)+pz^N]}{\Gamma(1-\alpha-N)}$$
$$+pN\,\Gamma(\alpha+N)\sum_{n=0}^{N-1}\binom{N-1}{n}\frac{(-1)^{N+1-n}z^n}{\Gamma(\alpha+n+1)}H_{\alpha+n,N}. \qquad (2\cdot9)$$

This recurrence relation is of a different nature from the relation

$$H_{\alpha+k+1,N} = p\,e^{i\pi(\alpha+k)}\sum_{n=0}^{\infty}(-1)^n\binom{\alpha+k}{n}\frac{\Gamma(N+1)\,z^{N+n-1-\alpha-k}}{\Gamma(N+n-\alpha-k)}H_{n,N} \qquad k\in N_0^\infty, \quad (2\cdot10)$$

which may be obtained from (2·1) by calculating $D^{\alpha+k}[z^{N-1}\exp(-pz^N)]$ with the help of the generalized Leibnitz rule

$$D_z^\alpha(fg) = \sum_{n=0}^{\infty}\binom{\alpha}{n}f^{(n)}D_z^{\alpha-n}g$$

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(it is easy to see that equations (1·8)–(1·10) as given in (1) are particular cases of our equation (2.10)). One may easily rewrite (2·9) in terms of ${}_NF_N$(see (2·2)), however such recurrence relations seem to be a mere curiosity at present.

In order to derive a differential equation for $H_{\alpha,N}$ one needs only to express $D_z^k[H_{\alpha,N}\exp(-pz^N)]$ in terms of $H_{\alpha,N}$ and insert relevant expressions into (2·5). This may be done easily with help of the formula

$$z^k[\exp(pz^N)]D_z^k[\exp(-pz^N)] = \Gamma(k+1)\sum_{r=0}^{k}(p)^r\frac{z^{rN}}{(1)_r}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\binom{sN}{k} \quad (2.10a)$$

(see (9), p. 27) and applying the Leibnitz rule to $D_z^k H_{\alpha,N}\exp(-pz^N)$. The final result is

$$\left[D^N+\sum_{n=0}^{N-1}P_{n,N}(\alpha,z)\,D^n\right]H_{\alpha,N}(p,z) = z^{-(N+\alpha)}\frac{\exp(i\pi\alpha+pz^N)}{\Gamma(1-\alpha-N)}, \quad (2·11)$$

where $P_{n,N}(\alpha,z)$ is a polynomial in $z$

$$P_{n,N}(\alpha,z) = z^n pN\sum_{k=0}^{N-n-1}\binom{N-n-1}{k}$$
$$\times\left[\binom{N-1}{n}\frac{\Gamma(\alpha+N)}{\Gamma(\alpha+n+k+1)}-\binom{N}{n}\frac{\Gamma(N)}{\Gamma(n+k+1)}\right]z^k e^{pz^N}D_z^k e^{-pz^N}. \quad (2·12)$$

When $\alpha\in N_0^\infty$ the term on the right-hand side of (2·11) vanishes and one finds that $H_{m,N}(p,z)$ satisfy the ordinary differential equation of the order $N$

$$\left[D^N+\sum_{n=0}^{N-1}P_{n,N}(m,z)\,D^n\right]H_{m,N}(p,z) = 0. \quad (2·13)$$

It is trivial to check that for $N=2$ equation (2·13) yields the well known Hermite equation $D^2H_m-2zDH_m+2nH_m=0$ (hereafter $H_m=H_{m,2}$). The equation (2·13) may be derived, in fact, without resorting to an analytic continuation. Since for $n\in N_0^\infty$

$$H_{n,N}(p,z) = (-1)^{n+k_0}(p)^{k_0}(1)_n\frac{\exp(pz^N)}{\Gamma(k_0+1)}\binom{k_0N}{n}\,{}_NF_N\left(\begin{matrix}c_{N-1},\,1\\ c_N-\dfrac{n}{N}\end{matrix}\middle|-pz^N\right), \quad (2·14)$$

where $c_i=k_0+i/N$, $i=1,\ldots,N$ and $k_0=\min.k$, $k\in\{k\colon kN\geqslant 0,\ k\in N_0^\infty\}$ (see (2·2)), equation (2·6) yields

$$z^{n-N-k_0N}\left[D^N+pN\,\Gamma(n+N)\sum_{k=0}^{N-1}\binom{N-1}{k}\frac{z^k}{\Gamma(n+k+1)}D^k\right]h_{n,N}(p,z)$$
$$= (-1)^{N+k_0}\frac{\Gamma(N+n-k_0N)}{\Gamma(n-k_0N)}(p)^{k_0}\frac{\Gamma(k_0N+1)}{\Gamma(k_0+1)\,\Gamma(k_0N+1-n)}. \quad (2·15)$$

But $N+n-k_0N>0$ and $n-k_0N=-s$, $s\in N_0^\infty$, hence the right-hand side of (2·15) vanishes identically and one easily obtains (2·13) from (2·15). We have derived initially equation (2·5) since its particular version ($N=2$) has some interesting interpretation in Quantum Mechanics (this point will be discussed elsewhere).

For we have defined $H_{\alpha,N}(p,z)$ only for $\operatorname{Re}\alpha\geqslant 0$ (see (2·1)) one may wonder how one could define $H_{\alpha,N}$ for $\operatorname{Re}\alpha<0$? The point to note is that although an extension of

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(2·1) to the half-plane $\mathrm{Re}\,\alpha < 0$ is possible (there exist a few concepts of $D_z^\alpha$ for $\mathrm{Re}\,\alpha < 0$ in the literature; see, for example, (3, 4, 8)) it is not unique unless additional assumptions are made. If one supposes that, for example, $H_{\alpha,N}(p,z)$ is holomorphic function of $\alpha$ for $\mathrm{Re}\,\alpha < 0$ and a fixed $N$, then (2·2) defines $H_{\alpha,N}$ in the whole $\alpha$-plane and such $H_{\alpha,N}$'s obey (2·11) for $\alpha \in \mathbb{C}$.

### 3. *Orthogonal properties*

It was claimed by Bell (see (1), (2·3)) that the generalized Hermite polynomials $\xi_n(a,x;N)$ defined by the relation

$$\left.\begin{aligned} &\xi_n(a,x;N) = e^{-ax^N} D^n e^{ax^N}, \quad a \in R^1, \quad N \in N_0^\infty \\ &\xi_{-n-1} = 0, \quad n > 0 \end{aligned}\right\} \tag{3·1}$$

satisfy an orthogonality condition

$$\int_{-\infty}^{\infty} dx\, e^{-ax^N} \xi_m(-a,x;2k)\,\xi_n(-a,x;2k) = 0, \quad m \neq n,\ a > 0,\ k \in N_1^\infty \tag{3·2}$$

(there is a misprint in equation (3·1) given in Bell's paper (1) namely, it should read

$$\zeta_n = \zeta_n(x,t;2r) = \left[\exp\left(-\frac{xt^{2r}}{2}\right)\right] \xi_n(-x,t;2r), \quad x > 0$$

instead of

$$\zeta_n = \zeta_n(x,t;2r) = [\exp(-xt^{2r/2})]\,\xi_n(-x,t;2r), \quad x > 0).$$

By changing the variable $x = ua^{-1/N}$ one transforms (3·2) into

$$a^{(n+m-1)/N} \int_{-\infty}^{\infty} du\, e^{-u^N} \xi_m(-1,u;2k)\,\xi_n(-1,u;2k) = 0,$$

and for $\xi_n(-1,x;N) = (-1)^n H_{n,N}(1,x)$ then (3·2) is equivalent to the condition

$$\int_{-\infty}^{\infty} dx\, e^{-x^N} H_{m,N}(x)\, H_{n,N}(x) = 0, \quad m \neq n, \quad N = 2k, \quad k \in N_1^\infty \tag{3·3}$$

(hereafter $H_{m,N}(x) \equiv H_{m,N}(1,x)$). However, simple calculations immediately reveal that (3·3) cannot hold for $k > 1$, i.e. for $N$ even and larger than 2. To demonstrate the point let us calculate e.g. the integral

$$\int_{-\infty}^{\infty} dx\, e^{-x^N} H_{1,N}(x)\, H_{3,N}(x) \quad \text{for} \quad N = 2k, \quad k \in N_2^\infty:$$

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, e^{-x^N} H_{1,N}(x)\, H_{3,N}(x) &= N \int_{-\infty}^{\infty} dx\, e^{-x^N} x^{N-1} H_{3,N}(x) \\ &= -N \int_{-\infty}^{\infty} dx\, x^{N-1} D^3 e^{-x^N} = N(N-1) \int_{-\infty}^{\infty} dx\, x^{N-2} D^2 e^{-x^N} \\ &= N(N-1)(N-2)(N-3) \int_{-\infty}^{\infty} dx\, x^{N-4} e^{-x^N} = 2\,\frac{\Gamma(N+1)}{\Gamma(N-3)} \int_0^{\infty} dx\, x^{(N-3)-1} e^{-x^N} \\ &= 2\,\frac{\Gamma(N+1)}{N\Gamma(N-3)}\,\Gamma\left(1-\frac{3}{N}\right) = 2\Gamma(N)\,\Gamma\left(1-\frac{3}{N}\right)/\Gamma(N-3). \end{aligned}$$

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Since the last expression never vanishes provided that $N \geqslant 3$, our conclusion that (3·2) cannot hold for $N = 2k$, $k \in N_2^\infty$ follows. One obtains, in fact, for $N = 2k$, $k \in N_1^\infty$:

$$(1)\quad m < n(N-1)$$

$$(H_{m,N};\, H_{n,N})_1 = \frac{\Gamma(m+1)\,\Gamma(n+1)}{N}\,[(-1)^m + (-1)^n] \sum_{k=0}^{m} \binom{kN-m}{n} \Gamma\left(k - \frac{m+n-1}{N}\right)$$

$$\cdot \sum_{i=0}^{k} (-1)^i \binom{iN}{m} \binom{k}{i} \Big/ (1)_k. \tag{3·4}$$

$$(2)\quad m = n(N-1)$$

$$(H_{m,N};\, H_{n,N})_1 = 2N^{N-1}\,\Gamma[1+n(N-1)]\,\Gamma(1/N). \tag{3·5}$$

$$(3)\quad m > n(N-1)$$

$$(H_{m,N};\, H_{n,N})_1 = 0, \tag{3·6}$$

where

$$(H_{m,N};\, H_{n,N})_1 = \int_{-\infty}^{\infty} dx\, e^{-x^N} H_{m,N}(x)\, H_{n,N}(x).$$

(3·4) may be easily obtained with help of (2·10$a$) and the equality

$$[1+\exp i\pi(z+n)] \int_0^\infty dx\, e^{-x^N} x^{kN-z} H_{n,N}(x)$$

$$= [1+\exp i\pi(z+n)]\,\Gamma(kN+1-z)\,\Gamma\left(k - \frac{z+n-1}{N}\right) \Big/ N\Gamma(kN+1-z-n).$$

Relevant formulae for a general case $p \neq 1$ follow immediately from (3·4)–(3·6) due to the relation:

$$\int_{-\infty}^{\infty} dx\, H_{m,N}(p,x)\, H_{n,N}(p,x) \exp(-px^N) = p^{(m+n-1)/N} (H_{m,N};\, H_{n,N})_1.$$

## *Conclusions*

We have derived the ordinary differential equation of order $N$ for the Hermite–Bell function $H_{\alpha,N}(p,z)$ (see (2·11)) which for $\alpha \in N_0^\infty$ becomes a differential equation for the Hermite–Bell polynomial $H_{n,N}(p,z)$, $n \in N_0^\infty$, $N \in N_2^\infty$ (see (2·13)).

We have also shown that HB polynomials are not orthogonal with weight $\exp(-pz^N)$ for $N$ even at the interval $-\infty < z < \infty$, contrary to Bell's claim.

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