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Minimization of field-aligned current and flow

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Abstract. The radial profiles of current and flow permitted by the relaxed states of a two-fluid cylindrical plasma system using double Beltrami formulation are presented. The relaxed states are then compared with those of the internal conductor plasma configuration. The results show that for appropriate boundary conditions and scale parameters, field-aligned currents and flows could be minimized to a greater extent in an internal conductor plasma system as compared with a simple cylindrical plasma configuration. Consequently, high beta relaxed states with minimum values of parallel components of current and flow could be created using internal conductor plasma configurations.

1. Introduction

In the absence of flow energy, the relaxed magnetic field **B** satisfies the 'Beltrami condition'

$$\nabla \times \mathbf{B} = \lambda \mathbf{B}, \tag{1.1}$$

where λ is a scalar field that must satisfy $\mathbf{B} \cdot \nabla \lambda = 0$ to ensure $\nabla \cdot \mathbf{B} = 0$. Since the associated Lorentz force ($\mathbf{j} \times \mathbf{B}$) is identically zero, the Beltrami magnetic field represents a force-free macroscopic state of the magnetofluid. Solutions of (1.1) have been intensively used to characterize the magnetic field lines in astrophysical, space and laboratory plasmas (Chandrasekhar and Kendall 1957; Taylor 1974, 1986). Beltrami fields are also useful in studying turbulence in plasmas (Montgomery et al. 1978) and considered to be self-organized through the energy relaxation (Hasegawa 1985; Brown 1997). Woltjer (1958) and Taylor (1974) derived (1.1) by minimizing the magnetic energy with the constraint that magnetic helicity is conserved. Taylor conjectured that in a slightly resistive system, minimization of the field energy with the global constraint of helicity leads to the 'relaxed state' characterized by a spatially homogeneous λ as given in (1.1), i.e. 'a constant-λ Beltrami field'.

Sudan (1979) presented the force-free ($\nabla p = 0$) equilibrium model for the two-fluid plasma based on the variational principle. Since the force-free relaxed state is of no interest for thermonuclear fusion, different attempts using different kinds of constraints have been made to impart finite flow and create pressure gradients in the relaxed state (Turner 1986; Avinash and Taylor 1991; Steinhaur and Ishida 1997; Dasgupta et al. 1998; Bhattacharyya et al. 2003).

The relaxation model for an ideal two species magnetofluid have also been presented by Mahajan and Yoshida (1998) and Yoshida and Mahajan (2002). By this

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model the relaxed state admitted exhibiting strong flow, and a finite pressure called the double curl Beltrami field can be described as a linear combination of two different Beltrami fields. The model has been applied to explain different physical phenomena (Yoshida et al. 2001; Iqbal et al. 2001).

The Taylor-type relaxed states, being force-free states, are characterized by large currents parallel to the magnetic field. The magnetic helicity is to remain conserved in plasma relaxation, but the parallel components cause its dissipation in a resistive plasma. Also, most of the plasma configurations with currents parallel to the field intended for confinement of the plasma for fusion require a means for current drive. The double Beltrami field, on the other hand, could lead to high beta relaxed states depending on different boundary conditions and Beltrami parameters. In this paper, high beta relaxed states with minimum values of field-aligned current and flows are investigated for different boundary conditions and Beltrami parameters for a simple cylindrical plasma configuration. This configuration is then compared with an internal conductor plasma system (Iqbal et al. 2001) to seek the relaxed states with an appreciable strength of perpendicular flows that are driven by charging the plasma or producing a pressure gradient.

In Sec. 2, the radial profiles of the field, flow and beta for a cylindrical plasma are presented based on the formulation derived by Yoshida et al. (2001) and Iqbal et al. (2001). In Sec. 3, the internal conductor plasma system is described briefly and a comparison of two configurations is presented. A brief summary is contained in Sec. 4.

2. Cylindrical plasma system

The simplest equilibrium solution of a two-fluid plasma system consists of two simultaneous Beltrami conditions implying the alignment of vorticities to the corresponding flows and can be expressed as follows (Yoshida et al. 2001):

$$\mathbf{B} = a(\mathbf{V} - \nabla \times \mathbf{B}) \tag{2.1}$$

$$(\mathbf{B} + \nabla \times \mathbf{V}) = b\mathbf{V}, \tag{2.2}$$

where a and b are constants. Combining (2.1) and (2.2) yields a second-order partial differential equation called a 'double-curl Beltrami equation'

$$\nabla \times (\nabla \times \mathbf{u}) - \alpha_1 \nabla \times \mathbf{u} + \alpha_2 \mathbf{u} = 0, \tag{2.3}$$

where $\mathbf{u} = \mathbf{B}$ or $\mathbf{V}$, and

$$\alpha_1 = b - 1/a, \quad \alpha_2 = 1 - b/a. \tag{2.4}$$

The mechanism for creating the pressure gradients in the relaxed states is given by the generalized Bernoulli equation

$$\frac{\beta}{2} + \frac{V^2}{2} = \text{constant}, \tag{2.5}$$

where β is the conventional beta ratio. Equation (2.5) reveal that an appropriate sheared velocity can sustain a desired pressure gradient. The general solution of (2.3) is given by the linear combination of the two Beltrami fields $G_{\pm}$ such that

$$\nabla \times \mathbf{G}_{\pm} = \lambda_{\pm} \mathbf{G}_{\pm}, \tag{2.6}$$

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where $\lambda_\pm = [\alpha_1 \pm (\alpha_1^2 - 4\alpha_2)^{1/2}]/2$. Then, for arbitrary constants $c_\pm$, the sum

$$\mathbf{u} = c_+\mathbf{G}_+ + c_-\mathbf{G}_- \tag{2.7}$$

solves (2.3).

The parallel and perpendicular components of flow and current are given by (Iqbal et al. 2001),

$$V_\parallel = \frac{B_z V_z + B_\theta V_\theta}{(B_z^2 + B_\theta^2)^{1/2}}, \tag{2.8}$$

$$j_\parallel = \frac{B_z V_z + B_\theta V_\theta}{(B_z^2 + B_\theta^2)^{1/2}} - \frac{(B_z^2 + B_\theta^2)^{1/2}}{a}, \tag{2.9}$$

$$V_\perp = j_\perp = \frac{B_z V_\theta - B_\theta V_z}{(B_z^2 + B_\theta^2)^{1/2}}, \tag{2.10}$$

where $\parallel$ and $\perp$ refer to the direction of magnetic field.

The double Beltrami field is a linear combination of two different Beltrami fields and can be described by the Chandrasekhar–Kendal functions in a cylindrical geometry. The axisymmetric solution for a one-dimensional cylinder is

$$\mathbf{B} = \begin{pmatrix} 0 \\ C_1 J_1(\lambda_+ r) \\ C_1 J_0(\lambda_+ r) \end{pmatrix} + \begin{pmatrix} 0 \\ C_2 J_1(\lambda_- r) \\ C_2 J_0(\lambda_- r) \end{pmatrix}, \tag{2.11}$$

where J_0 and J_1 are the ordinary Bessel functions. Applying boundary conditions, the values of constants C_1 and C_2 can be evaluated. Under the boundary conditions $(\nabla \times \mathbf{B})_z|_{r=0} = g$ and $\mathbf{B}_z|_{r=0} = s$, we obtain

$$C_1 = \frac{s - \lambda_- g}{\lambda_+ - \lambda_-}$$

$$C_2 = \frac{\lambda_+ g - s}{\lambda_+ - \lambda_-}.$$

The values of V_z and V_θ are given by

$$\mathbf{V}_z = \left(\frac{1}{a} + \lambda_+\right) C_1 J_0(\lambda_+ r) + \left(\frac{1}{a} + \lambda_-\right) C_2 J_0(\lambda_- r)$$

$$\mathbf{V}_\theta = \left(\frac{1}{a} + \lambda_+\right) C_1 J_1(\lambda_+ r) + \left(\frac{1}{a} + \lambda_-\right) C_2 J_1(\lambda_- r). \tag{2.12}$$

The radial profiles of magnetic fields, flows, pressure and components of flows and currents for the case of a one-dimensional simple cylindrical geometry are given in Figs 1–4 for different boundary conditions and Beltrami parameters using (2.11), (2.12), (2.5), (2.8), (2.9) and (2.10). Subject to appropriate boundary conditions and Beltrami parameters, the double Beltrami formulation provide paramagnetic and diamagnetic structures as is evident from the figures. The size of the system for all cases is taken to be $2\lambda_i$, where $\lambda_i = 0.072$ m for $n = 10^{19}$ m^{-3}, β at the edge $r = 2$ is taken to be zero and the velocities are measured in the units of central Alfvén speed.

Figure 1 shows the radial profiles of magnetic fields, velocities and components of flows and currents for the Beltrami parameters $a = -0.8$ and $b = -1.25$ with the boundary conditions $B_z = g = 1$ and $j_z = s = 0.75$. The normalized eigenvalues of the curl operators are $\lambda_+ = 0.75$ and $\lambda_- = -0.75$. The fields show paramagnetic

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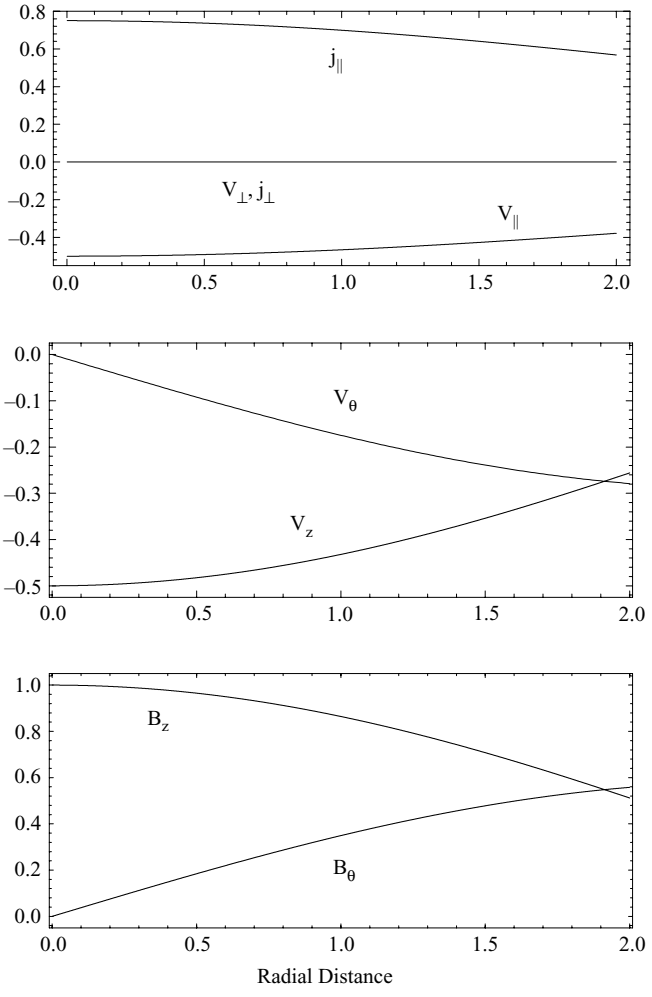


Figure 1. Radial profiles of B_z, B_θ, V_z, V_θ, $j_\parallel$, $j_\perp$, $V_\parallel$, $V_\perp$ and β for the Beltrami parameters $a = -0.8$ and $b = -1.25$. The eigenvalues are $\lambda_\pm = \pm 0.75$.

behavior and are minimized at the edge. For this case we have taken $s = \lambda_+ g$ as a boundary condition, which leads us to a single Beltrami system. $V_\perp$ and $j_\perp$ are zero and only parallel components of current and flow exist. This indicates the emergence of force-free magnetic fields as considered by Taylor (1974) and Sudan (1979) that exhibit only higher values of parallel components. This configuration produces very insignificant pressure confinement by the Bernoulli mechanism.

Figure 2 represents the radial profiles of fields, flows, components of flows and current and pressure for real values of $\lambda_\pm$. In this case we have taken $B_z = 1$ and $j_z = 0.52$ as boundary conditions. The Beltrami parameters are $a = -1.8$ and $b = -1.5$ implying eigenvalues of $\lambda_+ = -0.2349$ and $\lambda_- = -0.7096$. Fields and flows increase away from the center showing diamagnetism. Pressure is high at the center and decreases monotonically. $j_\parallel$ has a higher value at $r \sim 0$, then decreases smoothly towards the edge while $j_\perp = 0$ at $r = 0$, but increases smoothly and at the edge of

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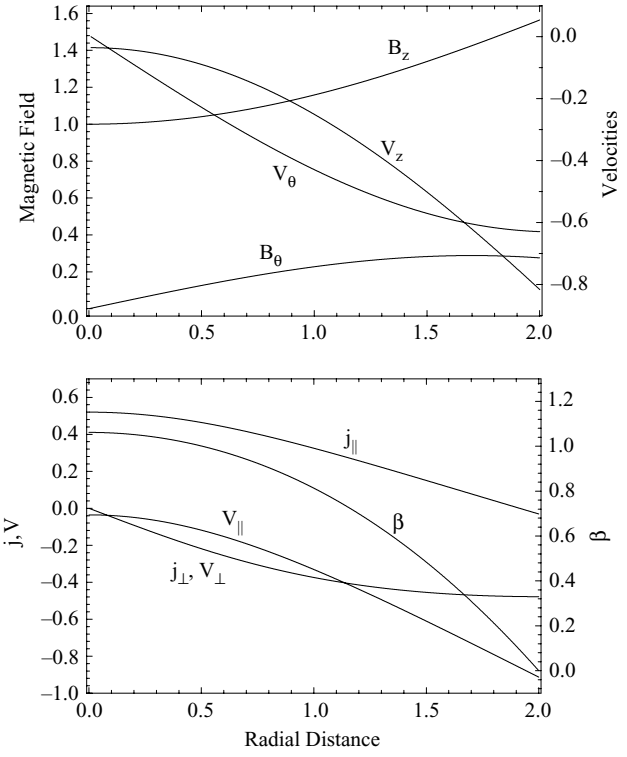


Figure 2. Radial profiles of B_z, B_θ, V_z, V_θ, $j_\parallel$, $j_\perp$, $V_\parallel$, $V_\perp$ and β for the Beltrami parameters $a = -1.8$ and $b = -1.5$. The corresponding eigenvalues are $\lambda_+ = -0.2349$ and $\lambda_- = -0.7096$.

the plasma $j_\perp > j_\parallel$. On the other hand, $V_\parallel > V_\perp$ from $r \sim 0$ to $r \sim 0.1$, then $V_\perp$ dominates $V_\parallel$ until $r \sim 1.1$, after that $V_\parallel$ increases and at the boundary $V_\parallel > V_\perp$.

Figure 3 depicts the profiles for pure imaginary values of $\lambda_\pm$. For this case, we have taken values of Beltrami parameters $a = -2.5$ and $b = -0.4$ that make $\alpha_1 = 0$ and $\alpha_2 > 0$. The boundary values are $B_z = 1$ and $j_z = 0.42$. The eigenvalues normalized to λ_i are $\lambda_\pm = \pm 0.9165i$. For these parameters and boundary conditions, the system reduces to London's equation and we can observe diamagnetic behavior as shown in Fig. 3. $j_\parallel$ remains constant throughout the system while $j_\perp$ increases sharply. Starting from $r \sim 0$–0.8, $j_\parallel > j_\perp$ and after that $j_\perp$ dominates $j_\parallel$ whereas $V_\perp$ is almost greater than $V_\parallel$ throughout the system.

In Fig. 4, the eigenvalues λ_+ and λ_- are complex conjugate. The Beltrami parameters are taken as $a = -4.9$ and $b = 0.0$, implying that $\lambda_\pm = 0.102 \pm 0.9948i$. The boundary conditions are $B_z = 1$ and $j_z = 0.0$. In this case, the magnetic fields and flows also show diamagnetism. $j_\parallel$ varies from 0.0 to 0.23 whereas $V_\parallel$ changes its value from 0.204 to 0.214 in the system. For this case the magnetic field becomes proportional to the vorticity of flow because of $b = 0.0$.

For this system, we observe that the parallel components of the flow and current are zero at the center if we take $j_z = -1/a$ and $j_z = s = 0.0$, respectively, as the boundary condition at the center, while the perpendicular components of flow and current are always zero at the center ($r = 0$).

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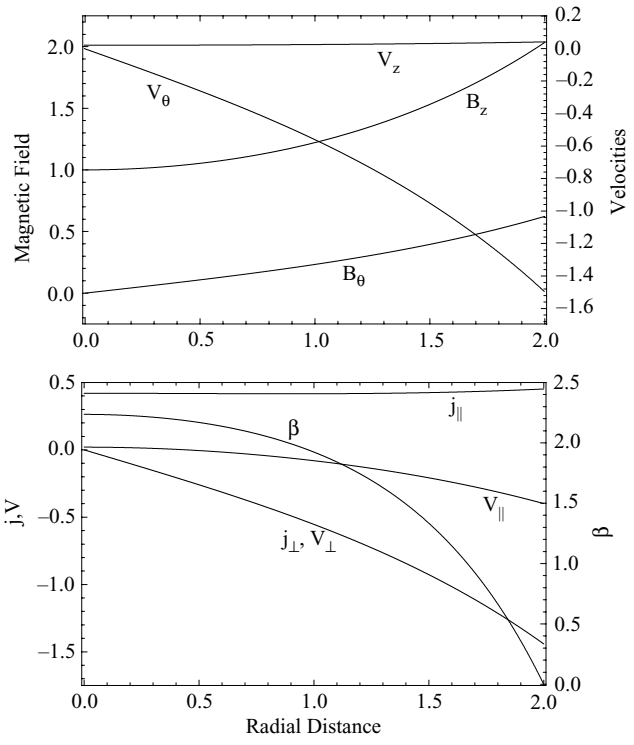


Figure 3. Radial profiles of B_z, B_θ, V_z, V_θ, $j_\parallel$, $j_\perp$, $V_\parallel$, $V_\perp$ and β for the Beltrami parameters $a = -2.5$ and $b = -0.4$. The imaginary eigenvalues normalized to λ_i are $\lambda_\pm = \pm 0.9165i$.

3. Internal conductor plasma system

In this section, to include the effect of poloidal field and current density, we examine an annular plasma configuration as described by Iqbal et al. (2001). The annular plasma configuration is a cylindrical plasma having an internal conductor at its center carrying a current I that produces a magnetic field in the azimuthal direction. The general solution for the 'double curl Beltrami equation' reads

$$\mathbf{B} = \begin{pmatrix} 0 \\ P_1 J_1(\lambda_+ r) + P_2 Y_1(\lambda_+ r) \\ P_1 J_0(\lambda_+ r) + P_2 Y_0(\lambda_+ r) \end{pmatrix} + \begin{pmatrix} 0 \\ Q_1 J_1(\lambda_- r) + Q_2 Y_1(\lambda_- r) \\ Q_1 J_0(\lambda_- r) + Q_2 Y_0(\lambda_- r) \end{pmatrix}, \tag{3.1}$$

where J and Y are Bessel functions of first and second kind, respectively, with the subscripts indicating their orders. P and Q are constants and can be evaluated by using boundary conditions. Applying boundary values at the conductor radius d, i.e. $|(\nabla \times \mathbf{B})_z|_{r=d} = j_z = s$, $|(\nabla \times \mathbf{B})_\theta|_{r=d} = j_\theta = k$ and $|\mathbf{B}_z|_{r=d} = g$, $|\mathbf{B}_\theta|_{r=d} = h$. The value of the constants after using the Wroskian relation $J_{m+1}(z)Y_m(z) - Y_{m+1}(z)J_m(z) = 2/\pi z$, become

$$P_1 = \frac{\pi \lambda_+ d}{2(\lambda_- - \lambda_+)}[(s - \lambda_- g)Y_1(\lambda_+ d) - (k - \lambda_- h)Y_0(\lambda_+ d)],$$

$$P_2 = \frac{\pi \lambda_+ d}{2(\lambda_- - \lambda_+)}[(k - \lambda_- h)J_0(\lambda_+ d) - (s - \lambda_- g)J_1(\lambda_+ d)],$$

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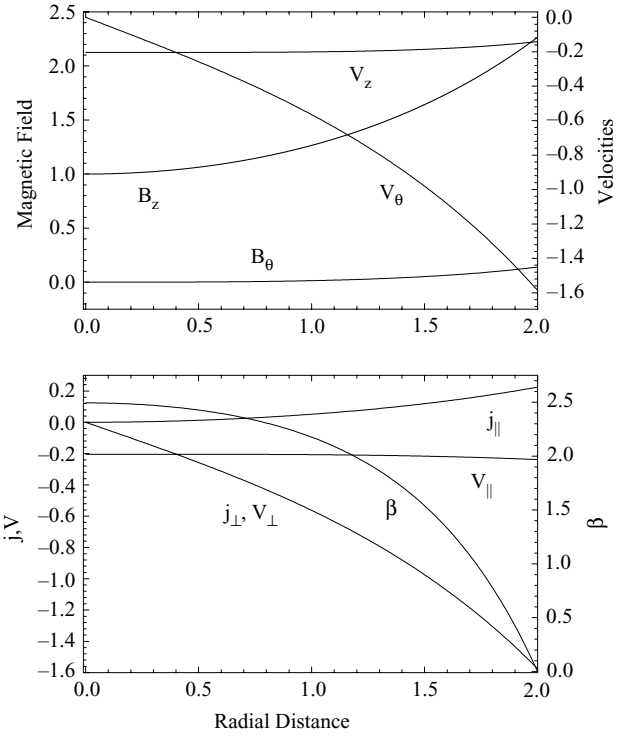


Figure 4. Radial profiles of B_z, B_θ, V_z, V_θ, $j_\parallel$, $j_\perp$, $V_\parallel$, $V_\perp$ and β for the Beltrami parameters $a = -4.9$ and $b = 0.0$. The eigenvalues normalized to λ_i are complex conjugate pair $\lambda_\pm = 0.102 \pm 0.9948i$.

$$Q_1 = \frac{\pi\lambda_- d}{2(\lambda_- - \lambda_+)}[(\lambda_+ g - s)Y_1(\lambda_- d) - (\lambda_+ h - k)Y_0(\lambda_- d)],$$

and

$$Q_2 = \frac{\pi\lambda_- d}{2(\lambda_- - \lambda_+)}[(\lambda_+ h - k)J_0(\lambda_- d) - (\lambda_+ g - s)J_1(\lambda_- d)]. \tag{3.2}$$

Using relation (2.1), expressions for V_z and V_θ are given by

$$\begin{aligned} V_z &= \left(\frac{1}{a} + \lambda_+\right)(P_1 J_0(\lambda_+ r) + P_2 Y_0(\lambda_+ r)) \\ &\quad + \left(\frac{1}{a} + \lambda_-\right)(Q_1 J_0(\lambda_- r) + Q_2 Y_0(\lambda_- r)), \\ V_\theta &= \left(\frac{1}{a} + \lambda_+\right)(P_1 J_1(\lambda_+ r) + P_2 Y_1(\lambda_+ r)) \\ &\quad + \left(\frac{1}{a} + \lambda_-\right)(Q_1 J_1(\lambda_- r) + Q_2 Y_1(\lambda_- r)). \end{aligned} \tag{3.3}$$

The pressure can be obtained by the use of a generalized Bernoulli condition as described in (2.5).

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The difference between the simple cylindrical plasma system and the internal conductor cylindrical plasma system is only the internal conductor. The internal conductor produces an azimuthal magnetic field in the internal conductor configuration, whereas for a simple cylindrical geometry there is no poloidal magnetic field at the center. In the case of a simple cylinder, the solution only consists of Bessel functions of the first kind because the Bessel functions of the second kind, denoted by $Y_m(r)$ are infinite at the center of the cylinder where $r = 0$. When an internal conductor of radius d is placed at the center, the solution consists of Bessel functions of both the first and second kind. Therefore, mathematically, the difference between these systems is a Bessel function of the second kind. The two systems become equivalent if $C_1 = P_1$, $C_2 = Q_1$ and P_2 and Q_2 are identically zero. For $C_1 = P_1$, we get the following condition:

$$2/\pi\lambda_+ d = \frac{k - \lambda_- h}{s - \lambda_- g} Y_0(\lambda_+ d) - Y_1(\lambda_+ d). \tag{3.4}$$

When C_2 is equal to Q_1, we get the following condition:

$$2/\pi\lambda_- d = \frac{\lambda_+ h - k}{\lambda_+ g - s} Y_0(\lambda_- d) - Y_1(\lambda_- d). \tag{3.5}$$

$P_2 = 0$ yields

$$\frac{k - \lambda_- h}{s - \lambda_- g} = \frac{J_1(\lambda_+ d)}{J_0(\lambda_+ d)}, \tag{3.6}$$

and $Q_2 = 0$ give the result

$$\frac{k - \lambda_+ h}{s - \lambda_+ g} = \frac{J_1(\lambda_- d)}{J_0(\lambda_- d)}. \tag{3.7}$$

Plugging in the values of (3.6) into (3.4), we obtain

$$J_0(\lambda_+ d) = 1. \tag{3.8}$$

This equation is satisfied only if $\lambda_+ d = 0$. For $\lambda_+ \neq 0$, we get $d = 0$. Similarly putting the values of (3.7) into (3.5), we obtain $d = 0$ taking $\lambda_- \neq 0$. This implies that if there is no internal conductor, i.e. $d = 0$, then both systems are the same under the conditions imposed above.

For a simple cylindrical plasma system, it is impossible to have $|j_\perp|$ and $|V_\perp|$ greater than $|j_\parallel|$ and $|V_\parallel|$, respectively, throughout the system. For this system $|j_\perp|(r = 0) = |V_\perp|(r = 0) = 0$ whereas $|j_\parallel|(r = 0) = s$ and $|V_\parallel|(r = 0) = s + 1/a$. Hence, the perpendicular components of current and flows are always zero at the center of the simple cylindrical plasma system, whereas for the case of the internal conductor system, perpendicular components may have some finite values and for certain boundary conditions and scale parameters could have higher values than the parallel components.

For the internal conductor system, we have got extra boundary conditions and as a consequence of extra degrees of freedom, we can control the whole system very effectively. Pressure inside the internal conductor system can be handled easily by varying the current in the internal conductor. Central pressure can be minimized at the radius of the internal conductor and the hollow profile of the pressure can be obtained very easily. On the other hand, a simple cylindrical plasma system having less boundary conditions cannot be controlled so effectively. Using

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the simple cylindrical configuration it is difficult to simultaneously minimize the pressure at the center and the edge with a high beta profile.

4. Summary

The plasma relaxation has been studied in a cylindrical configuration based on the double curl Beltrami equation and it has been shown that we can get the relaxed states with a dominance of perpendicular components throughout the system except at the center of the cylindrical plasma. However, we are interested in minimizing the parallel components of current and flow throughout the system. This is because an external agency is required to drive the parallel components in a dissipative system and in the presence of strong flow, parallel components generate shock waves that in turn could destabilize the system. Moreover, parallel components cause the magnetic helicity to dissipate in resistive plasma. We have therefore considered the internal conductor plasma configuration and compared it with cylindrical plasma. Our results show that with the help of an internal conductor configuration, one can obtain high beta relaxed states with a dominance of perpendicular components throughout the system for ceratin Beltrami parameters and boundary conditions. Such high beta relaxed states are important with regard to thermonuclear fusion.

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