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Creeping modes in a shadow

By F. URSELL
Department of Mathematics, University of Manchester

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Abstract. In the two-dimensional theory of diffraction by smooth curves there are certain canonical problems that can be solved explicitly in series form. The series converge slowly at short wavelengths but they can be transformed by Watson's transformation into another form (the residue series or creeping-mode expansion) which has been much used in shadow regions. It is found that throughout the shadow region the first few terms of the residue series are exponentially small and decrease rapidly, and these have often been used as an estimate of the wave potential without further justification. Leppington's recent work on the shadow of an ellipse has shown, however, that in part of the shadow some of the later terms of the residue series are exponentially large. In other words, the complete residue series in part of the shadow is even more slowly convergent than the original series.

In the present paper the Watson transformation is re-examined in the light of this result. The original series is expressed as the sum of a finite number of terms of the residue series and of a remainder. It is shown that throughout the shadow the remainder is at short wavelengths asymptotically of smaller order than the last term retained in the finite residue series. It follows that the residue series (which is a convergent infinite series) is also an asymptotic series, and this fact is sufficient to justify most of the usual applications. The proof is given in detail for the circle and in outline for the ellipse; it makes use of the theory of conformal mapping.

1. *Introduction. The Watson transformation.* In the mathematical theory of short-wave diffraction an important part is played by certain *canonical problems*, concerned with simple curved boundary configurations for which an explicit solution in the form of a singly infinite series can be found. At short wavelengths these series converge very slowly and are particularly useless in shadow regions where the sum is much smaller than the magnitude of individual terms. A shadow problem was first solved effectively in 1918 by Watson (12) in his classical work on the diffraction of radio waves by the earth. The Watson transformation transforms the original series into another series (the residue series or creeping-mode expansion) which is particularly convenient at short wavelengths and which enabled Watson to derive the law of shadow formation on the earth's surface. The transformation has since been applied to various two-dimensional shadow problems involving spheres, quadric surfaces of revolution, circles and conics, and it has often been verified that the convergence on the boundary is rapid. Because the first few terms of the residue series decrease rapidly, not merely

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on the curved boundary but throughout the shadow region, it has been thought that the region of rapid convergence of the infinite residue series also extends over the whole of the shadow, and this was verified by Ursell (9) for the shadow near a circle. (In the present paper we are not concerned with shadow boundaries and their neighbourhoods which are always supposed to be excluded; cf. Nussenzveig (6).) However, Leppington in his fundamental work on the ellipse (5) has shown that the circle is not typical: In part of the shadow of any ellipse (other than the circle) the transformed series contains exponentially large terms and is therefore more slowly convergent than the original series. A detailed study of the Watson transformation has therefore become desirable and is given in the present paper.

It is convenient to begin with one of the simplest cases of Watson's transformation, the formation of an acoustic shadow by a source on a circle. (Source potentials of this type are important in the rigorous asymptotic theory for arbitrary curves; Ursell (9), (10).) The treatment of this problem given in (9) will be summarized here, as Leppington's difficulty cannot be understood unless the reader has some acquaintance with Watson's transformation. Let us then suppose that short sound waves are emitted towards infinity by a time-periodic source situated at $\theta = 0$ on the circle $r = a$. Polar coordinates are chosen as in (9), the period is $2\pi/\omega$, the velocity of sound is c. We shall write $k = \omega/c, \rho = r/a, N = ka$. The velocity potential $\Phi(r, \theta)\, e^{-i\omega t}$ satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+k^2\right)\Phi(r,\theta) = 0,$$

with the boundary condition

$$\frac{\partial \Phi}{\partial r} = \frac{2i}{a}\delta(\theta) \quad \text{on} \quad r = a$$

and the radiation condition

$$r^{\frac{1}{2}}\left(\frac{\partial}{\partial r}-ik\right)\Phi \to 0 \quad \text{as} \quad r \to \infty.$$

Then it can be shown (see (9), appendix) that

$$\begin{aligned}\Phi(r,\theta) &= \frac{i}{\pi ka}\left[\frac{H_0(kr)}{H_0'(ka)}+2\sum_{m=1}^{\infty}\frac{H_m(kr)}{H_m'(ka)}\cos m\theta\right],\\ &= \Phi_c(kr,\theta;ka), \quad \text{say,} \end{aligned} \tag{1·1}$$

where $H_\nu(z)$ denotes the Hankel function $H_\nu^{(1)}(z)$ of the first kind (Watson (13), section 3·6). Equation (1·1) gives Φ in the form of a series, but when $N = ka$ is large the convergence is slow, and it is convenient to transform the series into a contour integral. Following Watson (12), consider

$$\frac{1}{\pi ka}\frac{H_\nu(ka)}{H_\nu'(ka)}\frac{\cos\nu(\pi-\theta)}{\sin\nu\pi}$$

as a function of the complex variable ν. It is an odd function of ν (cf. Watson (13), section 3·61, equation (7)) which has poles when $\sin\nu\pi = 0$ and when $H_\nu'(ka) = 0$. The zeros of $H_\nu'(ka)$ lie in the first and third quadrants and are never real. Write $\xi = \nu/N = \nu/ka$, and take

$$\int \frac{1}{\pi}\frac{H_{N\xi}(kr)}{H_{N\xi}'(ka)}\frac{\cos N(\pi-\theta)\,\xi}{\sin N\pi\xi}\,d\xi$$

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along the contour shown in Figure 1, which does not contain any zeros of $H'_{N\xi}(ka)$. The radius of the large circle is such that it never passes through a zero of $\sin N\pi\xi$. The open loop $\Xi_1\Xi_2$ passes between the real ξ-axis and the first zero $\xi_1(N)$ of $H'_{N\xi}(N)$, where

$$\xi_1(N) = 1 + 2^{-\frac{1}{3}} e^{\frac{1}{3}\pi i} \frac{|a'_1|}{N^{\frac{2}{3}}} + \ldots$$

(see equation (1·8) below). Suppose also to fix ideas that for large $|\xi|$ the zero locus (see (1·7) below) lies inside the open loop $\Xi_1\Xi_2$. The contribution to the integral from the

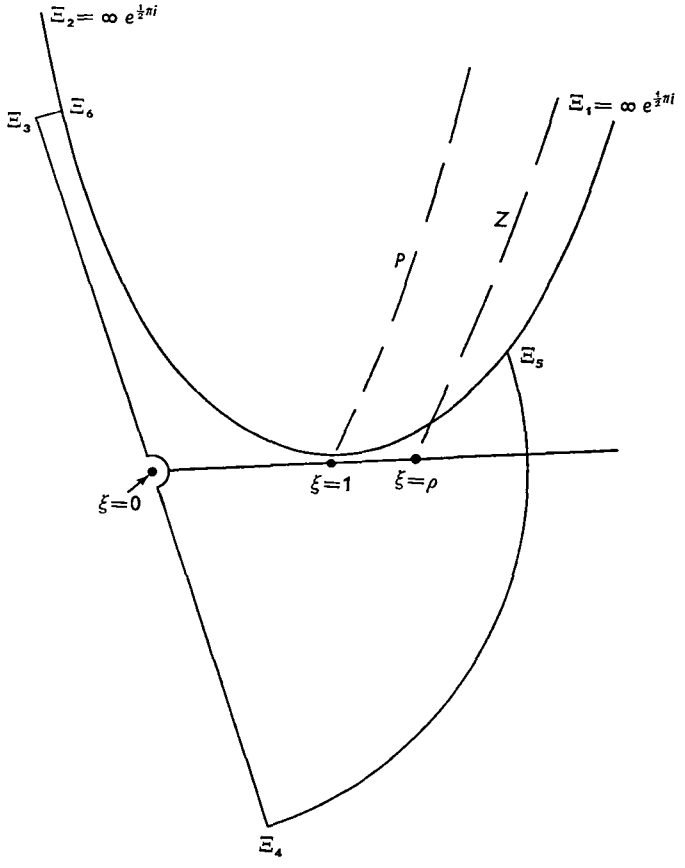


Fig. 1. The complex ξ-plane. The zeros of $H'_{N\xi}(N)$ lie near the pole locus P, the zeros of $H_{N\xi}(N\rho)$ lie near the zero locus Z. Near $\xi = 1$ the curve $\Xi_1\,\Xi_2$ passes between the first zero of $H'_{N\xi}(N)$ and the real ξ-axis.

radial parts of $\Xi_3\Xi_4$ vanishes since the integrand is an odd function; if $r > a$ and $0 < \theta < 2\pi$, then the contributions from the circular arcs $\Xi_4\Xi_5$ and $\Xi_6\Xi_3$ vanish in the limit as the radius of the contour tends to infinity. The poles inside the contour are at the zeros of $\sin N\pi\xi$, at $N\xi = 1, 2, 3, \ldots$. On equating the contour integral to $2\pi i$ times the sum of the residues at the poles (with an additional contribution from the

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indentation at $\xi = 0$) we find, in the limit as the radius of the contour tends to infinity, that

$$\frac{1}{\pi}\int_{\Xi_1}^{\Xi_2} \frac{H_{N\xi}(kr)}{H'_{N\xi}(ka)} \frac{\cos N(\pi-\theta)\,\xi}{\sin N\pi\xi}\,d\xi = \frac{i}{\pi ka}\left[\frac{H_0(kr)}{H'_0(ka)} + 2\sum_{m=1}^{\infty} \frac{H_m(kr)}{H'_m(ka)}\cos m\theta\right]$$
$$= \Phi_c(kr, \theta;\ ka) \qquad (1{\cdot}2)$$

from (1·1), and so the integral represents the potential in the region ($r > a, 0 < \theta < 2\pi$), and actually also in $r < a$ as was shown in (9). Now the contour $\Xi_1\Xi_2$ in equation (1·2) may also be regarded as a closed contour enclosing the zeros of $H'_\nu(N)$ provided that the integrand tends to zero suitably on a sequence of contours not passing through the zeros of $H'_\nu(N)$. (It can be shown that it does so in a region including the shadow region $r\cos\theta < a < r$.) Thus

$$\Phi_c = -\frac{2i}{ka}\sum_{\nu=\nu(m,\,ka)} \frac{H\nu_\nu(kr)}{(\partial/\partial\nu)\,H'_\nu(ka)} \frac{\cos\nu(\pi-\theta)}{\sin\nu\pi}, \qquad (1{\cdot}3)$$

where the summation takes place over the values $\nu(m, ka)$ of ν that satisfy $H'_\nu(ka) = 0$, ranked in order of increasing imaginary parts. The series (1·3) is the residue series mentioned above. Now we suppose that $N = ka$ is large, and write

$$\nu = N\xi, \quad \nu(m, N) = N\xi_m(N) = \nu_m(N).$$

Then the asymptotic theory of Hankel functions (Olver (7), see equation (A 1·2) below), show that for large N the functions $\xi_m(N)$ satisfy

$$\xi_m^{\frac{2}{3}}\zeta(1/\xi_m) = e^{\frac{1}{3}\pi i}\frac{|a'_m|}{N^{\frac{2}{3}}}(1 + o(1)), \qquad (1{\cdot}4)$$

where the function $\zeta(z)$, defined by

$$\tfrac{2}{3}\zeta^{\frac{3}{2}}(z) = \int_z^1 (1-\eta^2)^{\frac{1}{2}}\frac{d\eta}{\eta},$$

is regular near $z = 1$ and positive when $0 < z < 1$; and where a'_m denotes the mth zero (real and negative) of the derivative Ai′ of the Airy function. Equation (1·4) shows that the poles in the ξ-plane lie near the curve

$$\arg \xi^{\frac{2}{3}}\zeta(1/\xi) = \tfrac{1}{3}\pi, \qquad (1{\cdot}5)$$

i.e.
$$\Re\int_1^\xi (\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu} = 0, \qquad (1{\cdot}6)$$

which is independent of N; this curve will be referred to as the *pole locus*. Similarly the numerator $H_{N\xi}(kr) = H_{N\xi}(N\rho)$ vanishes at points lying near the *zero locus*

$$\arg \xi^{\frac{2}{3}}\zeta(\rho/\xi) = \tfrac{1}{3}\pi. \qquad (1{\cdot}7)$$

As N increases, the number of poles lying on a given segment of the pole locus increases, in particular the mth pole

$$\xi_m(N) = 1 + 2^{-\frac{1}{3}}e^{\frac{1}{3}\pi i}\frac{|a'_m|}{N^{\frac{2}{3}}} + \ldots \qquad (1{\cdot}8)$$

is seen to approach $\xi = 1$ when m is any fixed integer. It follows that

$$\nu_m(N) = N\xi_m(N) = N + 2^{-\frac{1}{3}}e^{\frac{1}{3}\pi i}N^{\frac{1}{3}}|a'_m| + \ldots. \qquad (1{\cdot}9)$$

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Let us first consider the residue series (1·3) on the boundary $r = a$ where it is of the form

$$\Sigma C_m(N) \frac{\cos \nu_m(\pi - \theta)}{\sin \nu_m \pi}, \tag{1·10}$$

and

$$C_m(N) \frac{\cos \nu_m(\pi - \theta)}{\sin \nu_m \pi} \sim -iC_m(N) \exp(i\nu_m(N)\theta), \tag{1·11}$$

where $\nu_m(N)$ is given by equation (1·4) for all m, or by equation (1·8) for the initial terms. Evidently each term $\exp(i\nu_m\theta - i\omega t)$ represents a damped progressive wave travelling along the boundary; the later terms in (1·10) being more strongly damped than the earlier terms. The individual terms in (1·10) are often described as creeping modes, and the residue series as the creeping-mode expansion.

Next, let us consider the series (1·3) when $r > a$. When N is large the Hankel functions can be approximated by Airy functions which can be further approximated by exponentials. Thus, when $r > a$, it can be shown (cf. (2·20) below) that the mth term in (1·3) is approximated by a slowly varying function multiplied by

$$\exp\left[iN \int_1^\rho (\mu^2 - \xi_m^2)^{\frac{1}{2}} \frac{d\mu}{\mu}\right] \exp[iN\xi_m\theta], \tag{1·12}$$

the product of a radial factor and an angular factor. The initial terms (m fixed as $N \to \infty$) can be further approximated by using (1·8), and the radial factor is then seen to increase with ρ for fixed m, the angular factor to decrease with θ for fixed m. The product remains exponentially small provided that (r, θ) lies in the shadow, i.e. $r\cos\theta < a$. Moreover, the product is then a decreasing function of m for fixed r and θ. Thus successive terms of (1·3) can still be properly described as creeping modes, and they decrease as m increases, *provided that the initial approximation* (1·8) *is valid.* Otherwise the approximation (1·4) must be used, and the order of magnitude of (1·12) is then much more difficult to determine. It has in fact been shown by a complicated argument (Ursell (9)) that in the shadow of a circle the exponential factor (1·12) continues to decrease as m increases, in other words that

$$\Re\mathfrak{F}_1(\xi; \rho, \theta) \equiv \Re\left[i \int_1^\rho (\mu^2 - \xi^2)^{\frac{1}{2}} \frac{d\mu}{\mu} + i\xi\theta\right] \tag{1·13}$$

decreases as ξ moves along the pole locus

$$\Re \int_1^\xi (\xi^2 - \mu^2)^{\frac{1}{2}} \frac{d\mu}{\mu} = 0 \tag{1·14}$$

from $\xi = 1$ to $\xi = i\infty$, provided that $\rho\cos\theta < 1 < \rho$. (In this form the problem does not involve N.) Thus the residue series for the circle can be approximated by its initial terms at any fixed point in the shadow. This concludes our summary of the treatment given in (9).

However, as has been noted above, the circle is not typical. The corresponding problem for a source on an ellipse (involving Mathieu functions) has been studied by Leppington (5). He found that in the shadow the initial terms of the residue series

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decrease as m increases; these therefore represent creeping modes. To show whether the later terms are negligible it was necessary to investigate the function

$$\Re\mathfrak{F}_1(\xi;\alpha,\alpha_0,\eta,\eta_0) \equiv \Re\left[\int_{\cosh\alpha_0}^{\cosh\alpha}\left(\frac{\xi^2-\mu^2}{\mu^2-1}\right)^{\frac{1}{2}}d\mu + i\int_{\cos\eta}^{\cos\eta_0}\left(\frac{\xi^2-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}}d\mu\right], \quad (1{\cdot}15)$$

where ξ lies on the pole locus

$$\Re\int_{\cosh\alpha_0}^{\xi}\left(\frac{\xi^2-\mu^2}{\mu^2-1}\right)^{\frac{1}{2}}d\mu = 0, \quad (1{\cdot}16)$$

cf. (1·13), (1·14) above. Leppington found that in parts of the shadow the function (1·15) first decreases as ξ moves along the pole locus (as has already been noted), it then increases again to become positive, and thereafter it decreases again. In other words, certain intermediate terms of the residue series are exponentially large, and it can no longer be inferred that in these parts of the shadow (which are independent of N) the series is dominated by its initial terms. It cannot even be inferred that the amplitude in these parts of the shadow is exponentially small. (Note: in both (1·13) and (1·15) it is supposed that the pole locus and the zero locus do not intersect. This can be proved; Ursell (11).) The residue series is therefore more unsuitable than the original series for obtaining asymptotic estimates of the potential, such as are needed in developing the rigorous theory for arbitrary curves (see (9)). It is the purpose of the present paper to obtain such asymptotic estimates in the shadow. We have already expressed the potential as an integral along an open contour, see (1·2) above, but now this contour will not be closed at infinity. Instead, the open contour will be deformed across a finite number of poles to give a finite residue series with a remainder (a modified contour integral). It will be shown in section 2 below that the integrand in (1·2) can be closely approximated by exponentials provided that the poles of the integrand are suitably excluded. The exponentials change discontinuously across the pole locus and across the zero locus. In section 3 the contour of integration will be chosen so that the exponential bounds take their greatest value on the pole locus; this idea is familiar in the method of steepest descents, but in our problem we are dealing with discontinuous functions, and in fact the proof of the existence of suitable curves of descent is here the most difficult part of the argument. It is now a simple matter to find bounds (in section 4) for the remainder in the finite asymptotic series, and to show that it is smaller than the last term retained in the series if the number of terms is kept fixed while $N = ka$ tends to infinity. The proof is given in detail for the circle. It is, however, equally applicable to the ellipse. In other words, we find that the residue series obtained by Watson's transformation gives a valid asymptotic estimate, provided that it is cut off while the terms are exponentially small. The later terms (including the exponentially large ones) may be omitted.

2. *Statement of results. Exponential bounds.* We shall prove

THEOREM 1. *Let M be any fixed integer, and let (r,θ) be any fixed point in the shadow $r\cos\theta < a \leqslant r$. Then*

$$\Phi_c(kr,\theta;\,ka) = -\frac{2i}{ka}\sum_{\substack{m=1\\ \nu=\nu(m)}}^{M}\frac{H_\nu(kr)}{(\partial/\partial\nu)H_\nu'(ka)}\frac{\cos\nu(\pi-\theta)}{\sin\nu\pi} + R(r,\theta;\,M,ka), \quad (2{\cdot}1)$$

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where (asymptotically as $ka \to \infty$*) each term in the series is of smaller magnitude than the preceding one, and the remainder* R *is of smaller magnitude than the last term retained in the series, and where the summation is taken over the first* M *zeros*

$$\nu = \nu_m(ka) \quad (m = 1, 2, \ldots, M)$$

of the Hankel function derivative $H'_\nu(ka)$.

In fact the remainder is uniformly smaller than the last term retained if (r, θ) lies in any closed bounded region within the shadow.

The proof of this theorem is given later.

THEOREM 2. *The theorem analogous to Theorem* 1 *holds for the ellipse.*

The proof of this theorem is very similar to the proof of Theorem 1, and only an outline will be given, in section 5 below.

To explain the idea underlying the proof of Theorem 1 we observe that there is an explicit expression for the remainder $R(r, \theta; M, ka)$. For let the contour $\Xi_1\Xi_2$ of Figure 1 be moved across the lowest M poles; then

$$\Phi_c = -2\pi i \,(\text{sum of lowest } M \text{ residues}) + \int_{C_M(N)}, \tag{2·2}$$

where the contour $C_M(N) = C_M(ka)$, going from Ξ_1 to Ξ_2, crosses the pole locus between $\xi_M(ka)$ and $\xi_{M+1}(ka)$. Thus clearly

$$R(r, \theta; M, N) = \int_{C_M(N)} \frac{1}{\pi} \frac{H_{N\xi}(N\rho)}{H'_{N\xi}(N)} \frac{\cos N(\pi - \theta)\xi}{\sin N\pi\xi} d\xi, \tag{2·3}$$

where $N = ka$, $\rho = r/a$. Let the contour be chosen so as not to pass too near the poles of the integrand. The argument consists of two steps. First, the integrand can be approximated by an exponential of the form $\exp(N\mathfrak{F}(\xi; \rho, \theta))$ where, however, $\mathfrak{F}(\xi)$ takes different forms according as ξ lies to the left or right of the pole locus or of the zero locus. Secondly the contour $C_M(N)$ can be chosen to consist of three arcs such that the exponential bound has a sharp maximum at the pole locus. Theorem 3 will show how such a choice can be made. To show that the quotient $H_{N\xi}(N\rho)/H'_{N\xi}(N)$ can be approximated by exponentials we use the asymptotic formulae (see e.g. Olver (7))

$$H_\nu(\nu z) \sim \frac{2e^{-\frac{1}{3}\pi i}}{\nu^{\frac{1}{3}}} F_1(z) \operatorname{Ai}(\nu^{\frac{2}{3}}\zeta(z)\, e^{\frac{2}{3}\pi i}), \tag{2·4}$$

$$H'_\nu(\nu z) \sim \frac{2e^{\frac{1}{3}\pi i}}{\nu^{\frac{2}{3}}} \zeta'(z) F_1(z) \operatorname{Ai}'(\nu^{\frac{2}{3}}\zeta(z)\, e^{\frac{2}{3}\pi i}), \tag{2·5}$$

where the symbols are defined in the Appendix. Equations (2·4) and (2·5) are uniformly valid in unbounded regions of z when ν is large. It is next shown that the Airy functions Ai and Ai′ can be approximated by exponentials. For the function in the numerator this is almost obvious since for large $|z|$

$$\operatorname{Ai}(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \exp(-\tfrac{2}{3}z^{\frac{3}{2}}) \quad (|\arg z| \leqslant \pi - \delta); \tag{2·6}$$

$$\operatorname{Ai}(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \{\exp(-\tfrac{2}{3}z^{\frac{3}{2}}) + i \exp(\tfrac{2}{3}z^{\frac{3}{2}})\} \quad (\tfrac{2}{3}\pi < \arg z < \tfrac{4}{3}\pi). \tag{2·7}$$

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Also $\mathrm{Ai}(z)$ is regular near $z = 0$. On combining these results it is readily seen that

$$|\mathrm{Ai}(z)| \leqslant \frac{A_1}{1+|z|^{\frac{1}{4}}} |\exp(-\tfrac{2}{3}z^{\frac{3}{2}})| \quad (-\pi \leqslant \arg z \leqslant \pi), \tag{2·8}$$

for (2·6) shows that when z is large this holds when $|\arg z| \leqslant \frac{2}{3}\pi$, while

$$|\exp(\tfrac{2}{3}z^{\frac{3}{2}})| \leqslant \exp(-\tfrac{2}{3}z^{\frac{3}{2}})| \quad \text{when} \quad \tfrac{2}{3}\pi \leqslant \arg z \leqslant \pi;$$

also (2·8) holds, trivially, when z is not large. Similarly

$$|\mathrm{Ai}'(z)| \leqslant A_2(1+|z|^{\frac{1}{4}})|\exp(-\tfrac{2}{3}z^{\frac{3}{2}})| \quad (-\pi \leqslant \arg z \leqslant \pi). \tag{2·9}$$

In the denominator the function $H'_{N\xi}(N)$ is approximated by the Airy-function derivative Ai′ (see (2·5) above), and for this a positive *lower bound* is required which is possible only if the zeros of Ai′ are suitably avoided. Suppose, then, that certain non-overlapping neighbourhoods of all the zeros $z = -|a'_m|$ $(m = 1, 2, 3, \ldots)$ are removed from the z-plane. If these neighbourhoods are suitably chosen, it can be shown that

$$|\mathrm{Ai}'(z)| > A_3(1+|z|^{\frac{1}{4}})|\exp(-\tfrac{2}{3}z^{\frac{3}{2}})| \quad (-\pi \leqslant \arg z \leqslant \pi) \tag{2·10}$$

in the cut-out plane; a detailed proof is given in the appendix. The equations (2·9) and (2·10) together show that $(1+|z|^{\frac{1}{4}})|\exp(-\frac{2}{3}z^{\frac{3}{2}})|$ gives the true order of magnitude of $|\mathrm{Ai}'(z)|$. The definition of the cut-out plane can be varied considerably without changing the form of the results (2·10). The cut-out neighbourhoods need only to be small enough not to overlap, and large enough for $\mathrm{Ai}'(z)$ to be suitably bounded away from zero. (See appendix 2.) We observe that the exponent in (2·10) is not a single-valued function of z as z crosses the line of zeros $\arg z = \pi$. We find by the foregoing argument that, for instance,

$$|\mathrm{Ai}'(z)| < A_2(1+|z|^{\frac{1}{4}})|\exp(\tfrac{2}{3}z^{\frac{3}{2}})| \quad (\pi \leqslant \arg z \leqslant 3\pi). \tag{2·11}$$

Note the change in sign from (2·10) to (2·11) when the line of zeros is crossed.

These arguments can be applied to the Hankel functions. It is only necessary to take the argument of the Airy functions on the right of (2·4) and (2·5) as the new independent variable. Details are given in the appendix.

The factor $\cos N(\pi-\theta)\,\xi/\sin N\pi\xi$ in (2·3) remains to be considered. Let the circles $|N\xi - m| < \frac{1}{4}$ $(m = 0, \pm 1, \pm 2, \ldots)$ be removed from the complex ξ-plane. Then in the upper half $\Im\xi > 0$ of the cut-out ξ-plane we have

$$A_4|\exp(iN\theta\xi)| < \left|\frac{\cos N(\pi-\theta)\,\xi}{\sin N\pi\xi}\right| < A_5|\exp(iN\theta\xi)| \tag{2·12}$$

when $0 < \theta < \pi$. The proof is elementary.

We now introduce some definitions. Consider the first quadrant Q in the ξ-plane. Equation (2·5) shows that the zeros $\xi_m(N)$ of $H'_{N\xi}(N)$ lie near the pole locus P on which $\arg \xi^{\frac{3}{2}}\zeta(1/\xi) = \frac{1}{3}\pi$, i.e. on which

$$\Re\int_1^{\xi} (\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu} = 0; \tag{2·13}$$

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this is independent of N. Similarly the zeros of $H_{N\xi}(N\rho)$ lie near the zero locus Z on which

$$\Re\int_{\rho}^{\xi}(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu}=0. \tag{2·14}$$

The cut-out quadrant $Q_{-}(N)$ is defined by excluding from Q the neighbourhoods $|N\xi-m|<\frac{1}{4}(m=0,1,2,\ldots)$ of the zeros of $\sin N\pi\xi$, and also suitable non-overlapping neighbourhoods of the zeros $\xi_m(N)\,(m=1,2,3,\ldots)$ of $H'_{N\xi}(N)$. The neighbourhoods are chosen so large that $\sin N\pi\xi$ and $H'_{N\xi}(N)$ do not become too small in $Q_{-}(N)$ for the integrand of (2·3) to be approximated by exponential bounds; and they are chosen so small that there is a large enough gap between successive neighbourhoods for the contour to be suitably drawn. That such a construction is possible is shown in detail in the appendix. We next define the regions $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ (see Figure 2) and the corresponding cut-out regions $D^{(1)}_{-}(N)$, $D^{(2)}_{-}(N)$, $D^{(3)}_{-}(N)$. The regions $D^{(1)}$ and $D^{(1)}_{-}(N)$ are

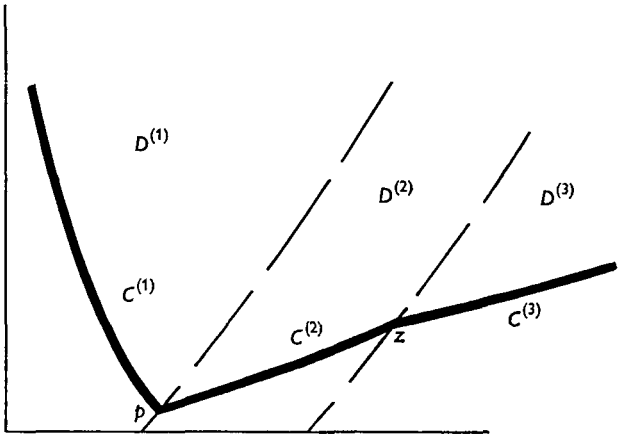


Fig. 2. The regions $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, and the arcs $C^{(1)}$, $C^{(2)}$, $C^{(3)}$.

respectively that part of Q and of $Q_{-}(N)$ lying to the left of both the pole locus P and the zero locus Z; the region $D^{(2)}$ (the region $D^{(2)}_{-}(N)$) is the part of Q (the part of $Q_{-}(N)$) lying to the right of P and to the left of Z, and below their lowest point of intersection (if any); the region $D^{(3)}$ (the region $D^{(3)}_{-}(N)$) is the part of Q (the part of $Q_{-}(N)$) lying to the right of both P and Z. It is known that in fact P and Z do not intersect (Ursell (11)) but we do not need to make use of this rather difficult result.

We now suppose that the contour $C_M(N)$ of equation (2·3) lies in the cut-out quadrant $Q_{-}(N)$.

The integrand

$$\begin{aligned}\frac{1}{\pi}\frac{H_{N\xi}(N\rho)}{H'_{N\xi}(N)}\frac{\cos N(\pi-\theta)\,\xi}{\sin N\pi\xi}&\\ \sim\frac{1}{\pi}(N\xi)^{\frac{1}{3}}e^{-\frac{2}{3}\pi i}\frac{F_1(\rho/\xi)\,\mathrm{Ai}\,(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(\rho/\xi)\,e^{\frac{2}{3}\pi i})\cos N(\pi-\theta)\,\xi}{\zeta'(1/\xi)\,F_1(1/\xi)\,\mathrm{Ai}'\,(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(1/\xi)\,e^{\frac{2}{3}\pi i})\sin N\pi\xi}&\\ =\frac{e^{\frac{1}{3}\pi i}}{2\pi}N^{\frac{1}{3}}\xi^{-\frac{2}{3}}F_1(1/\xi)\,F_1(\rho/\xi)\frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(\rho/\xi)\,e^{\frac{2}{3}\pi i})\cos N(\pi-\theta)\,\xi}{\mathrm{Ai}'\,(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(1/\xi)\,e^{\frac{2}{3}\pi i})\sin N\pi\xi},&\end{aligned} \tag{2·15}$$

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as in Ursell (9), Appendix 1. The function $F_1(1/\xi)$ is regular and non-zero near $\xi = 1$, and varies more slowly than any power of ξ near $\xi = \infty$. And from (2·8) we have

$$|\mathrm{Ai}\,(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(\rho/\xi)\,e^{\frac{2}{3}\pi i})| < \frac{A_1}{1+|N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(\rho/\xi)|^{\frac{1}{4}}}\,|\exp\{\pm\tfrac{2}{3}N\xi\zeta^{\frac{3}{2}}(\rho/\xi)\}| \tag{2·16}$$

$$< A_1\left|\exp\left\{\pm N\int_\rho^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu}\right\}\right|, \tag{2·17}$$

where the sign depends on whether ξ lies to the left or right of the zero locus. Similarly in $Q_-(N)$ we have

$$|\mathrm{Ai}'\,(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(1/\xi)\,e^{\frac{2}{3}\pi i})| > A_3\,(1+|N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(1/\xi)|^{\frac{1}{4}})\left|\exp\left\{\pm N\int_1^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu}\right\}\right|$$

$$> A_3\left|\exp\left\{\pm N\int_1^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu}\right\}\right|, \tag{2·18}$$

and from (2·12) we have

$$\left|\frac{\cos N(\pi-\theta)\,\xi}{\sin N\pi\xi}\right| < A_5\,|\exp(iN\theta\xi)|.$$

Thus finally in $Q_-(N)$

$$\frac{1}{\pi}\left|\frac{H_{N\xi}(N\rho)}{H'_{N\xi}(N)}\,\frac{\cos N(\pi-\theta)\,\xi}{\sin N\pi\xi}\right| \tag{2·19}$$

$$< \frac{A_6N^{\frac{1}{3}}\,|\xi^{-\frac{2}{3}}F_1(1/\xi)\,F_1(\rho/\xi)|}{(1+|N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(\rho/\xi)|^{\frac{1}{4}})\,(1+|N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(1/\xi)|^{\frac{1}{4}})}$$

$$\times\left|\exp\left\{N\left(i\theta\xi\pm\int_\rho^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu}\pm\int_1^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu}\right)\right\}\right|, \tag{2·20}$$

where the signs are appropriately chosen. Our argument shows that (2·20) in fact gives the true order of magnitude of (2·19) in $Q_-(N)$. (Actually for most purposes the denominator may be omitted.) Further the expression (2·20) is seen to be the product of a slowly varying function of ξ, and of a rapidly varying exponential. It is the latter factor that dominates the magnitude of the integrand (2·19), and we shall not need to mention the former factor until equation (4·1) below.

The appropriate signs in the exponential (2·20) still remain to be determined from the asymptotic formulae for the Airy functions in (2·4) and (2·5), and this is readily done. (The details are omitted.) The results are as follows.

In the cut-out region $D_-^{(1)}(N)$ the magnitude of the exponential (2·20) is

$$|\exp(N\mathfrak{F}_1(\xi))| = \exp(\Re N\mathfrak{F}_1(\xi)),$$

where

$$\mathfrak{F}_1(\xi;\rho,\theta) = i\xi\theta + i\int_\xi^\rho(\mu^2-\xi^2)^{\frac{1}{2}}\frac{d\mu}{\mu} - i\int_\xi^1(\mu^2-\xi^2)^{\frac{1}{2}}\frac{d\mu}{\mu}, \tag{2·21}$$

and where the square roots are real and positive when $0 < \xi < 1$. Similarly in the cut-out regions $D_-^{(2)}(N)$ and $D_-^{(3)}(N)$ the magnitude of the exponential is given by $|\exp(N\mathfrak{F}_2(\xi))|$ and $|\exp(N\mathfrak{F}_3(\xi))|$ respectively, where

$$\mathfrak{F}_2(\xi;\,\rho,\theta) = i\xi\theta + i\int_\xi^\rho(\mu^2-\xi^2)^{\frac{1}{2}}\frac{d\mu}{\mu} - \int_1^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu} \tag{2·22}$$

and

$$\mathfrak{F}_3(\xi;\rho,\theta) = i\xi\theta + \int_\rho^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu} - \int_1^\xi(\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu} \tag{2·23}$$

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and where the square roots in (2·22) and (2·23) are positive when $1 < \xi < \rho$ and when $\rho < \xi < \infty$ respectively. Note that $\Re\mathfrak{F}_1(\xi) = \Re\mathfrak{F}_2(\xi)$ along the pole locus, and that $\Re\mathfrak{F}_2(\xi) = \Re\mathfrak{F}_3(\xi)$ along the zero locus.

3. *Curves of descent.* Now let the contour $C_M(N)$ defined at the beginning of section 2 be deformed (if necessary) into a contour $C_M^{(1)}(N) + C_M^{(2)}(N) + C_M^{(3)}(N)$, consisting of 3 arcs: $C_M^{(1)}(N)$ lying in $D^{(1)}(N)$, $C_M^{(2)}(N)$ lying in $D^{(2)}(N)$, and $C_M^{(3)}(N)$ lying in $D^{(3)}(N)$, as shown in Figure 2. Suppose that this contour crosses the pole locus at the point $p = p_M(N)$ between $\xi_M(N)$ and $\xi_{M+1}(N)$, and the zero locus at the point $z_M(N)$. Then it will be shown (and this is the decisive step in the argument) that the arcs can be chosen so that $\Re\mathfrak{F}_1(\xi)$ decreases monotonically on $C^{(1)}$ as ξ varies from p to ∞, and that $\Re\mathfrak{F}_2(\xi)$ decreases on $C^{(2)}$ as ξ varies from p to z, and that $\Re\mathfrak{F}_3(\xi)$ decreases on $C^{(3)}$ as ξ varies from z to ∞. Thus the integrands (2·19) contain rapidly decreasing factors of the form $\exp(N\mathfrak{F}(\xi))$, as in the well-known method of steepest descents (see e.g. Watson (13), section 8·3; or Jeffreys (2), section 2·3), and it follows that the main contribution to the integral (2·3) comes from the neighbourhood of the point p. However, p is not a saddle point of either $\mathfrak{F}_1(\xi)$ or $\mathfrak{F}_2(\xi)$, nor are $C^{(2)}$ and $C^{(3)}$ curves of steepest descent. The conventional steepest-descents argument is not applicable because the exponent (2·20) changes in a non-analytic manner when the contour of integration crosses the pole locus or the zero locus. We now proceed to give proofs of the statements made above. The theorem is divided into three parts, relating to the regions $D^{(1)}$, $D^{(2)}$ and $D^{(3)}$ respectively.

THEOREM 3·1. *Suppose that (r, θ) is a fixed point in the shadow, and suppose that the point p lies on the pole locus sufficiently near to the point $\xi = 1$. Then in the region $D^{(1)}$ there exist infinitely many arcs $C^{(1)}$ (starting at p and ending at ∞) such that $\Re\mathfrak{F}_1(\xi)$ decreases monotonically to $-\infty$ as $\xi \to \infty$ on $C^{(1)}$.*

If N is large enough and if p is further restricted to the part of the pole locus lying on the boundary of the cut-out region $D^{(1)}(N)$ between $\xi_M(N)$ and $\xi_{M+1}(N)$, then such arcs $C_M^{(1)}(N)$ can be chosen in infinitely many ways to lie in the cut-out region $D^{(1)}(N)$.

Two proofs will be given. The first proof depends on the remarkable fact that the function $\mathfrak{F}_1(\xi)$ is univalent in the quadrant Q; i.e. that $\mathfrak{F}_1(\xi^*) \neq \mathfrak{F}_1(\xi^{**})$ if ξ^*, ξ^{**} are distinct points of the quadrant. Thus the function $\mathfrak{F}_1(\xi)$ gives a conformal *mapping* of the first quadrant of the ξ-plane on a quadrantal region in the $\mathfrak{F}_1$-plane. It is a simple matter to sketch the configuration of the boundary and the proof of the theorem is then geometrically obvious. (Similarly in the other two cases.) The second proof depends on the fact (evident from the first proof) that a suitable arc $C^{(1)}$ can be chosen arbitrarily close to the real and imaginary ξ-axes. A local mapping is used.

Proof of Theorem 3·1.

First proof. By a well-known theorem of conformal mapping (Titchmarsh (8), section 6·45) the mapping of the regions is (1, 1) if the mapping of the boundaries is (1, 1). Consider the function

$$\mathfrak{F}_1(\xi) = i\xi\theta + i\int_1^{\rho} (\mu^2 - \xi^2)^{\frac{1}{2}} \frac{d\mu}{\mu},$$

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where the positive square root is taken when $0 < \xi < 1$. This defines a function regular in the quadrant.

On bc, Figure 3,

$$\Re\mathfrak{F}_1(\xi) = 0, \quad \mathfrak{F}_1'(\xi) = i\theta - i\int_1^\rho \frac{\xi}{(\mu^2-\xi^2)^{\frac{1}{2}}}\frac{d\mu}{\mu},$$

$$\mathfrak{F}_1''(\xi) = -i\int_1^\rho \frac{\mu}{(\mu^2-\xi^2)^{\frac{3}{2}}}d\mu.$$

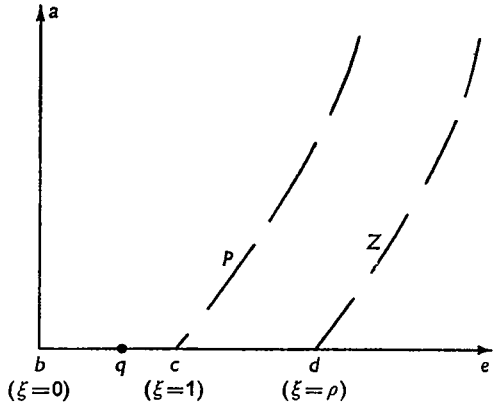
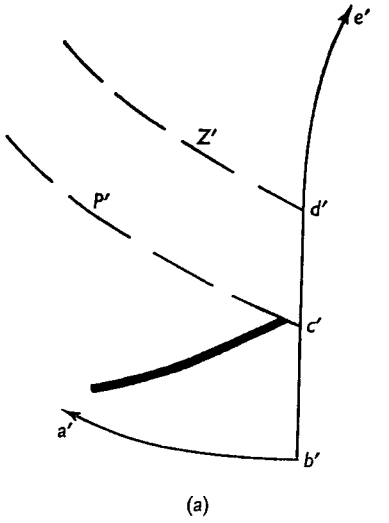


Fig. 3. The ξ-plane. The point q is a zero of $\mathfrak{F}_2'(\xi)$.



(a)

Fig. 4(a). The $\mathfrak{F}_1$-plane. Points on P' near c' can be joined to a' by arcs on which $\Re\mathfrak{F}_1$ decreases, e.g. the arc shown.

The last equation shows that $\Im\mathfrak{F}_1''(\xi) < 0$, i.e. that $\Im\mathfrak{F}_1'(\xi)$ is a decreasing function of ξ on bc. Also $\Im\mathfrak{F}_1(0) = \theta > 0$, whence $\Im\mathfrak{F}_1'(\xi) > 0$ on bc if $\Im\mathfrak{F}_1'(1) > 0$; this turns out to be the condition that (r, θ) lies in the shadow. Thus

$$\Im\mathfrak{F}_1'(\xi) > 0 \quad \text{on} \quad bc(0 < \xi < 1). \tag{3·1}$$

On cd,

$$\mathfrak{F}_1(\xi) = i\xi\theta + \int_1^\xi (\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu} + i\int_\xi^\rho (\mu-\xi^2)^{\frac{1}{2}}\frac{d\mu}{\mu},$$

where the sign of the second term is fixed by the condition that $\mathfrak{F}_1(\xi) = \mathfrak{F}_1(\xi + i0)$. Thus

$$\Re\mathfrak{F}_1'(\xi) = \int_1^\xi \frac{\xi}{(\xi^2-\mu)^{\frac{1}{2}}}\frac{d\mu}{\mu} > 0 \quad \text{on} \quad cd.$$

On de,

$$\mathfrak{F}_1(\xi) = i\xi\theta + \int_1^\rho (\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu},$$

whence

$$\Re\mathfrak{F}_1'(\xi) = \int_1^\rho \frac{\xi}{(\xi^2-\mu^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} > 0 \quad \text{on} \quad de, \tag{3·2}$$

$$\Im\mathfrak{F}_1'(\xi) = \theta > 0 \quad \text{on} \quad de.$$

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On ab,
$$\mathfrak{F}_1(i\eta) = -\eta\theta + i\int_1^\rho (\mu^2+\eta^2)^{\frac{1}{2}}\frac{d\mu}{\mu},$$

whence
$$\frac{\partial}{\partial\eta}\Re\mathfrak{F}_1(i\eta) = -\theta < 0 \quad \text{on} \quad ab \tag{3·3}$$

$$\frac{\partial}{\partial\eta}\Im\mathfrak{F}_1(i\eta) = \int_1^\rho \frac{\eta}{(\mu^2+\eta^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} > 0 \quad \text{on} \quad ab. \tag{3·4}$$

At infinity,
$$\mathfrak{F}_1(\xi) \sim i\xi\theta + \xi\int_1^\rho \frac{d\mu}{\mu}$$

is linear, so the mapping is regular there. These results show that $\mathfrak{F}_1'(\xi)$ does not vanish on the boundary, and the equations (3·1), (3·2), (3·3) and (3·4) show that the boundary *abcde* of Figure 3 is mapped on the curve $a'b'c'd'e'$ of Figure 4 (a) (closed at infinity), which does not intersect itself. Thus the mapping of the boundaries is (1, 1), and the quadrant in the ξ-plane is mapped on the region lying to the left of and above the boundary of figure 4 (a). In particular the pole locus P and the zero locus Z are mapped on curves P', Z' lying in this region. It is easily shown that P and Z do not intersect the coordinate axes in the ξ-plane, and that at ∞ their distances from ab tend to infinity. The corresponding statements therefore hold for their images in Figure 4 (a). We further note that the pole locus P' makes an angle $\frac{2}{3}\pi$ with $c'b'$ at c'.

Now let us start on P', at any point p' sufficiently near c'. It is obvious that in Figure 4 (a) we can draw a curve C' lying between $b'a'$ and P', such that $\Re\mathfrak{F}_1$ decreases steadily on it, and such that it does not approach P' closely at ∞. (In figure 4 (a) the point p' must clearly lie to the right of any vertical tangent to the curve P'.) Having drawn a suitable curve C' in Figure 4 (a) we obtain a curve $C^{(1)}$ in the ξ-plane by drawing its inverse image in Figure 3. The construction can evidently be carried out in infinitely many ways and depends essentially on the fact that $\Re\mathfrak{F}_1 \to -\infty$ monotonically along $b'a'$.

The corresponding statements about $C_M^{(1)}(N)$ can be proved in the same way. It is only necessary to choose the parameter N so large that $\xi_{M+1}(N)$ lies sufficiently close to $\xi = 1$. Also $C_M^{(1)}(N)$ can be drawn to make a positive non-zero angle ($\geqslant \frac{1}{4}\pi$, say) with the direction cb.

Second proof. This is a localized variant of the first proof, and is only given in outline. The first proof shows that a suitable curve C' can be drawn to lie close to $c'b'a'$ (if p' lies close to c'); it may actually be made to coincide with $b'a'$ except near $c'b'$. It is therefore sufficient to show that $\mathfrak{F}_1(\xi)$ gives a mapping of a strip near bc, as will now be done. We have

$$\Im\mathfrak{F}_1'(\xi) = \theta - \int_1^\rho \frac{\xi}{(\mu^2-\xi^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} \quad \text{on} \quad bc, \quad \text{Figure 3},$$

$$\Im\mathfrak{F}_1'(\xi) = \theta - \int_\xi^\rho \frac{\xi}{(\mu^2-\xi^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} \quad \text{on} \quad cd,$$

i.e. $\Im\mathfrak{F}_1'(\xi) > B_1 > 0$ on a segment $0 \leqslant \Re\xi \leqslant 1+B_2$ of the real ξ-axis. (Here $B_1, B_2, \ldots$ denote positive constants.) By continuity $\Im\mathfrak{F}_1'(\xi) > \frac{1}{2}B_1 > 0$ in a rectangle

$$0 \leqslant \Re\xi \leqslant 1+B_2, \quad 0 \leqslant \Im\xi \leqslant B_3;$$

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this rectangle clearly contains the lower part of the pole locus. The function $\mathfrak{F}_1(\xi)$ is univalent in the rectangle, for let ξ^*, ξ^{**} denote any two distinct points in the rectangle, then

$$\mathfrak{F}_1(\xi^{**}) - \mathfrak{F}_1(\xi^*) = \int \frac{d\mathfrak{F}_1}{d\xi} d\xi = \int \frac{d\mathfrak{F}_1}{d\xi} |d\xi| \, e^{i\alpha},$$

when the integration is taken along the straight line of slope α joining ξ^* and ξ^{**}. Thus

$$\Im[e^{-i\alpha}\{\mathfrak{F}_1(\xi^{**}) - \mathfrak{F}_1(\xi^*)\}] = \int \left(\Im \frac{d\mathfrak{F}_1}{d\xi}\right) |d\xi| > \tfrac{1}{2}B_1|\xi^{**} - \xi^*| > 0,$$

and it follows that $\mathfrak{F}_1$ maps the interior of the rectangle on a strip-like region lying in the bottom corner of Figure 4 (a). Since the image $b'c'$ is vertical, and since P' makes an angle $\frac{2}{3}\pi$ with $c'b'$ at c', it is evident (cf. Figure 4 (a)) that any point near enough to b' on the lower edge $a'b'$ can be joined to any point near enough to c' on P' by a curve on which $\Re\mathfrak{F}_1$ decreases and which lies in the strip. The inverse image of this curve in the ξ-plane may be taken as part of the curve $C^{(1)}$. The curve can be continued to infinity along the imaginary ξ-axis on which $\Re\mathfrak{F}_1(\xi)$ is known to decrease (see Figure 4 (a)).

The second part of the theorem follows as before.

THEOREM 3·2. *Suppose that (r, θ) is a fixed point in the shadow, and suppose that the point p lies on the pole locus sufficiently near to the point $\xi = 1$. Then in the region $D^{(2)}$ there exist infinitely many arcs $C^{(2)}$ (starting at p and ending at a point z near the lower end of the zero locus) such that $\Re\mathfrak{F}_2(\xi)$ decreases monotonically as goes from p to z on the arc. As in Theorem 3·1 the arcs $C^{(2)}$ can be chosen to lie in the cut-out region $D^{(2)}_-(N)$.*

THEOREM 3·3. *Similarly in the region $D^{(3)}$ there exist infinitely many arcs $C^{(3)}$ (starting at z near the lower end of the zero locus and ending at ∞) such that $\Re\mathfrak{F}_3(\xi)$ decreases monotonically to $-\infty$ on the arc. The arcs can be chosen to lie in $D^{(3)}_-(N)$.*

Proof of Theorems 3·2 and 3·3. These proceed on the same lines as for Theorem 3·1. From (2·22)

$$\mathfrak{F}_2(\xi; \rho, \theta) = i\xi\theta + i\int_\xi^\rho (\mu^2 - \xi^2)^{\frac{1}{2}} \frac{d\mu}{\mu} - \int_1^\xi (\xi^2 - \mu^2)^{\frac{1}{2}} \frac{d\mu}{\mu}.$$

It can be shown, as before, that when $0 < \xi < 1$, $\Im\mathfrak{F}_2'(\xi)$ is an increasing function of ξ. When $\xi = 0$, its value is $\theta - \pi < 0$, when $\xi = 1$, its value is $\theta - \cos^{-1}\rho > 0$. Thus $\Im\mathfrak{F}_2'(\xi)$ has one zero q on the segment $0 < \xi < 1$; at this point in the image of the segment turns back on itself. When $1 < \xi < \rho = r/a$,

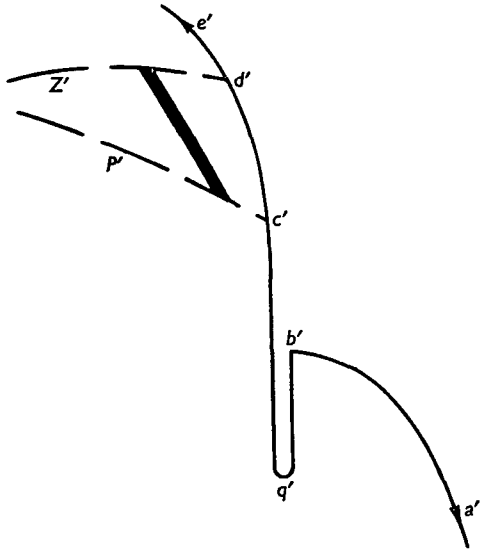
$$\Re\mathfrak{F}_2(\xi) = -\int_1^\xi (\xi^2 - \mu^2)^{\frac{1}{2}} \frac{d\mu}{\mu};$$

when $\rho < \xi < \infty$, $\Re\mathfrak{F}_2(\xi) = -\int_1^\xi (\xi^2 - \mu^2)^{\frac{1}{2}} \frac{d\mu}{\mu} - \int_\rho^\xi (\xi^2 - \mu^2)^{\frac{1}{2}} \frac{d\mu}{\mu}.$

Both these are evidently decreasing functions of ξ. At $\xi = 0$ a more detailed investigation shows that $\mathfrak{F}_2(\xi)$ is a regular mapping, and that $\Re\mathfrak{F}_2(\xi)$ increases along ba. At ∞, $\mathfrak{F}_2(\xi)$ is comparable to $\xi \ln \xi$. Thus $\mathfrak{F}_2(\xi)$ maps the first quadrant of the ξ-plane into the quadrantal region lying below the curve shown in Figure 4 (b). It is now obvious

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from the figure that points on P' near $c' = \mathfrak{F}_2(1)$ can be joined to points on Z' near $d' = \mathfrak{F}_2(\rho)$ by arcs C' on which $\Re\mathfrak{F}_2$ decreases. (This depends essentially on the fact that $\Re\mathfrak{F}_2$ decreases on $c'd'$.) The inverse image of such a curve is an arc $C^{(2)}$ in the ξ-plane which satisfies the conditions of Theorem 3·2. Evidently C' can be chosen to lie close to $c'd'$, and an alternative proof using local mappings can be based on this.



(*b*)

Fig. 4(*b*). The $\mathfrak{F}_2$-plane. Points on P' near c' can be joined to points on Z' near d' by arcs on which $\Re\mathfrak{F}_2$ decreases, e.g. the arc shown.

The same argument can be applied in the cut-out plane, provided that the Mth and $(M+1)$th cut-out neighbourhoods are small enough to allow a curve of negative slope to pass between them. The construction given in the appendix shows that this can be done, and that the slope (necessarily $< \frac{1}{3}\pi$) may be chosen to exceed $\frac{1}{4}\pi$.

Turning now to Theorem 3·3, and the function

$$\mathfrak{F}_3(\xi) = i\xi\theta - \int_\xi^\rho (\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu} + \int_\xi^1 (\xi^2-\mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu},$$

see equation (2·23) above, we observe that

on bc, $$\Im\mathfrak{F}_3'(\xi) = \theta + \int_1^\rho \frac{\xi}{(\mu^2-\xi^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} > 0,$$

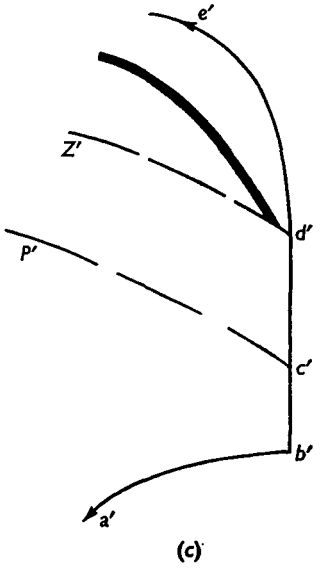
on cd, $$\Re\mathfrak{F}_3'(\xi) = -\int_1^\xi \frac{\xi}{(\xi^2-\mu^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} < 0, \quad \Im\mathfrak{F}_3'(\xi) = \theta + \int_\xi^\rho \frac{\xi}{(\mu^2-\xi^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} > 0,$$

on de, $$\Re\mathfrak{F}_3'(\xi) = -\int_1^\rho \frac{\xi}{(\xi^2-\mu^2)^{\frac{1}{2}}}\frac{d\mu}{\mu} < 0, \quad \Im\mathfrak{F}_3'(\xi) = \theta > 0.$$

Also, along ab both $\Re\mathfrak{F}_3(\xi)$ and $\Im\mathfrak{F}_3(\xi)$ can be shown to increase. Thus the first quadrant of the ξ-plane is mapped on a quadrantal region in the $\mathfrak{F}_3$-plane, as shown in Figure 4(*c*).

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Evidently points on Z' near $d' = \mathfrak{F}_3(\rho)$ can be joined by arcs C' on which $\Re\mathfrak{F}_3$ decreases. This depends essentially on the fact that $\Re\mathfrak{F}_3$ decreases on $d'e'$. Once again the inverse image of any such curve is an arc C_3 in the ξ-plane satisfying the conditions of Theorem 3·3, and here also an alternative proof can be based on local mappings.



(c)

Fig. 4 (c). The $\mathfrak{F}_3$-plane. Points on Z' near d' can be joined to e' by arcs on which $\Re\mathfrak{F}_3$ decreases, e.g. the arc shown.

This concludes the proof of the three parts of Theorem 3. Two methods of proof were given. Without doubt the first method is the more powerful, and it has other applications (e.g. in proving that the pole locus for a circle or an ellipse has no double points, see appendix below). Nevertheless, the success of this method seems to be a lucky accident, for there is no obvious reason why the mapping functions $\mathfrak{F}_1(\xi)$, $\mathfrak{F}_2(\xi)$, $\mathfrak{F}_3(\xi)$ should be univalent in *the whole* of the first quadrant. The alternative proof makes use only of local properties that are clearly relevant to the geometrical properties under discussion.

4. *Proof of Theorem* 1. We are now in a position to make an asymptotic estimate of the contour integral in (2·3). Suppose that the contour $C = C_M^{(1)}(N) + C_M^{(2)}(N) + C_M^{(3)}(N)$ intersects the pole locus at a point $\xi_P(M, N)$ lying in the gap between the poles $\xi_M(N)$ and $\xi_{M+1}(N)$, and that it intersects the zero locus at $\xi_Z(M, N)$. If M is a fixed integer, and if N is large enough, the points $\xi_P(M, N)$, $\xi_Z(M, N)$ lie near enough to the real ξ-axis for Theorems 3·1, 3·2, and 3·3 to apply. We can therefore suppose that $C_M^{(1)}(N)$ is a line of descent from ξ_P for the function $\mathfrak{F}_1(\xi)$, that $C_M^{(2)}(N)$ is a line of descent from ξ_P for the function $\mathfrak{F}_2(\xi)$, and that $C_M^{(3)}(N)$ is a line of descent from ξ_Z for the function $\mathfrak{F}_3(\xi)$. Then, from (2·20),

$$\left|\int_{C^{(1)}}\right| \leqslant \int_{\xi_P}^{\infty} N^{\frac{1}{3}} |F(\xi, N, \rho)| \exp\left(N\Re\mathfrak{F}_1(\xi)\right) ds(\xi), \tag{4·1}$$

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where $F(\xi, N, \rho)$ is a slowly varying function of ξ and N. This integral can be estimated by standard methods. Let $\Re\mathfrak{F}_1(\xi)$ be taken as a new variable of integration. Since $-(d/ds)\,\Re\mathfrak{F}_1(\xi) > A > 0$ near ξ_P, we find that

$$\left|\int_{C^{(1)}}\right| < \frac{A}{N^{\frac{2}{3}}}\,|F^*(\xi_P, N, \rho)|\exp\{N\Re\mathfrak{F}_1(\xi_P)\},$$

where F^* is another slowly varying function. Also $N = N(M)$ was chosen so that $\Re\mathfrak{F}_1(\xi)$ decreases on the pole locus between $\xi = 1$ and $\xi = \xi_{M+1}(N)$, and it can be seen from (1·8) that

$$\exp\{N\Re\mathfrak{F}_1(\xi_P)\}/\exp\{N\Re\mathfrak{F}_1(\xi_M)\} < \exp\{-N^{\frac{1}{3}}A(M;\rho,\theta)\},$$

where $A(M;\rho,\theta)$ is positive and independent of N. But the Mth residue is of order $\exp\{N\Re\mathfrak{F}_1(\xi_M)\}$, except possibly for powers of N. Thus the integral along $C_M^{(1)}(N)$ is negligible compared to the Mth residue. Similarly the contributions from $C_M^{(2)}(N)$ and from $C_M^{(3)}(N)$ are negligible. By replacing M by $(M+1)$ we readily see that the remainder (2·3) is actually of the same order as the $(M+1)$th residue. This concludes the proof of Theorem 1.

We observe that the foregoing argument is still valid if ξ_P is near enough to $\xi = 1$ without tending to $\xi = 1$. In other words, Theorem 1 is still valid with any

$$M \leqslant NA(\rho,\theta),$$

where $A(\rho,\theta)$ is a suitable bound.

5. *Proof of Theorem* 2. This is the corresponding theorem for the ellipse. The details of the proof will not be given here. It has been verified that the same arguments are applicable. We have

$$\mathfrak{F}_1(\xi) = \int_{\cosh\alpha_0}^{\cosh\alpha}\left(\frac{\xi^2-s^2}{s^2-1}\right)^{\frac{1}{2}}ds + i\int_{\cos\eta}^{\cos\eta_0}\left(\frac{\xi^2-s^2}{1-s^2}\right)^{\frac{1}{2}}ds,$$

$$\mathfrak{F}_2(\xi) = -\int_{\cosh\alpha_0}^{\xi}\left(\frac{\xi^2-s^2}{s^2-1}\right)^{\frac{1}{2}}ds + i\int_{\xi}^{\cosh\alpha}\left(\frac{s^2-\xi^2}{s^2-1}\right)^{\frac{1}{2}}ds + i\int_{\cos\eta}^{\cos\eta_0}\left(\frac{\xi^2-s^2}{1-s^2}\right)^{\frac{1}{2}}ds,$$

$$\mathfrak{F}_3(\xi) = -\int_{\cosh\alpha_0}^{\cosh\alpha}\left(\frac{\xi^2-s^2}{s^2-1}\right)^{\frac{1}{2}}ds + i\int_{\cos\eta}^{\cos\eta_0}\left(\frac{\xi^2-s^2}{1-s^2}\right)^{\frac{1}{2}}ds.$$

There are 3 different cases, according as $0 < \cos\eta < \cos\eta_0$, or $0 < -\cos\eta < \cos\eta_0$, or $0 < \cos\eta_0 < -\cos\eta$. The details are complicated since, e.g., the mapping $\mathfrak{F}_2(\xi)$ has singularities at the 8 boundary points $\xi = \cosh\alpha, \cosh\alpha_0, \xi_q, 1, \cos\eta_0, \cos\eta, 0, \infty$. ($\xi_q$ is the real point at which $\mathfrak{F}_2'(\xi_q) = 0$.) In each case the variation of $\mathfrak{F}'(\xi)$ along the boundary is traced as in the proof of Theorem 1; when considering $\mathfrak{F}_2'(\xi)$ it is sometimes convenient to eliminate the variable limit of integration by means of the identity

$$\int_{\xi}^{\infty}\frac{\xi\,ds}{(\xi^2-s^2)^{\frac{1}{2}}(s^2-1)^{\frac{1}{2}}} = \int_0^1\frac{\xi\,dv}{(\xi^2-v^2)^{\frac{1}{2}}(1-v^2)^{\frac{1}{2}}},$$

obtained by putting $s = \xi v^{-1}$.

It is found that each of the functions $\mathfrak{F}_1(\xi)$, $\mathfrak{F}_2(\xi)$, $\mathfrak{F}_3(\xi)$ gives a mapping of the first quadrant on to a quadrantal region. Alternatively local mappings may be used, as in Theorem 3·1 above.

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Discussion. In the present paper we have considered the residue series obtained by Watson's transformation. This transformation has often been applied to the shadow regions formed by wave refraction (Jones (4), ch. 6) and wave diffraction ((4), ch. 8). In the earlier work this series received two different kinds of treatment. Either the series was truncated after its first few terms and the remaining terms were arbitrarily ignored; this is obviously unsatisfactory. Or the complete residue series was written down without detailed investigation. Recent work (5) has shown that some of its terms can be exponentially large in certain parts of the shadow, where the complete transformed series is thus even more slowly convergent than the original series and cannot provide any useful estimate of the magnitude of the potential.

In the present paper it has been shown that at any fixed point inside the shadow the sum of the series is in fact approximated correctly by its first few terms, and that the error (obtained explicitly in the form of a contour integral) is of smaller order than the last term retained in the truncated series. It is not difficult to see that this result remains valid in any fixed bounded region inside the shadow, and possibly even in certain regions lying near shadow boundaries and depending on N. Such regions are important in rigorous short-wave asymptotics (Ursell (9), Appendix A 4, (10)) and need further study. Residue series have also been used ((9), Appendix A 3) in the immediate neighbourhood of a source, and here too further study is needed. Only for a circle were these problems adequately treated in (9). In this atypical case the complete residue series could be used, but the methods were much more complicated than the methods of the present paper. The argument of the present paper may have other applications. The contour integral (1·2) and its analogues represent the field in the illuminated region (see e.g. Nussenzveig (6), p. 42) but now the exponential factor has saddle points on the real axis which are believed to account for the dominant terms in the asymptotic expansion. A rigorous proof of this assumption may now be possible. Further, the work of D. S. Jones on refraction ((3), p. 353, (4), p. 383) has disclosed a residue series which is applicable only in part of the shadow. His difficulty may be related to ours.

Finally, the work of Goodrich and Kazarinoff (1) deserves to be mentioned. They have demonstrated the convergence of the residue series, i.e. the behaviour of the terms for large ξ. Our work is concerned with the difficulties arising from exponentially large terms at intermediate values of ξ, and it suggests that for most applications the asymptotic properties are more important than the convergence properties Nevertheless, their study of convergence is of great intrinsic interest and of considerable difficulty.

I am indebted to Dr F. Leppington for bringing his discovery to my attention, and for a helpful correspondence.

APPENDICES

1. *Asymptotic results.* We have

$$H_\nu(\nu z) \sim \frac{2e^{-\frac{1}{3}\pi i}}{\nu^{\frac{1}{3}}} F_1(z)\,\mathrm{Ai}\,(\nu^{\frac{2}{3}}\zeta(z)\,e^{\frac{2}{3}\pi i}), \tag{A 1·1}$$

$$H'_\nu(\nu z) \sim \frac{2e^{\frac{1}{3}\pi i}}{\nu^{\frac{2}{3}}} \zeta'(z)\,F_1(z)\,\mathrm{Ai}'\,(\nu^{\frac{2}{3}}\zeta(z)\,e^{\frac{2}{3}\pi i}). \tag{A 1·2}$$

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These are uniformly valid in z when ν is large (see, for example, Olver (7)). In these expressions the function $\zeta(z)$ is regular near $z = 1$,

$$\tfrac{2}{3}\zeta^{\frac{3}{2}}(z) = \ln\frac{1+\sqrt{(1-z^2)}}{z} - \sqrt{(1-z^2)} = -\int_1^z \sqrt{(1-\eta^2)}\frac{d\eta}{\eta}; \tag{A 1·3}$$

$$F_1(z) = \left(\frac{4\zeta(z)}{1-z^2}\right)^{\frac{1}{4}} \quad \text{is real for real } z, \quad F_1(1) = 2^{\frac{1}{3}} \neq 0; \tag{A 1·4}$$

$$\zeta'(z) = -2z^{-1}\{F_1(z)\}^{-2}.$$

The Airy function is defined by

$$\mathrm{Ai}\,(z) = \frac{z^{\frac{1}{2}}}{3^{\frac{1}{2}}\pi} K_{\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}}), \tag{A 1·5}$$

where $K_{\frac{1}{3}}$ is the Bessel function of imaginary argument (Watson (13), p. 78). The Airy function (cf. ref. (7)) satisfies the equation

$$\mathrm{Ai}''(z) = z\,\mathrm{Ai}\,(z), \tag{A 1·6}$$

and is therefore a regular function for all finite z. An alternative representation

$$\mathrm{Ai}\,(z) = \frac{1}{\pi}\int_0^\infty \cos\,(sz + \tfrac{1}{3}s^3)\,ds \tag{A 1·7}$$

is valid for real z. The zeros of $\mathrm{Ai}\,(z)$ and of $\mathrm{Ai}'(z)$ are all real and negative. The zeros of $\mathrm{Ai}'\,(z)$ are denoted by a'_m, where $|a'_1| < |a'_2| < \ldots$. For large m

$$a'_m \sim -(\tfrac{3}{2}\pi m)^{\frac{2}{3}}, \quad a'_m - a'_{m+1} \sim \left(\frac{2\pi^2}{3m}\right)^{\frac{1}{3}}, \tag{A 1·8}$$

2. *The cut-out region for the Airy-function derivative* $\mathrm{Ai}'\,(z)$. Suitable neighbourhoods of the real negative zeros $a'_m\,(m = 1, 2, 3, \ldots)$ are to be removed in such a way that in the remainder of the z-plane the inequality

$$|\mathrm{Ai}'\,(z)| > A_3(1+|z|^{\frac{1}{4}})\,|\exp\,(-\tfrac{2}{3}z^{\frac{3}{2}})| \quad (-\pi \leqslant \arg z \leqslant \pi) \tag{A 2·1}$$

is satisfied. Now when $|z| \to \infty, |z_1| \to \infty$, we have the following asymptotic relations (Olver (7), Appendix)

$$\mathrm{Ai}'(z) \sim -\tfrac{1}{2}\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\exp\,(-\tfrac{2}{3}z^{\frac{3}{2}}) \quad (-(\pi-\delta) \leqslant \arg z \leqslant (\pi-\delta)), \tag{A 2·2}$$

$$\mathrm{Ai}'(-z_1) \sim \pi^{-\frac{1}{2}}z_1^{\frac{1}{4}}[\cos\,(\tfrac{2}{3}z_1^{\frac{3}{2}} - \tfrac{3}{4}\pi)\,R(\tfrac{2}{3}z_1^{\frac{3}{2}}) + \sin\,(\tfrac{2}{3}z_1^{\frac{3}{2}} - \tfrac{3}{4}\pi)\,S(\tfrac{2}{3}z_1^{\frac{3}{2}})] \quad (-\tfrac{1}{2}\pi \leqslant \arg z_1 \leqslant \tfrac{1}{2}\pi), \tag{A 2·3}$$

where

$$|R(\tfrac{2}{3}z_1^{\frac{3}{2}}) - 1| < \frac{A}{|z_1|^3}, \quad |S(\tfrac{2}{3}z_1^{\frac{3}{2}})| < \frac{A}{|z_1|^{\frac{3}{2}}}.$$

$$\tfrac{2}{3}|a'_n|^{\frac{3}{2}} - \tfrac{1}{4}\pi(4n-3) \to 0 \quad \text{as} \quad n \to \infty. \tag{A 2·4}$$

Clearly the inequality (A 2·1) holds trivially in a neighbourhood of the origin, say $|z| < \frac{1}{4}|a'_1|$. Also (A 2·2) shows that (A 2·1) holds when $-\frac{5}{6}\pi \leqslant \arg z \leqslant \frac{5}{6}\pi$ and $|z| \geqslant \frac{1}{4}|a'_1|$. Thus only the region $(-\frac{1}{6}\pi < \arg z_1 < \frac{1}{6}\pi,\ |z_1| \geqslant \frac{1}{4}|a'_1|)$ containing the zeros remains to be considered, where we have written $z_1 = -z$.

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In this region, consider the product $z_1^{-\frac{1}{4}}\operatorname{Ai}'(-z_1)$. Put $Z = \frac{2}{3}z_1^{\frac{3}{2}}$; this maps the region into the region

$$-\tfrac{1}{4}\pi < \arg Z < \tfrac{1}{4}\pi, \quad |Z| \geqslant \tfrac{1}{12}|a_1'|^{\frac{3}{2}}.$$

In the Z-plane the distance between successive zeros tends to π; see (A 2·4) above. Choose any l so small that the circular neighbourhoods $|Z - \frac{2}{3}|a_n'|^{\frac{3}{2}}| < l$ do not overlap, and cut from the Z-plane the neighbourhoods $|Z - \frac{2}{3}|a_n'|^{\frac{3}{2}}| < l$, where $n > n_0(l)$ and $n_0(l)$ is chosen so that the nth neighbourhood contains the smaller circular neighbourhoods $|Z - n\pi + \frac{3}{4}\pi| < \frac{1}{2}l$; cf. (A 2·4). Outside the latter neighbourhoods (and therefore outside the former neighbourhoods) we have $|e^{iZ}\cos(Z - \frac{3}{4}\pi)| > A_c(l)$ when $0 < \arg Z < \frac{1}{4}\pi$, $\Re Z \geqslant (n_0 - \frac{3}{4})\pi$, as is easily seen from periodicity. In the same region

$$|e^{iZ}\sin(Z - \tfrac{3}{4}\pi)| < A_s(l).$$

Next choose $X(x) \geqslant (n_0 - \frac{3}{4})\pi$ so large that $|R(Z)| > \frac{1}{2}$ and $|S(Z)| < \frac{1}{4}A_s(l)/A_c(l)$ whenever $\Re Z > X(l)$ and $0 < \arg Z < \frac{1}{4}\pi$. It follows from (A 2·3) that

$$|z_1^{-\frac{1}{4}}\operatorname{Ai}'(-z_1)| > \tfrac{1}{4}A_c(l)|e^{-iZ_1}|$$

when Z lies in the cut-out plane and $0 < \arg Z < \frac{1}{4}\pi$ and $\Re Z > X(l)$. The corresponding inequality holds when $-\frac{1}{4}\pi < \arg Z < 0$ and $\Re Z > X(l)$.

We have now shown that the inequality (A 2·1) holds when $-\frac{1}{4}\pi < \arg Z < \frac{1}{4}\pi$, except when both $|Z| > \frac{2}{3}|a_1'|^{\frac{3}{2}}$ and $\Re Z \leqslant X(l)$. In this remaining region there is a finite number ($\leqslant n_0(l)$) of zeros. Let these be excluded by arbitrary neighbourhoods; then clearly an inequality of the form (A 2·1) holds in the cut-out sector. Finally, transform back to the z-plane. This concludes the proof of the inequality.

We note that with a finite number of exceptions all the neighbourhoods in the Z-plane have the same radius l. The constant A_3 in (A 2·1) depends on l, which can be chosen arbitrarily small. The construction may be modified so that the first $(M+1)$ zeros (say) are excluded by *circular* neighbourhoods in the z-plane. Evidently various other modifications are possible.

3. *The cut-out region for the Hankel function* $H_{N\xi}'(N)$. Since N is large it is sufficient to consider the cut-out region for the Airy function $\operatorname{Ai}'(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(1/\xi)\,e^{\frac{2}{3}\pi i})$. The construction proceeds on the lines of the previous appendix. Apply consecutively the mappings

$$z_{(1)} = \tfrac{2}{3}\xi\zeta^{\frac{3}{2}}(1/\xi) = \int_1^{\xi}(\xi^2 - \mu^2)^{\frac{1}{2}}\frac{d\mu}{\mu}, \quad z_{(N)} = Nz_{(1)}.$$

The first mapping maps the quadrant Q on a quadrantal region $Q_{(1)}$ in the $z_{(1)}$-plane. (To show this, put $\theta = 0$, $\rho = 1$ in the proof of Theorem 3·2 above.) The second mapping is a simple magnification. Suppose that $N \geqslant 1$. Clearly the image $z_{(N)}(m)$ of the mth zero of $\operatorname{Ai}'(N^{\frac{2}{3}}\xi^{\frac{2}{3}}\zeta(1/\xi)\,e^{\frac{2}{3}\pi i})$ lies on the positive imaginary $z_{(N)}$-axis at the point $z_{(N)}(m) = \frac{2}{3}i|a_m'|^{\frac{3}{2}}$ which is independent of N. For large m successive zeros tend to become equidistant. Now cut out neighbourhoods of the zeros as in the Z-plane of the previous appendix which corresponds to the present $z_{(N)}$-plane. If the neighbourhoods in the N-plane are circular, their images in the ξ-plane become more and more nearly circular for fixed m as $N \to \infty$. If we wish we can make the first $(M+1)$ neighbourhoods

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in the $z_{(N)}$-plane exactly circular, and the corresponding radii can be made so small that contours making an angle smaller than $\frac{1}{6}\pi$ (say) with the pole locus can be made to pass through the gaps between the first $(M+1)$ neighbourhoods.

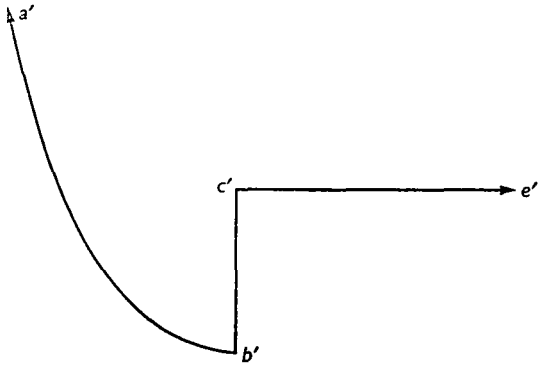


Fig. 5. The $z_{(1)}$-plane. The points a', b', c', e' are the images of the points a, b, c, e of Figure 3. The point c' is the origin $z_{(1)} = 0$ of the $z_{(1)}$-plane.

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